2/17 Mon

Test 1 - Weds, 3/4
covers chapter 7

8.3 continued...

Given $\sum_{k=1}^{\infty} a_k$ define the partial sums

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$\vdots \qquad \vdots$

$s_n = a_1 + a_2 + \dots + a_n$

$\sum_{k=}^{\infty}$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

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Geometric sum formula

If $r \neq 1$ then

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Why is this formula true?

$$(1 + r + r^2 + \cdots + r^n)(r - 1)$$

$$\begin{aligned} = \quad & r + r^2 + r^3 + \cdots + r^n + r^{n+1} \\ & -1 - r - r^2 - r^3 - \cdots - r^n \\ \hline & -1 + r^{n+1} \end{aligned}$$

So, $(1 + r + r^2 + \cdots + r^n)(r - 1) = r^{n+1} - 1$

Then divide by $r - 1$.

Ex: Does

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$ converge

or diverge?

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

partial sum

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \left(\frac{1}{2}\right)^2$$

$$S_3 = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$$

$$S_n = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n$$

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$$s_n = \frac{1}{2}\left[1+\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^{n-1}\right]$$

$$\boxed{r=\frac{1}{2}}$$

$$= \frac{1}{2}\left[\frac{\left(\frac{1}{2}\right)^n-1}{\frac{1}{2}-1}\right] = \frac{1}{2}\left[\frac{\left(\frac{1}{2}\right)^n-1}{\left(-\frac{1}{2}\right)}\right]$$

$$= -\left(\frac{1}{2}\right)^n+1 = 1-\left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k = \lim_{n\to\infty}\left[1-\left(\frac{1}{2}\right)^n\right] = 1-0 = 1.$$

$-1<\frac{1}{2}<1$

$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$

So, $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ converges to 1.

Geometric sums

Consider $\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \cdots$

where r is a constant.

The partial sums are

$$S_n = 1 + r + r^2 + \cdots + r^n$$

Case 1: $r = 1$

$$S_n = 1 + r + r^2 + \cdots + r^n$$
$$= \underbrace{1 + 1 + 1 + \cdots + 1}_{n} = n$$

So, $\lim_{n \to \infty} S_n = \lim_{n \to \infty} n = \infty$

Thus, $\sum_{k=0}^{\infty} 1^k = 1 + 1 + 1 + \cdots$ diverges

Ca

$S_n =$

$=$

$\lim_{n \to \infty}$

nges

Case 2: $r \neq 1$

$$s_n = 1 + r + r^2 + \cdots + r^n$$

$$= \frac{r^{n+1}-1}{r-1}$$

$$\lim_{n\to\infty} r^{n+1} = \begin{cases} 0, & \text{if } -1<r<1 \\ \text{diverges}, & \text{otherwise} \end{cases}$$

So,

$$\lim_{n\to\infty} \frac{r^{n+1}-1}{r-1} = \begin{cases} \frac{1}{1-r}, & -1<r<1 \\ \text{diverges}, & \text{otherwise} \end{cases}$$

$$\boxed{\frac{0-1}{r-1} = \frac{1}{1-r}}$$

Geometric series

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots = \frac{1}{1-r}$$

if $-1<r<1$. Otherwise the infinite sum diverges

Ex: $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots$

converge or diverge?

$r = \frac{2}{3}$, $-1 < r < 1$. So converges.

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-\frac{2}{3}} = \boxed{3}$$

If $-1 < r < 1$,
$1 + r + r^2 + \cdots = \frac{1}{1-r}$

Ex

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Ex: $\sum_{k=0}^{\infty}\left(\frac{\pi}{e}\right)^k = 1 + \frac{\pi}{e} + \left(\frac{\pi}{e}\right)^2 + \left(\frac{\pi}{e}\right)^3 + \cdots$

$r = \frac{\pi}{e} \approx 1.1557$

$r > 1$

So, $\sum_{k=0}^{k}\left(\frac{\pi}{e}\right)^k$ diverges

Ex: $\sum_{k=0}^{\infty}\left(-\frac{10}{3}\right)^k = 1 - \frac{10}{3} + \left(\frac{10}{3}\right)^2 - \left(\frac{10}{3}\right)^3 + \cdots$

$r = -\frac{10}{3} < -1$

diverges

Ex:

$= 1$

$S_1 =$

$S_2 =$

$S_3 =$

$S_4 =$

$S_5 =$

$\lim_{n \to \infty}$

$)^3 + \cdots$

<u>Ex:</u> $\sum_{k=0}^{\infty} (-1)^k$

$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - \cdots$

$s_1 = 1$

$s_2 = 1 - 1 = 0$

$s_3 = 1 - 1 + 1 = 1$

$s_4 = 1 - 1 + 1 - 1 = 0$

$s_5 = 1 - 1 + 1 - 1 + 1 = 1$

$\lim_{n \to \infty} s_n$ does not exist

$\sum_{k=0}^{\infty} (-1)^k$ diverges