2/25
Tuesday
week 6

8.5 continued...

Ratio test (From last time)

$\sum a_k$ is an infinite sum
where a_k is positive for all k.

Let $r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$

If $0 \le r < 1$, $\sum a_k$ converges.

If
If

$\lim_{k\to}$

e)

k.

ges

If $r=1$, the test is inconclusive.
If $r>1$ (including ∞), $\sum a_k$ diverges

Ex: Use the ratio test on $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+2} = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\left(\frac{1}{(k+1)+2}\right)}{\left(\frac{1}{k+2}\right)}$$

$$\lim_{k\to\infty} \left(\frac{k+2}{1}\right)\left(\frac{1}{k+3}\right)$$

$$= \lim_{k\to\infty} \frac{k+2}{k+3}$$

divide by k top/bottom

$$= \lim_{k\to\infty} \frac{1+\frac{2}{k}}{1+\frac{3}{k}}$$

$$= \frac{1+0}{1+0} = 1$$

Thus the test is inconclusive

Note that this series must diverge.

Suppose $\sum_{k=1}^{\infty} \frac{1}{k+2}$ converged to some number S.

Then

$$S = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

Then

$$1 + \frac{1}{2} + S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

But then

$$\frac{3}{2} + S = \sum_{k=1}^{\infty} \frac{1}{k}.$$

This

implies that the harmonic series converges, which we know is loco (the harmonic series diverges).

Thus, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+2}$ must diverge.

Ex: Does

$$\sum_{k=1}^{\infty} e^{-k}(k^2+4)$$

converge or diverge?

Note that

$$\underbrace{e^{-k}}_{>0}\underbrace{(k^2+4)}_{\geq 4} > 0 \quad \text{for all } k.$$

So we can use the ratio test.

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{e^{-(k+1)}\left((k+1)^2+4\right)}{e^{-k}(k^2+4)}$$

$$= \lim_{k\to\infty} \frac{e^{k}(k^2+2k+5)}{e^{k+1}(k^2+4)} = \frac{1}{e}\left(\lim_{k\to\infty} \frac{k^2+2k+5}{k^2+4}\right)$$

$$\boxed{e^{k}\cdot e^{1} = e^{k+1}}$$

$$= \frac{1}{e}\left(\lim_{k\to\infty} \frac{1+\frac{2}{k}+\frac{5}{k^2}}{1+\frac{4}{k^2}}\right) = \frac{1}{e}\left(\frac{1+0+0}{1+0}\right)$$

divide top/bottom by k^2

$$\frac{1}{e}(1) = \frac{1}{e} \approx 0.3678\ldots = r$$

Since $0 \le r < 1$

by the ratio test

$$\sum_{k=1}^{\infty} e^{-k}(k^2+4) \text{ converges.}$$

Totally sketchy intuition

If when k is large,

$$\frac{a_{k+1}}{a_k} \approx r, \text{ then}$$

$$a_{k+1} \approx r a_k \text{ and}$$

$$a_1 + a_2 + a_3 + \cdots \approx a_1 + r a_1 + r^2 a_1 + r^3 a_1 + \cdots$$

$$= a_1(1 + r + r^2 + \cdots)$$

Converges when $-1 < r < 1$.

Comparison Test

Let $\sum a_k$ and $\sum b_k$ be infinite series, both series only contain positive terms.

① If $0 < a_k \leq b_k$ for all k and $\sum b_k$ converges, then $\sum a_k$ converges.

② If $0 < b_k \le a_k$ for all k, and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Word version

① If you're smaller than a convergent series, then you converge.

② If you're bigger than a divergent series then you diverge.

Ex: Consider

$$\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$$

$\boxed{\text{divide top/bottom by } k^4}$

Divergence thm: $\lim_{k\to\infty} \frac{k^3}{2k^4-1} = \lim_{k\to\infty}\left(\frac{\frac{1}{k}}{2-\frac{1}{k^4}}\right)$

$= \frac{0}{2-0} = \frac{0}{2} = 0.$

So, the divergence thm doesn't apply.

Spider sense:

If k is large then

$$\frac{k^3}{2k^4-1} \approx \frac{k^3}{2k^4} = \frac{1}{2k}$$

So, $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$ should behave like $\sum_{k=1}^{\infty} \frac{1}{2k}$

$$\frac{k^3}{2k^4-1} > \frac{k^3}{2k^4} = \frac{1}{2k}.$$

$$\boxed{2k^4-1 < 2k^4}$$

And $\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

So, by the comparison test $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$ diverges.