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Thursday
Week 6

8.5 continued...

Limit comparison test

Let $\sum a_k$ and $\sum b_k$ be infinite series both $\underbrace{\text{with positive terms}}$

$a_k > 0$, $b_k > 0$ for all k

Let

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

① If $0 < L < \infty$
then either
both $\sum a_k$ and $\sum b_k$ converge

OR both $\sum a_k$ and $\sum b_k$ diverge

② If $L = 0$
and $\sum b_k$ converges,
then $\sum a_k$ converges.

③ If $L = \infty$
and $\sum b_k$ diverges
then $\sum a_k$ diverges.

Intuition behind ①

Suppose $L = \lim_{k\to\infty} \frac{a_k}{b_k}$
and $0 < L < \infty$.
This says that
when k is large

then $L \approx \frac{a_k}{b_k}$.

So, when k is large,

$$a_k \approx L\, b_k.$$

So, $\sum a_k \approx L \sum b_k$

only for large k

So, $\sum a_k$ and $\sum b_k$
either both converge
or both diverge.

Ex: Does $\sum_{k=1}^{\infty} \frac{42k^8 + 22k^6 - 1}{82k^{10} + 30{,}422}$ converge or diverge?

Does the divergence test apply?

$$\lim_{k\to\infty} \frac{42k^8 + 22k^6 - 1}{82k^{10} + 30{,}422} = \lim_{k\to\infty} \frac{\frac{42}{k^2} + \frac{22}{k^4} - \frac{1}{k^{10}}}{82 + \frac{30{,}422}{k^{10}}}$$

$\boxed{\text{divide top/bottom by } k^{10}}$

$$= \frac{0 + 0 - 0}{82 + 0} = 0$$

So the test doesnt apply.

Spider-sense

When k is large

$$\frac{42k^8 + 22k^6 - 1}{82k^{10} + 30{,}422} \approx \frac{42k^8}{82k^{10}} \approx \frac{21}{41} \cdot \frac{1}{k^2}$$

So our series probably behaves like $\sum \frac{1}{k^2}$.

Let's use the limit comparison test.

$$\lim_{k\to\infty} \frac{\left(\dfrac{42k^8+22k^6-1}{82k^{10}+30{,}422}\right)}{\left(\dfrac{1}{k^2}\right)}$$

$$= \lim_{k\to\infty} \left(\frac{k^2}{1}\right)\left(\frac{42k^8+22k^6-1}{82k^{10}+30{,}422}\right)$$

$$= \lim_{k\to\infty} \frac{42k^{10}+22k^8-k^2}{82k^{10}+30{,}422} = \lim_{k\to\infty} \frac{42+\frac{22}{k^2}-\frac{1}{k^8}}{82+\frac{30{,}422}{k^{10}}}$$

divide top/bottom by k^{10}

$$= \frac{42+0-0}{82+0}$$

$$= \frac{42}{82} = \boxed{\frac{21}{41} = L}$$

Since $0 < L < \infty$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges [p-series $p=2>1$]

this test shows that $\sum_{k=1}^{\infty} \frac{42k^8+22k^6-1}{82k^{10}+30{,}422}$ converges.

#46 **Ex:** Does $\sum_{k=1}^{\infty} \frac{2^k}{e^k - 1}$ converge or diverge?

Spider sense

When k is large,

$$\frac{2^k}{e^k-1} \approx \frac{2^k}{e^k} = \left(\frac{2}{e}\right)^k \approx (0.7357\ldots)^k$$

And $\sum_{k=1}^{\infty}\left(\frac{2}{e}\right)^k$ converge since its a geometric series with $r=\frac{2}{e}$ and $-1<r<1$.

Limit comparison test

$$\lim_{k\to\infty} \frac{\frac{2^k}{e^k-1}}{\left(\frac{2}{e}\right)^k} = \lim_{k\to\infty}\left(\frac{e^k}{\cancel{2^k}}\cdot\frac{\cancel{2^k}}{e^k-1}\right)$$

$$\lim_{k\to\infty}\frac{e^k}{e^k-1}$$

$$= \lim_{k\to\infty}\frac{1}{1-\frac{1}{e^k}}$$

$$= \frac{1}{1-0} = \boxed{1=L}$$

divide top/bottom by e^k

Since $0 < L < \infty$

and $\sum \left(\frac{2}{e}\right)^k$ converges,

by the limit comparison test

$$\sum_{k=1}^{\infty} \frac{2^k}{e^k - 1} \text{ converges.}$$