3/11
Weds
Week 8

9.2
continued

Ex: Find a power series representation and radius of convergence of

$$f(x) = \frac{x^3}{x+2}$$

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots$$

$$|r| < 1$$

f

power series
nd radius
of

$$f(x) = \frac{x^3}{x+2} = x^3\left[\frac{1}{2+x}\right] = x^3 \cdot \frac{1}{2}\left[\frac{1}{1+\frac{x}{2}}\right]$$

$$= \frac{x^3}{2}\left[\frac{1}{1-\left(-\frac{x}{2}\right)}\right] = \frac{x^3}{2}\left[1+\left(-\frac{x}{2}\right)+\left(-\frac{x}{2}\right)^2+\left(-\frac{x}{2}\right)^3+\cdots\right]$$

$$\left|-\frac{x}{2}\right| < 1 \qquad |x| < 2$$

$$\frac{|x|}{|2|} < 1 \qquad -2 < x < 2$$

$$\boxed{\frac{1}{1-r} = 1 + r + r^2 + \cdots \qquad |r| < 1}$$

$$= \frac{x^3}{2}\left[1 - \frac{x}{2} + \frac{x^2}{2^2} - \frac{x^3}{2^3} + \frac{x^4}{2^4} - \cdots\right]$$

$$= \boxed{\frac{x^3}{2} - \frac{x^4}{2^2} + \frac{x^5}{2^3} - \frac{x^6}{2^4} + \frac{x^7}{2^5} - \cdots}$$

$$\boxed{\begin{array}{l}\text{radius of convergence } R=2 \\ \quad -2 < x < 2\end{array}}$$

Other way:

$$f(x) = \frac{x^3}{2}\left[\frac{1}{1-\left(-\frac{x}{2}\right)}\right]$$

as before
$|x|<2$

$$= \frac{x^3}{2}\sum_{k=0}^{\infty}\left(\frac{-x}{2}\right)^k = \frac{x^3}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}x^k = \boxed{\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+1}}x^{k+3}}$$

$-2 < x < 2$

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \qquad |r|<1$$

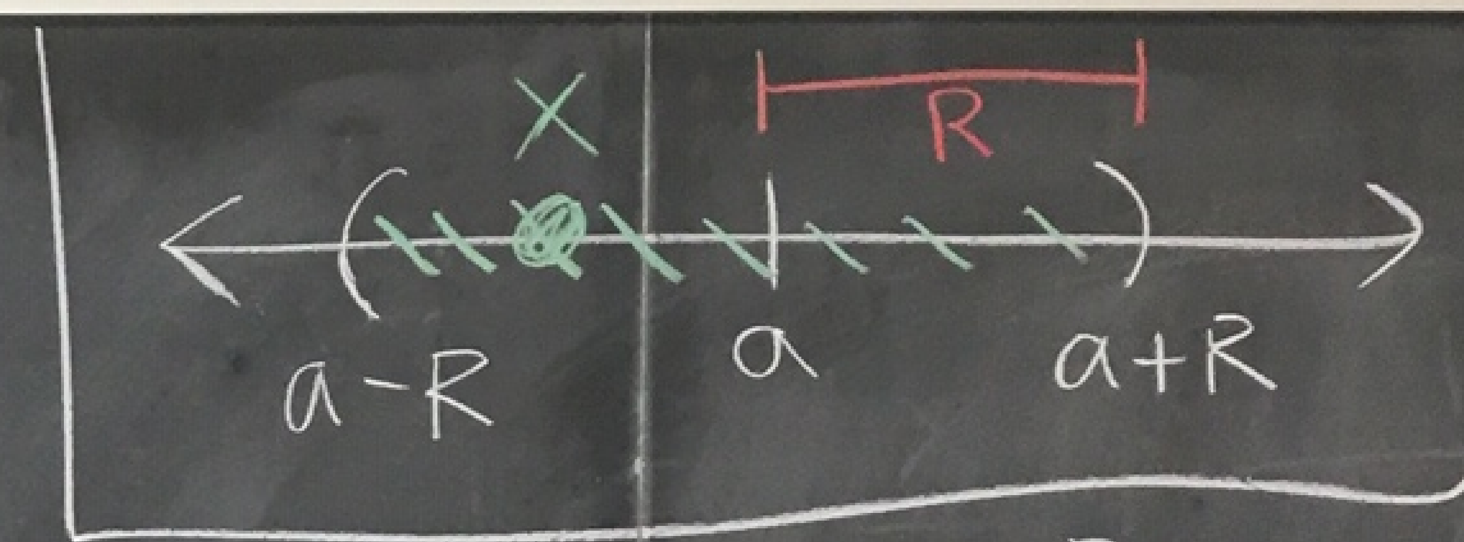
$$(ab)^k = a^k b^k \qquad \left(\frac{a}{b}\right)^k = \frac{a^k}{b^k}$$

Theorem: Let $f(x)$ be represented by a power series,

$$f(x)=\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with radius of convergence R.

Then f is differentiable and continuous on $(a-R, a+R)$ and



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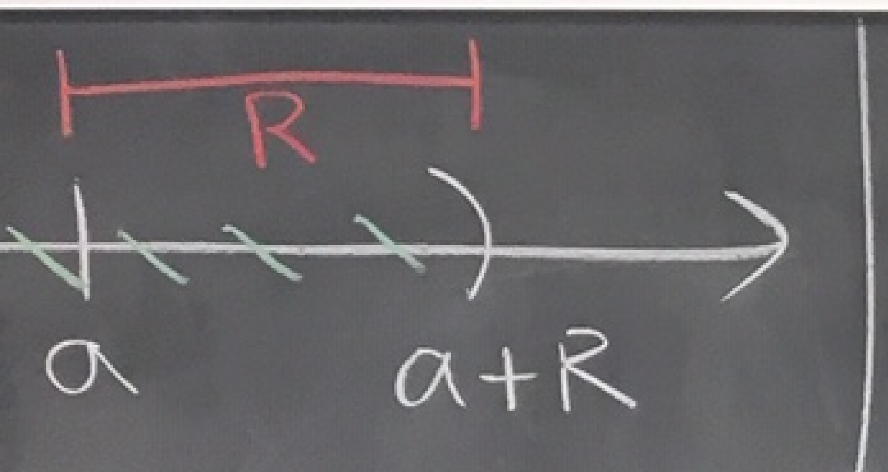
* Moreover, th

in ① & ② ha

of convergenc



$\cdots_3(x-a)^3 + \cdots$

① $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

$\qquad = \sum_{k=1}^{\infty} k \cdot c_k (x-a)^{k-1}$

② $\int f(x)\,dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots$

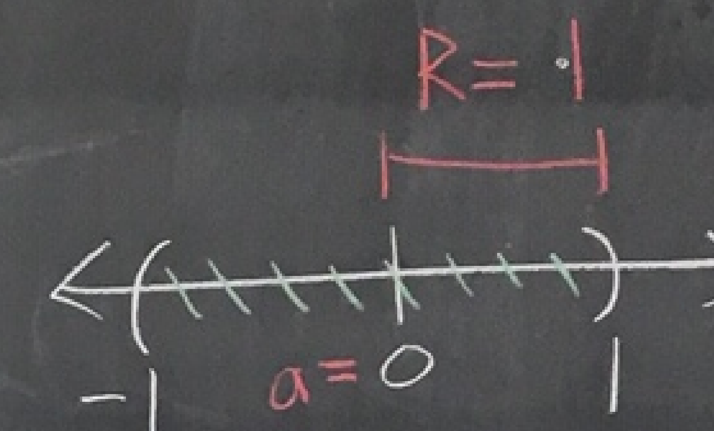
$\qquad = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}$ *

* Moreover, the series in ① & ② have radius of convergence R

Ex: Notice that $\int \frac{dx}{1-x} = -\ln|1-x| + C$

We know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{k=0}^{\infty} x^k \quad \text{when } -1 < x < 1$$



Integrating we get

$$-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots = C + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

this will still converge when $-1 < x < 1$

when $-1 < x < 1$

$0 < 1 - x$

so we don't need abs. values in log

So,

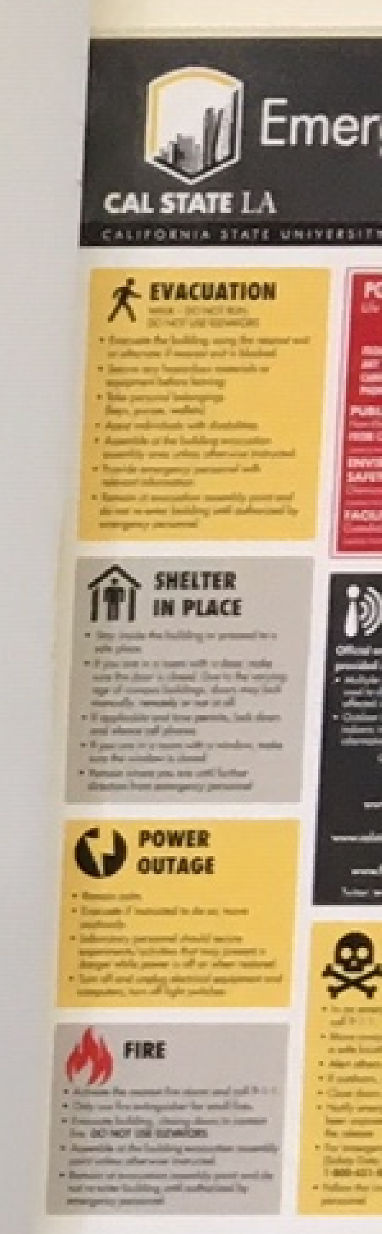
$$\ln(1-x) = -C - \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

$$= -C - \sum_{k=1}^{\infty} \frac{x^{k}}{k}$$

Find C. Plug in $x=0$ to get

$$0 = \ln(1) = \ln(1-0) = -C - \sum_{k=1}^{\infty} \frac{0^k}{k} = -C$$

So, $C = 0$.



Thus,

$$\boxed{\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{when } -1 < x < 1.}$$