# Math 2120

3/17/20

Tuesday

week 9

1. Our first video recording is on canvas. Log into my.calstatela.edu and click on canvas and then on Math 2120.
2. I put the 3/15 notes on our website.
3. Test out sending files via chat as a possibility during workshop.

# 9.3 – Taylor series

**Theorem:** Suppose $f$ has a power series representation at $a$, that is

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where it converges with some radius of convergence $R$.

Then

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

**Ex:** $f(x) = \ln(1-x)$

Previously, we saw that

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$= \underbrace{0}_{k=0} \underbrace{- x}_{k=1} \underbrace{- \frac{x^2}{2}}_{k=2} \underbrace{- \frac{x^3}{3}}_{k=3} - \cdots$$

converged for $|x| < 1$ that is $-1 < x < 1$

$$= c_0 + c_1(x-0) + c_2(x-0)^2 + c_3(x-0)^3 + \cdots$$

here $a = 0$

So,
$c_0 = 0$
$c_1 = -1$
$c_2 = -\frac{1}{2}$
$c_3 = -\frac{1}{3}$
$\vdots \quad \vdots$

Is this consistent with $c_k = \dfrac{f^{(k)}(a)}{k!}$ ?

$$f^{(0)}(x) = f(x) = \ln(1-x)$$

$$f^{(1)}(x) = f'(x) = \frac{1}{1-x} \cdot (-1) = \frac{-1}{1-x}$$

$$f^{(2)}(x) = f''(x) = \left[-(1-x)^{-1}\right]' = -\left[-(1-x)^{-2}(-1)\right] = -\frac{1}{(1-x)^2}$$

$$f(x) = \ln(1-x)$$

$$f^{(1)}(x) = \frac{-1}{1-x}$$

$$f^{(2)}(x) = \frac{-1}{(1-x)^2} = -(1-x)^{-2}$$

$$f^{(3)}(x) = -\left[(-2)(1-x)^{-3}(-1)\right] = \frac{-2}{(1-x)^3}$$

$$\boxed{\begin{aligned} c_0 &= 0 \\ c_1 &= -1 \\ c_2 &= -\tfrac{1}{2} \\ c_3 &= -\tfrac{1}{3} \end{aligned}}$$

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Check that $c_k = \dfrac{f^{(k)}(a)}{k!} = \dfrac{f^{(k)}(0)}{k!}$

$\boxed{a=0}$

$$\frac{f^{(0)}(0)}{0!} = \frac{f(0)}{1} = \frac{\ln(1-0)}{1} = \frac{0}{1} = 0 = c_0 \checkmark$$

$$\frac{f^{(1)}(0)}{1!} = \frac{f'(0)}{1} = \frac{\left(\frac{-1}{1-0}\right)}{1} = -1 = c_1 \checkmark$$

$$\frac{f^{(2)}(0)}{2!} = \frac{f''(0)}{2} = \frac{-\frac{1}{(1-0)^2}}{2} = \frac{-1}{2} = c_2 \checkmark$$

$$\frac{f^{(3)}(0)}{3!} = \frac{f'''(0)}{3\cdot 2\cdot 1} = \frac{\frac{-2}{(1-0)^3}}{3\cdot 2\cdot 1} = \frac{-2}{3\cdot 2\cdot 1} = \frac{-1}{3} = c_3 \checkmark$$

If you kept going the formula would keep working.

Def: Suppose $f(x)$ has derivatives of all orders at $a$. That is, suppose $f^{(k)}(a)$ exists for all $k \geq 0$.

Then we call the series

$$\sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(a)}{k!}}_{c_k} (x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the Taylor series for f at a. In the special case when $a=0$, the series is called the Maclaurin series for f.

Ex: $f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f^{(k)}(x) = e^x$ for all $k \geq 0$.

Let $a = 0$.

Then $c_k = \dfrac{f^{(k)}(a)}{k!} = \dfrac{f^{(k)}(0)}{k!} = \dfrac{e^0}{k!}$

$= \boxed{\dfrac{1}{k!}}$

The Taylor series for $f(x) = e^x$ centered at $a = 0$ is

also called the Maclaurin series since $a = 0$

$$\sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(0)}{k!}}_{c_k} (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$