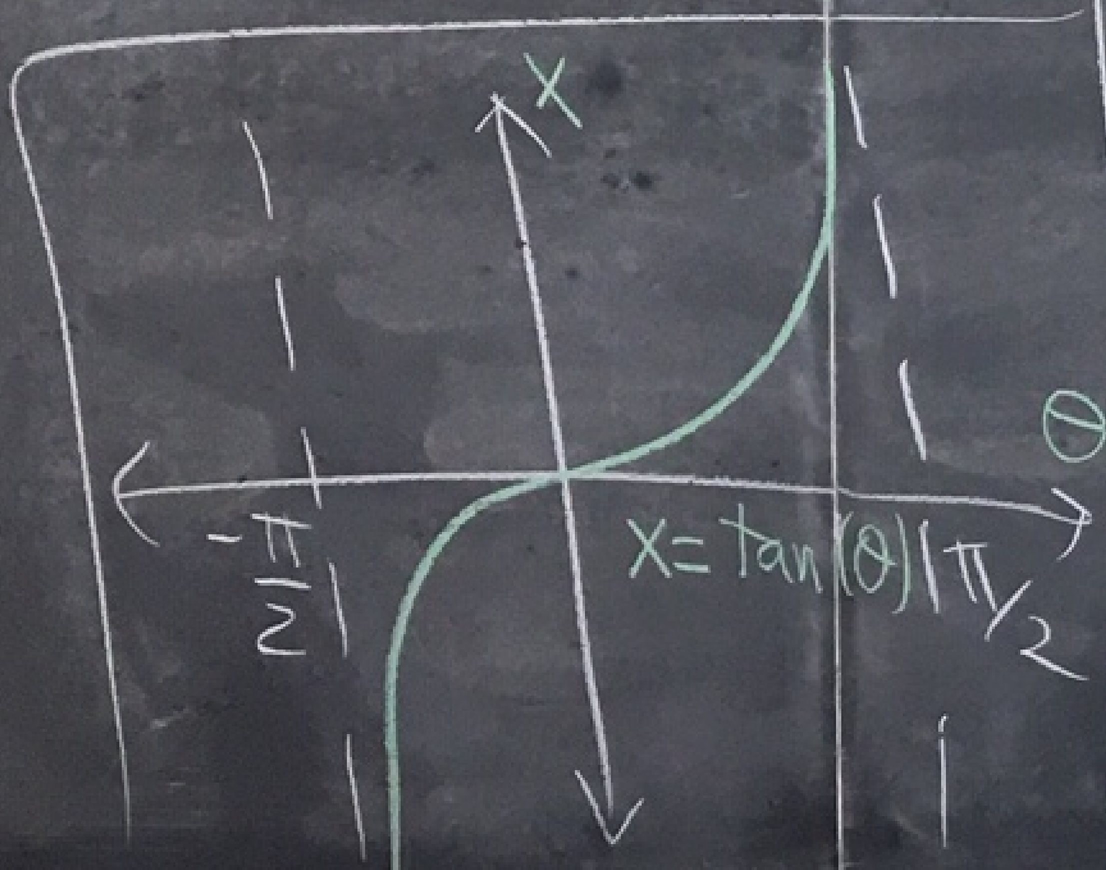
3/2
Monday
week 7

7.4

47 $$\int_0^1 \frac{dx}{\sqrt{x^2+16}} = \int_0^{\tan^{-1}(\frac{1}{4})} \frac{4\sec^2(\theta)}{\sqrt{16\tan^2(\theta)+16}}\, d\theta = \int_0^{\tan^{-1}(\frac{1}{4})} \frac{\cancel{4}\sec^2(\theta)}{\cancel{\sqrt{16}}\sqrt{\tan^2(\theta)+1}}\, d\theta$$

When $x=0$, $0=4\tan(\theta)$, $0=\tan(\theta)$, $\theta=0$	$\sqrt{x^2+a^2}$ $x=a\tan(\theta)$
When $x=1$, $1=4\tan(\theta)$, $1/4=\tan(\theta)$, $\tan^{-1}(1/4)=\theta$	$x=4\tan(\theta)$ $dx=4\sec^2(\theta)d\theta$ $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$



$$= \int_0^{\tan^{-1}(\frac{1}{4})} \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\, d\theta = \int_0^{\tan^{-1}(\frac{1}{4})} \frac{\sec^2(\theta)}{\sec(\theta)}\, d\theta$$

$\tan^2(\theta)+1 = \sec^2(\theta)$

$$= \int_0^{\tan^{-1}(\frac{1}{4})} \sec(\theta)\, d\theta$$

$$\int \ln\left|\sec(\theta)+\tan(\theta)\right| \Bigg|_{0}^{\tan^{-1}\left(\frac{1}{4}\right)}$$

$$= \ln\left|\sec\left(\tan^{-1}\left(\tfrac{1}{4}\right)\right)+\tan\left(\tan^{-1}\left(\tfrac{1}{4}\right)\right)\right|$$

$$- \ln\left|\underbrace{\sec(0)}_{\frac{1}{\cos(0)}=1}+\underbrace{\tan(0)}_{0}\right|$$

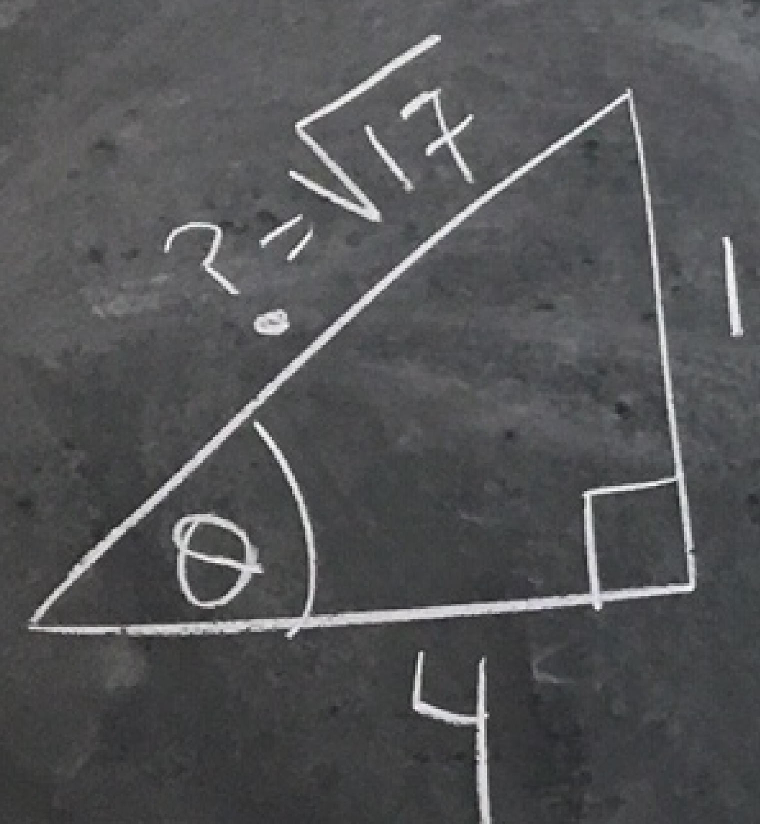
$$\ln(1)=0$$

$$= \ln\left| \underbrace{\sec\left(\tan^{-1}\left(\tfrac{1}{4}\right)\right)}_{\sqrt{17}/4} + \underbrace{\tan\left(\tan^{-1}\left(\tfrac{1}{4}\right)\right)}_{1/4} \right| = \ln\left|\frac{\sqrt{17}}{4} + \frac{1}{4}\right|$$

$$= \ln\left|\frac{\sqrt{17}+1}{4}\right|$$

Let $\theta = \tan^{-1}\left(\frac{1}{4}\right)$

Then $\tan(\theta) = \frac{1}{4} = \frac{\text{opp}}{\text{adj}}$



$$1^2 + 4^2 = ?^2$$

$$? = \sqrt{17}$$

$$\sec\left(\tan^{-1}\left(\tfrac{1}{4}\right)\right)$$

$$= \sec(\theta)$$

$$= \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{17}}{4}$$

50 (modified)

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\sqrt{4-4\sin^2(\theta)}} = \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\cdot 2\cos(\theta)}$$

$$= \frac{1}{4}\int \frac{d\theta}{\sin^2(\theta)}$$

$$= \frac{1}{4}\int \csc^2(\theta)\,d\theta$$

$\sqrt{a^2-x^2}$

$x = a\sin(\theta)$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$x = 2\sin(\theta)$

$dx = 2\cos(\theta)\,d\theta$

$\sqrt{4-4\sin^2(\theta)}$

$= \sqrt{4}\sqrt{1-\sin^2(\theta)}$

$= 2\sqrt{\cos^2(\theta)}$

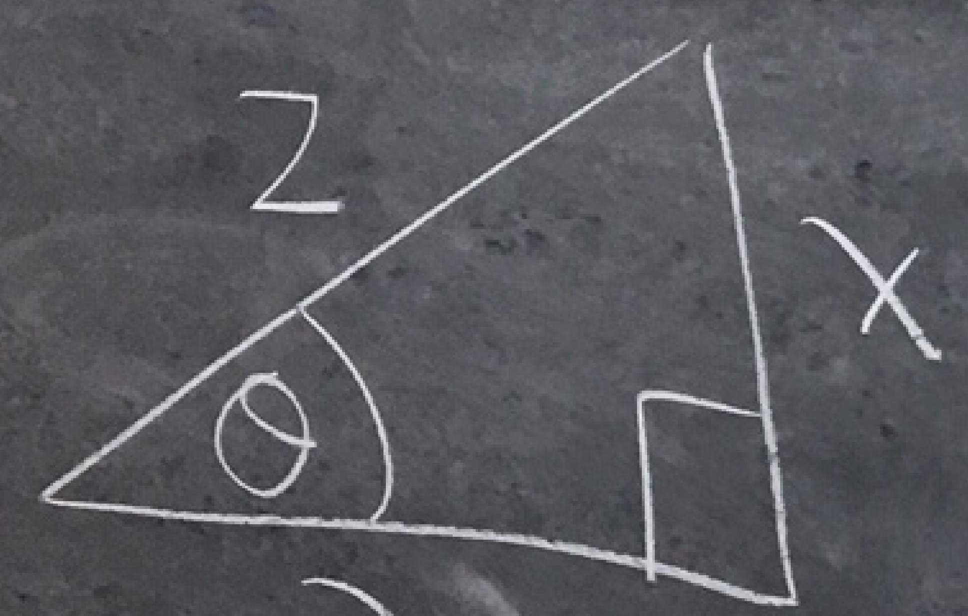
$= 2\cos(\theta)$

$$= -\frac{1}{4}\cot(\theta) + C = -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C$$

$$x = 2\sin(\theta)$$

$$\sin(\theta) = \frac{x}{2} = \frac{\text{opp}}{\text{hyp}}$$

$$\cot(\theta) = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{4-x^2}}{x}$$



$$? = \sqrt{4-x^2}$$

$$?^2 + x^2 = 2^2$$

$$? = \sqrt{4-x^2}$$

7.8 Does $\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}}$ converge or diverge? If it converges, what does it converge to?

38 $\int_{-3}^{1} \frac{dx}{(2x+6)^{2/3}} = \lim_{t \to -3^+} \int_{t}^{1} (2x+6)^{-2/3}\, dx$

$\frac{1}{(2x+6)^{2/3}}$ is undefined at $x = -3$ since $\frac{1}{(2(-3)+6)^{2/3}} = \frac{1}{0^{2/3}}$

$= \lim_{t \to -3^+} \left[\frac{3}{2}(2x+6)^{1/3} \Big|_{t}^{1} \right]$

$\int (2x+6)^{-2/3}\, dx = \frac{1}{2} \int u^{-2/3}\, du$

$u = 2x+6$

$du = 2dx$

$1/2\, du = dx$

$= \frac{1}{2} \frac{u^{1/3}}{(1/3)} + C$

$= \frac{3}{2} u^{1/3} + C = \frac{3}{2}(2x+6)^{1/3} + C$

$$= \lim_{t \to -3^+} \left[\frac{3}{2}(2(1)+6)^{1/3} - \frac{3}{2}(2t+6)^{1/3} \right]$$

$$= \frac{3}{2}(8)^{1/3} - \frac{3}{2}(2(-3)+6)^{1/3}$$

$$= \frac{3}{2} \cdot 2 - 0 = \boxed{3}$$

So, $\int_{-3}^{1} \frac{dx}{(2x+6)^{3/2}}$ converges to 3.

7.8

14 $\displaystyle\int_2^\infty \frac{dy}{y\ln(y)} = \lim_{t\to\infty}\int_2^t \frac{dy}{y\ln(y)} = \lim_{t\to\infty}$

$=$

$$\boxed{\int \frac{dy}{y\ln(y)} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(y)| + C}$$

$u = \ln(y)$

$du = \frac{1}{y}\,dy$

$$= \lim_{t\to\infty} \ln(\ln(y))\Big|_2^t = \lim_{t\to\infty}\Big[\underbrace{\ln(\ln(t))}_{\downarrow\ \infty} - \underbrace{\ln(\ln(2))}_{\#}\Big] = \infty$$

So, $\displaystyle\int_2^\infty \frac{dy}{y\ln(y)}$ diverges.

7.5

17 $\displaystyle\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx$

$$\frac{5x}{x^2-x-6} = \frac{5x}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\boxed{5x = A(x+2) + B(x-3)}$$

$\boxed{x=3}$

$5(3) = A(5) + B(0)$

$\boxed{3 = A}$

$\boxed{x=-2}$

$5(-2) = A(0) + B(-5)$

$\boxed{2 = B}$

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C}$$

$$\int_{-1}^{2} \frac{5x}{x^2-x-6}\,dx$$

$$= \int_{-1}^{2} \left[\frac{3}{x-3} + \frac{2}{x+2}\right] dx$$

$$= \left(3\ln|x-3| + 2\ln|x+2|\right)\Big|_{-1}^{2}$$

$$= \Big[3\ln|2-3| + 2\ln|2+2|$$

$$- \Big(3\ln|-1-3| + 2\ln|-1+2| \Big) \Big]$$

$$= 3\underbrace{\ln(1)}_{0} + 2\ln(4)$$

$$- 3\ln(4) - 2\underbrace{\ln(1)}_{0} = -\ln(4) = \ln(4^{-1}) = \ln\left(\frac{1}{4}\right)$$

7.3

$$\int \cos^4(\theta)\,d\theta = \int \left(\cos^2(\theta)\right)^2 d\theta$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)^2 d\theta$$

$$\boxed{\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)}$$

$$= \int \left[\frac{1}{4} + 2\cdot\frac{1}{4}\cos(2\theta) + \frac{1}{4}\underbrace{\cos^2(2\theta)}_{\frac{1}{2}+\frac{1}{2}\cos(4\theta)}\right] d\theta$$

$$= \int \left[\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{8} + \frac{1}{8}\cos(4\theta)\right] d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta) + \frac{1}{8}\theta + \frac{1}{8}\cdot\frac{1}{4}\sin(4\theta) + C$$

$$= \frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C$$

$\boxed{7.3}$

$$\int \sin^{11}(x)\cos^{10}(x)\,dx = \int \sin^{10}(x)\cos^{10}(x)\,\underset{\sim\sim\sim}{\sin(x)\,dx}$$

$$= \int \left(\sin^2(x)\right)^5 \cos^{10}(x)\sin(x)\,dx = \int \left(1-\cos^2(x)\right)^5 \cos^{10}(x)\sin(x)\,dx$$

$$= -\int (1-u^2)^5 u^{10}\,du = -\int \Big[1^5 + 5\cdot 1^4\cdot(-u^2) + 10\cdot 1^3\cdot(-u^2)^2$$

$$+ 10\cdot 1^2\cdot(-u^2)^3 + 5\cdot 1^1\cdot(-u^2)^4$$

$$+ 1\cdot(-u^2)^5\Big]\cdot u^{10}\,du$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

$$(a+b)^5 = 1 \cdot a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1 \cdot b^5$$

$$= -\int \left[u^{10} - 5u^{12} + 10u^{14} - 10u^{16} + 5u^{18} - u^{20}\right] du$$

$$= -\frac{u^{11}}{11} + 5\frac{u^{13}}{13} - 10\frac{u^{15}}{15} + 10\frac{u^{17}}{17} - 5\frac{u^{19}}{19} + \frac{u^{21}}{21} + C$$

$$= -\frac{1}{11}\cos^{11}(x) + \frac{5}{13}\cos^{13}(x) - \frac{2}{3}\cos^{15}(x) + \frac{10}{17}\cos^{17}(x) - \frac{5}{19}\cos^{19}(x) + \frac{1}{21}\cos^{21}(x) + C$$

$\sin^{11}$

7.2

22 $\int \underbrace{x}_{dv} \cdot \underbrace{\tan^{-1}(x^2)}_{u} \, dx = \underbrace{\frac{1}{2} x^2 \tan^{-1}(x^2)}_{uv} - \underbrace{\int \frac{2x}{1+x^4} \cdot \frac{1}{2} x^2 \, dx}_{\int v\,du} =$

LIATE

$u = \tan^{-1}(x^2) \quad du = \frac{1}{1+(x^2)^2} \cdot (2x)\, dx$

$dv = x\, dx \quad v = \frac{1}{2} x^2$

$\frac{d}{dt} \tan^{-1}(t) = \frac{1}{1+t^2}$

$$\frac{2x}{1+x^4} \cdot \frac{1}{2}x^2\,dx = \frac{1}{2}x^2\tan^{-1}(x^2) - \int \frac{x^3}{1+x^4}\,dx$$

$$- \int v\,du$$

$$= \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\int \frac{1}{u}\,du$$

$$u = 1+x^4$$

$$du = 4x^3\,dx$$

$$\frac{1}{4}du = x^3\,dx$$

$$= \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{2}x^2\tan^{-1}(x^2) - \frac{1}{4}\ln|1+x^4| + C$$