3/3
Tuesday
week 7

8.6 – Alternating series

Alternating series test

If

① $0 < a_{k+1} \leq a_k$ for all k [ie the a_k are non-increasing]

and ② $\lim_{k\to\infty} a_k = 0$ then $\sum_k (-1)^k a_k$ converges.

You could also have $\sum (-1)^{k+1} a_k$ or $\sum (-1)^{k-1} a_k$

Ex: Does $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

converge or diverge?

$\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \underbrace{\frac{1}{k}}_{a_k}$

Let $a_k = \frac{1}{k}$.

① $0 < \frac{1}{k+1} \leq \frac{1}{k}$ ✓

② $\lim \frac{1}{k} = 0$

By the alt. series test $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ converges.

Ex: Does $\sum_{k=2}^{\infty} (-1)^k \underbrace{\frac{\ln(k)}{k}}_{a_k}$ converge or diverge ?

Let $a_k = \frac{\ln(k)}{k}$.

① We want that $0 < \frac{\ln(k+1)}{k+1} \leq \frac{\ln(k)}{k}$.

Definitely the terms a_k are positive when $k \geq 2$.

Why is $a_k = \frac{\ln(k)}{k}$ not increasing ?

Let $f(x) = \frac{\ln(x)}{x}$

Then,

$$f'(x) = \frac{\frac{1}{x}\cdot x - 1\cdot\ln(x)}{x^2} = \frac{1-\ln(x)}{x^2} < 0 \quad \text{when } x > e \approx 2.71\ldots$$

So, $f(x) = \frac{\ln(x)}{x}$ is decreasing when $x > e$.

When $x > e$, $\ln(x) > 1$.
So when $x > e$, $1 - \ln(x) < 0$

Thus, $0 < \frac{\ln(k+1)}{k+1} < \frac{\ln(k)}{k}$ when $x \geqslant 3$.

② $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln(k)}{k} \underset{\uparrow}{\overset{L'H}{=}} \lim_{k\to\infty} \frac{1/k}{1} = \lim_{k\to\infty} \frac{1}{k} = 0$

$\boxed{\text{“}\frac{\infty}{\infty}\text{”}}$

So, by the alternating series test

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln(k)}{k}$ converges

Thm: If $\sum |a_k|$ converges,

then $\sum a_k$ converges.

Ex: Consider $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$

You could use the alternating series test, but we can istead use the above theorem.

Note that $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^2}\right| = \sum_{k=1}^{\infty} \frac{|(-1)^k|}{|k^2|} = \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\boxed{(-1)^k = \pm 1}$

And $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges ($p=2$ series, $p>1$)

So, by the thm, since $\displaystyle\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^2}\right|$ converges,

we have $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges.

<u>Ex:</u> Consider $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$.

Here we have $\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges (Harmonic series)

But $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

Remember:
If $\sum |a_k|$ converges
then $\sum a_k$ converges

Def:

If $\sum |a_k|$ converges,
then we say $\sum a_k$ converges absolutely.

If $\sum |a_k|$ diverges but $\sum a_k$ converges,
then we say $\sum a_k$ converges conditionally.

ges
erges

$$
\begin{tikzcd}
 & \text{$\sum |a_k|$ converges} \\
\text{$\sum a_k$ converges} \arrow[ur] \arrow[dr] & \\
 & \text{$\sum |a_k|$ diverges}
\end{tikzcd}
$$

absolute convergence of $\sum a_k$

conditional convergence of $\sum a_k$

Ex:

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges & $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

So, $\sum \frac{(-1)^k}{k^2}$ converges absolutely.

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges but $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

So, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges conditionally.