Thursday
3/5
week 7

9.2 – Properties of power series

power series are infinite polynomials.
We use them to represent functions

Some examples that we will see are:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$ which holds for $-1 < x < 1$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$ which holds for all x.

We will have power series for $\sin(x)$, $\cos(x)$, $\ln(x)$, ...

Def: A <u>power series centered at a constant a</u> is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where c_k is a constant and x is a variable.

Given a power series we ask, what x's can we plug in that make the series converge?

Here $a=0$

$x^k=(x-0)^k$

Ex: For what x does

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

converge?

$$\frac{x^0}{0!}=\frac{1}{1}=1$$

We can plug in $x=0$ and the power series converges to

$$1+\underbrace{\frac{0}{1!}+\frac{0^2}{2!}+\frac{0^3}{3!}+\cdots}_{0}=1$$

Let's use the ratio test to see what x's make the series converge.

Ratio test Given $\sum a_k$, $a_k>0$

$$L=\lim_{k\to\infty}\frac{a_{k+1}}{a_k}$$

$0\le L<1 \rightarrow$ converges

$L>1 \rightarrow$ diverges

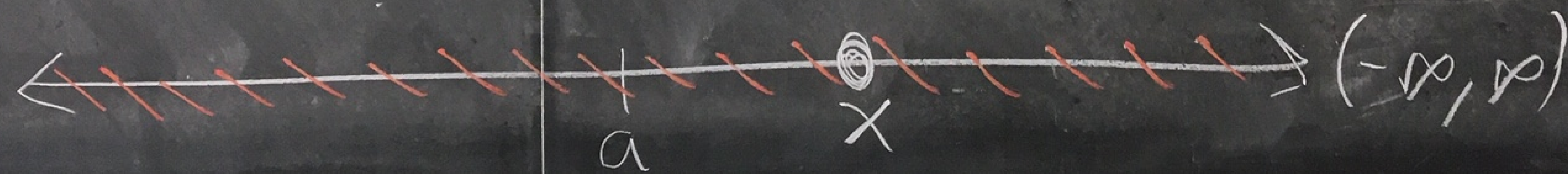
$L=1 \rightarrow$ inconclusive

Before we do this. Power series have a special property.

Convergence of power series

Let $\sum_{k=0}^{\infty} c_k (x-a)^k$ be a power series. There are 3 possibilities for convergence.

1. The series converges for all x.
   Here $(-\infty, \infty)$ is called the interval of convergence
   The radius of convergence is $R=\infty$

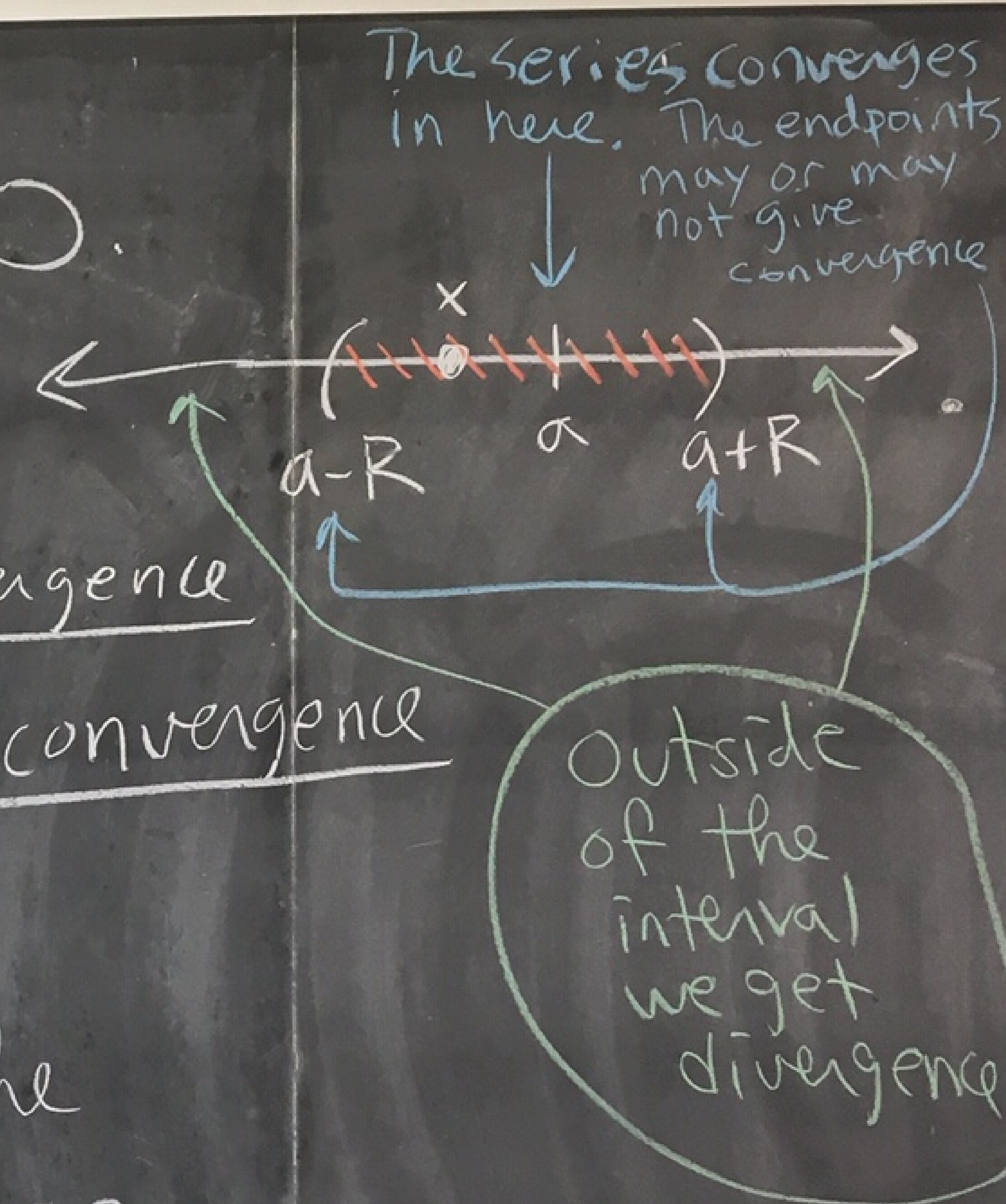


(2) The series converges for all x where $\underbrace{|x-a| < R}_{a-R < x < a+R}$ for some $R > 0$.

Here R is called the <u>radius of convergence</u>

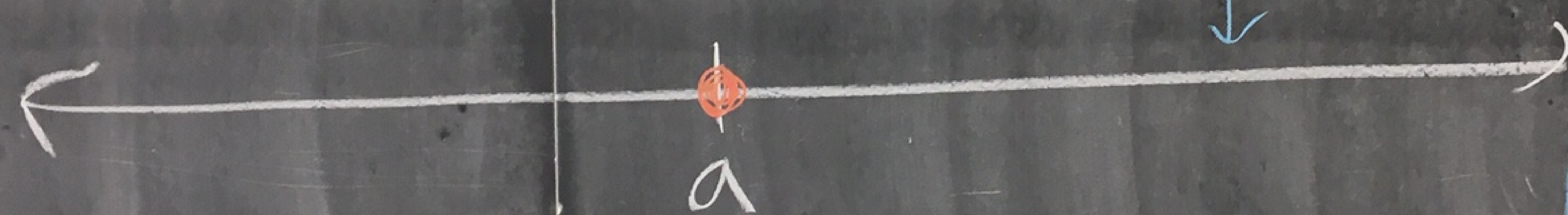
And $(a-R, a+R)$ is the <u>interval of convergence</u>

And the series diverges outside of the interval of convergence. At the endpoints it may converge or diverge.



(3) The series only converges at $x=a$ and diverges everywhere else.

Here $R=0$ is the **radius of convergence**



the series only converges for $x=a$.

The series diverges for all other x

To test $\sum c_k(x-a)^k$ for the x that makes it converge, we usually test where it converges absolutely, ie we test where

$$\sum \left| c_k(x-a)^k \right|$$

converges. This series will have convergence on the same radius of convergence as without the absolute values.

$|ab| = |a||b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

L

...ten what x is.

...onverges for all x.

Ex: Where does $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converge?

Use the ratio test with absolute values to make the terms positive

$$\lim_{k\to\infty} \frac{\left|\frac{x^{k+1}}{(k+1)!}\right|}{\left|\frac{x^k}{k!}\right|} = \lim_{k\to\infty} \frac{|k!|}{|x^k|} \cdot \frac{|x^{k+1}|}{|(k+1)!|}$$

$$= \lim_{k\to\infty} \frac{k!}{(k+1)!} \cdot \left| \frac{x^{k+1}}{x^k} \right|$$

x is fixed in the limit

$$|ab| = |a||b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

$$= \lim_{k\to\infty} \frac{k!}{(k+1)\cdot k!} |x| = \lim_{k\to\infty} \frac{|x|}{(k+1)} = 0 = L$$

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5(4!)$$

$$(k+1)! = (k+1)[k!]$$

$L = 0 < 1$ no matter what x is.

So, the series $\sum \frac{x^k}{k!}$ converges for all x.