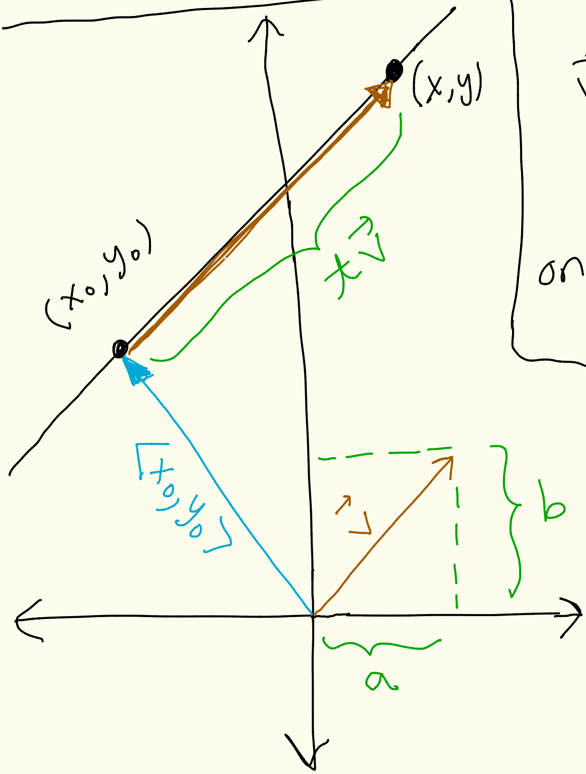
Math 2120

4/15/20

10.1 - continued...

Suppose we have a line through (x_0, y_0) with slope $\frac{b}{a}$ (for now assume its not a vertical line, ie $a \neq 0$). Let



$\vec{v} = \langle a, b \rangle$. Then $\vec{v}$ has slope $\frac{b}{a}$ and is parallel to the line. Any point (x,y) on the line can be described as

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\vec{v}$$

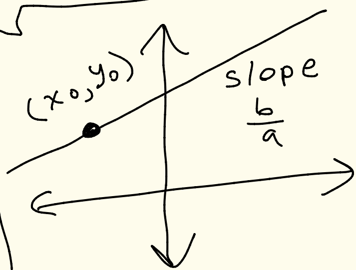
$$= \langle x_0, y_0 \rangle + t\langle a, b \rangle$$

$$= \langle x_0 + at, y_0 + bt \rangle$$

for some number t.

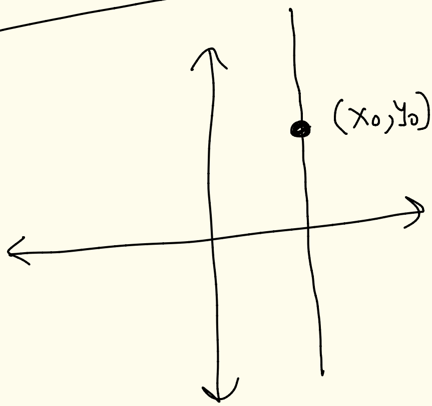
So, the line with slope $\frac{b}{a}$ through (x_0, y_0) is given by the parametric equations

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}}$$



If the line is vertical, that means $a = 0$ and b can be anything and the formula is

$$\boxed{\begin{aligned} x &= x_0 \\ y &= y_0 + t \end{aligned}}$$

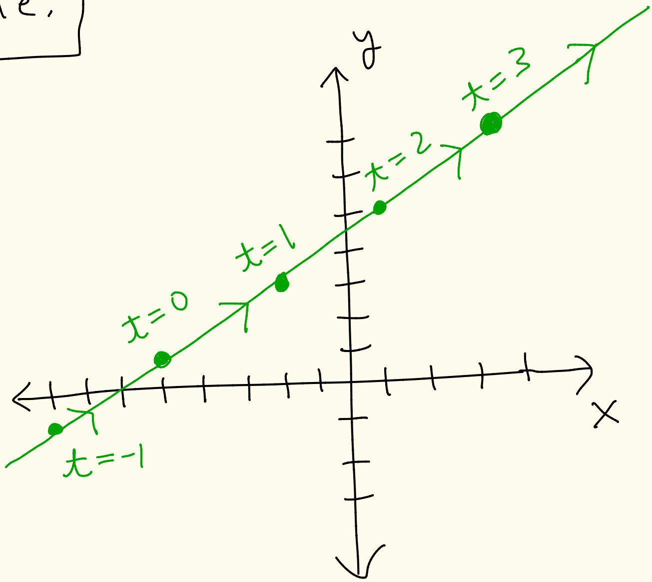


Ex: Sketch the line given by

$$x = 3t - 5, \quad y = 2t + 1$$

Then eliminate the variable t to find a Cartesian (xy) equation for the line.

t	x	y
-2	-11	-3
-1	-8	-1
0	-5	1
1	-2	3
2	1	5
3	4	7



Eliminate t:

$$\boxed{\begin{aligned} x &= 3t - 5 \\ y &= 2t + 1 \end{aligned}}$$

$$t = \frac{1}{2}y - \frac{1}{2}$$

$$x = 3\left[\frac{1}{2}y - \frac{1}{2}\right] - 5 = \frac{3}{2}y - \frac{3}{2} - 5$$

$$x = \frac{3}{2}y - \frac{13}{2}$$

$$\boxed{2x = 3y - 13}$$

OR

$$\boxed{y = \frac{2}{3}x + \frac{13}{3}}$$

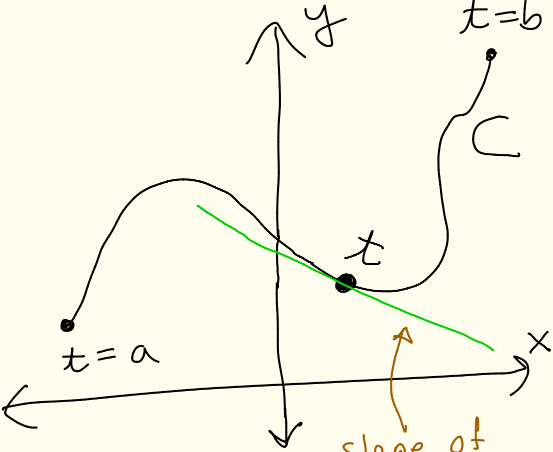
Thm: Let

$$x = f(t), \quad y = g(t) \qquad a \le t \le b$$

be parametric equations of a curve C where f and g are differentiable in the interval $[a, b]$. Then the slope of the tangent line to the curve C at the point corresponding to t is

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

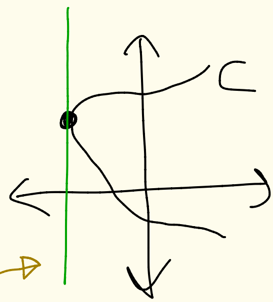
provided $f'(t) \neq 0$.



slope of this tangent line is $\dfrac{g'(t)}{f'(t)}$

Note:

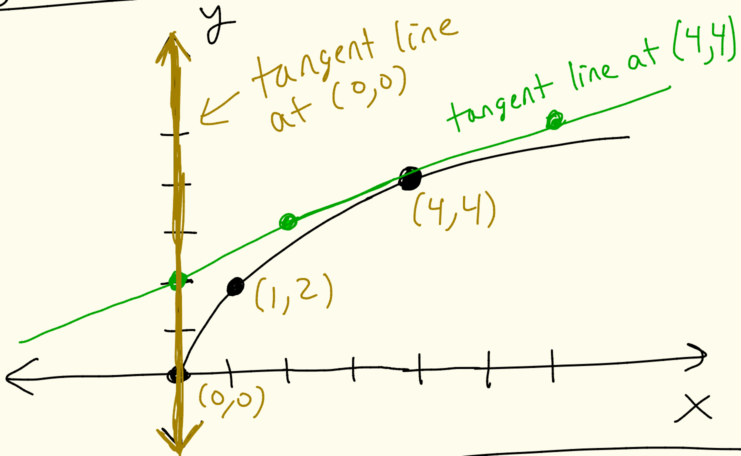
If $f'(t) = 0$ and $g'(t) \neq 0$, then you have a vertical tangent line



Ex: Consider the parametric equations given by

$$x = t \qquad y = 2\sqrt{t} \quad \text{for } t \geq 0$$

t	x	y
0	0	0
1	1	2
4	4	4



$x = f(t) = t$

$y = g(t) = 2t^{1/2}$

$f'(t) = 1$

$g'(t) = 2\left(\frac{1}{2}t^{-1/2}\right) = \frac{1}{\sqrt{t}}$

$$\frac{g'(t)}{f'(t)} = \left(\frac{\frac{1}{\sqrt{t}}}{1}\right) = \frac{1}{\sqrt{t}}$$

At $t=4$, ie $(x,y)=(4,4)$ the slope of the tangent line is $\frac{g'(4)}{f'(4)} = \frac{1}{\sqrt{4}} = \frac{1}{2}$.

So the tangent line at $(4,4)$ is $y-4 = \frac{1}{2}(x-4)$

OR $\boxed{y = \frac{1}{2}x + 2}$

What's the tangent line at $(0,0)$? $t=0$

You can't plug $t=0$ into $\frac{1}{\sqrt{t}}$

But $\lim_{t\to 0^+} \frac{1}{\sqrt{t}} = \infty$. So the tangent line at $(0,0)$ is vertical. It's $x=0$ or the y-axis.

Workshop

11.3

48 Find all unit vectors orthogonal to $\vec{v} = \langle 3,4,0 \rangle$.

Let $\vec{w} = \langle a,b,c \rangle$. Then $\vec{w}$ is orthogonal to $\vec{v}$ if and only if $\vec{w} \cdot \vec{v} = 0$.

That is, $\langle a,b,c \rangle \cdot \langle 3,4,0 \rangle = 0$.

That is, $3a + 4b + 0c = 0$

i.e. $3a + 4b = 0$

So, $\boxed{a = -\frac{4}{3}b}$ where b and c can be any real numbers.

So, $\vec{w} = \langle a, b, c \rangle$

$= \langle -\frac{4}{3}b, b, c \rangle$

where b, c can be any real #s.

<u>Ex:</u> $b = 3$, $c = 1$ we get $\vec{w} = \langle -4, 3, 1 \rangle$

and $\vec{w} \cdot \vec{v} = \langle -4, 3, 1 \rangle \cdot \langle 3, 4, 0 \rangle$

$= (-4) \cdot 3 + 3 \cdot 4 + 1 \cdot 0 = 0$

The above formula gives all vectors orthogonal to $\vec{v}$, We just want the unit vectors. So we need to divide $\vec{w}$ by its length to scale it to have length 1 (ie make it a unit vector)

$$\vec{u} = \frac{1}{|\vec{w}|}\vec{w} = \frac{1}{\sqrt{(-\frac{4}{3}b)^2 + b^2 + c^2}} \langle -\frac{4}{3}b, b, c \rangle$$

$$= \boxed{\frac{1}{\sqrt{\frac{25}{9}b^2 + c^2}} \langle -\frac{4}{3}b, b, c \rangle \quad \text{where } b \neq 0 \text{ and } c \neq 0}$$

(43) Find a parametric formula for the line through the points $P(-1,-3)$ and $Q(6,-16)$.

Method 1 $\quad (x_0,y_0) = (-1,-3)$

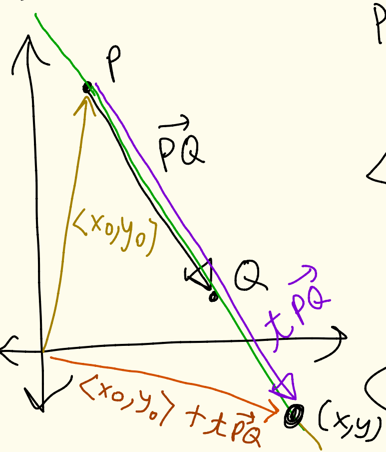
$$\text{slope} = \frac{-16-(-3)}{6-(-1)} = \frac{-13}{7} = \frac{b}{a}$$

$$x = x_0 + at = -1 + 7t$$
$$y = y_0 + bt = -3 - 13t$$

t can be any real number

Method 2 (vector method)

$$\vec{PQ} = \langle 6-(-1), -16-(-3)\rangle$$
$$= \langle 7, -13\rangle$$



$$\langle x,y\rangle = \langle x_0,y_0\rangle + t\,\vec{PQ}$$
$$= \langle -1,-3\rangle + t\langle 7,-13\rangle$$
$$= \langle -1+7t, -3-13t\rangle$$

$$x = -1+7t,\ y = -3-13t$$

9.2 Find interval of convergence. pg 9

12 $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}$

$(x-1)^k(x-1)$

$$L = \lim_{k\to\infty} \frac{\left|\frac{(x-1)^{k+1}}{(k+1)!}\right|}{\left|\frac{(x-1)^k}{k!}\right|} = \lim_{k\to\infty} \left|\frac{k!}{(x-1)^k} \cdot \frac{(x-1)^{k+1}}{(k+1)!}\right|$$

$(k+1)\cdot k!$

$$= \lim_{k\to\infty}\left|\frac{x-1}{k+1}\right| = \underbrace{|x-1|}_{\text{Constant w/ respect to } k} \lim_{k\to\infty}\left|\frac{1}{k+1}\right|$$

$$= |x-1| \cdot 0 = 0 \quad \text{for all } x.$$

Since $0 \le L < 1$ no matter what x is, the series converges for all x. So the interval of convergence is $(-\infty, \infty)$
And the radius of convergence is ∞.