Math 2120

4/21/20

Test 3

Ch. 9

11.1 – 11.4

Weds

4/29

10.3 – Calculus in Polar Coordinates

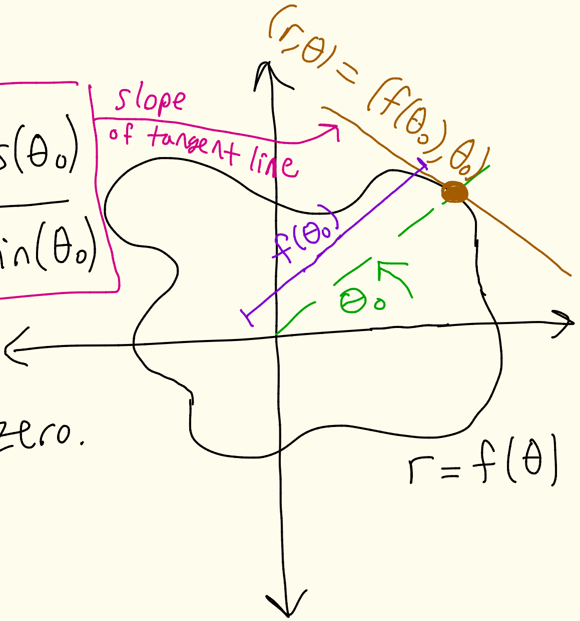
Theorem

Suppose we have a polar equation $r = f(\theta)$. Suppose f is differentiable at θ_0. Then the slope of the tangent line at $(r,\theta) = (f(\theta_0), \theta_0)$ is

$$\boxed{\frac{f'(\theta_0)\sin(\theta_0) + f(\theta_0)\cos(\theta_0)}{f'(\theta_0)\cos(\theta_0) - f(\theta_0)\sin(\theta_0)}}$$

slope of tangent line

provided the denominator is not zero.



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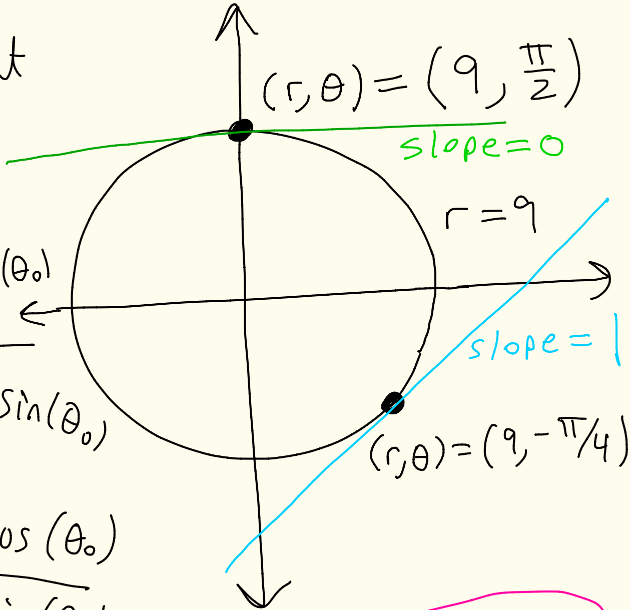
Ex: Consider

$$r = 9, \quad \left[r = f(\theta) \right] \quad \left. \right\} \begin{array}{l} f(\theta) = 9 \\ f'(\theta) = 0 \end{array}$$

Slope of the tangent line at θ_0 is

$$\frac{f'(\theta_0)\sin(\theta_0) + f(\theta_0)\cos(\theta_0)}{f'(\theta_0)\cos(\theta_0) - f(\theta_0)\sin(\theta_0)}$$



$$= \frac{0 \cdot \sin(\theta_0) + 9\cos(\theta_0)}{0\cos(\theta_0) - 9\sin(\theta_0)}$$

$$= -\frac{\cos(\theta_0)}{\sin(\theta_0)} = \boxed{-\cot(\theta_0)} \leftarrow \boxed{\text{slope of tangent line at } \theta = \theta_0}$$

$\boxed{\theta_0 = \frac{\pi}{2}}$: slope is $-\cot\left(\frac{\pi}{2}\right) = -\frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = -\frac{0}{1} = 0$

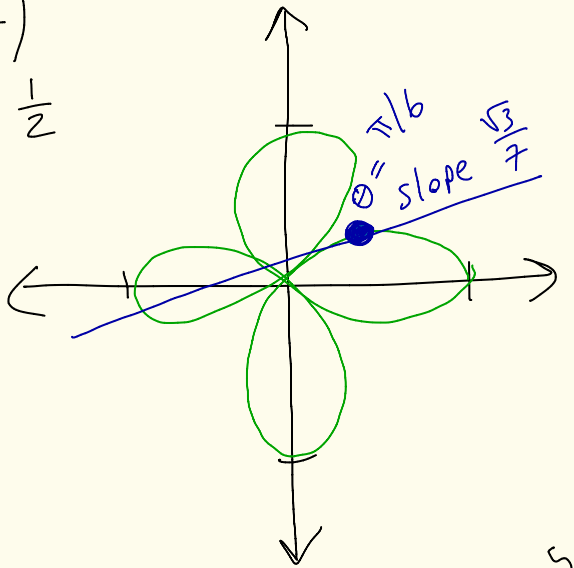
$\boxed{\theta_0 = \frac{-\pi}{4}}$: slope is $-\cot\left(\frac{-\pi}{4}\right) = -\frac{\cos(-\pi/4)}{\sin(-\pi/4)} = -\left[\frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}\right] = 1$

Ex: Last time we graphed

$$r = \underbrace{\cos(2\theta)}_{f(\theta)=\cos(2\theta)}$$

When $\theta = \frac{\pi}{6}$, $r = \cos\left(2 \cdot \frac{\pi}{6}\right)$

$$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$



What is the slope of the tangent line at $\theta_0 = \frac{\pi}{6}$?

$$f(\theta) = \cos(2\theta)$$

$$f'(\theta) = -2\sin(2\theta)$$

slope at θ_0 is

$$\frac{f'(\theta_0)\sin(\theta_0) + f(\theta_0)\cos(\theta_0)}{f'(\theta_0)\cos(\theta_0) - f(\theta_0)\sin(\theta_0)}$$

$$= \frac{-2\sin(2\theta_0)\sin(\theta_0) + \cos(2\theta_0)\cos(\theta_0)}{-2\sin(2\theta_0)\cos(\theta_0) - \cos(2\theta_0)\sin(\theta_0)}$$

$\boxed{\text{At } \theta_0 = \frac{\pi}{6}}$:

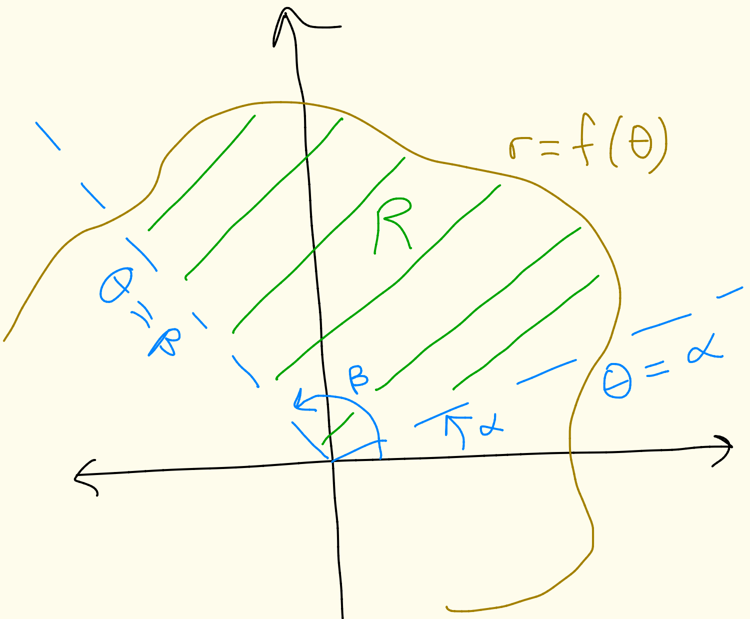
$$\frac{-2\left[\frac{\sqrt{3}}{2}\right]\left[\frac{1}{2}\right] + \left[\frac{1}{2}\right]\left[\frac{\sqrt{3}}{2}\right]}{-2\left[\frac{\sqrt{3}}{2}\right]\left[\frac{\sqrt{3}}{2}\right] - \left[\frac{1}{2}\right]\left[\frac{1}{2}\right]} = \frac{-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}}{-\frac{3}{2} - \frac{1}{4}} = \frac{-\frac{\sqrt{3}}{4}}{-\frac{7}{4}} = \frac{\sqrt{3}}{7} \approx 0.25$$

Finding areas

Suppose R is a region, bounded by the polar curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$. Suppose $f(\theta)$ is positive and continuous and $0 < \beta - \alpha \leq 2\pi$.

Then the area of R is given by

$$\int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$



$\boxed{\alpha \text{ is alpha, } \beta \text{ is beta}}$

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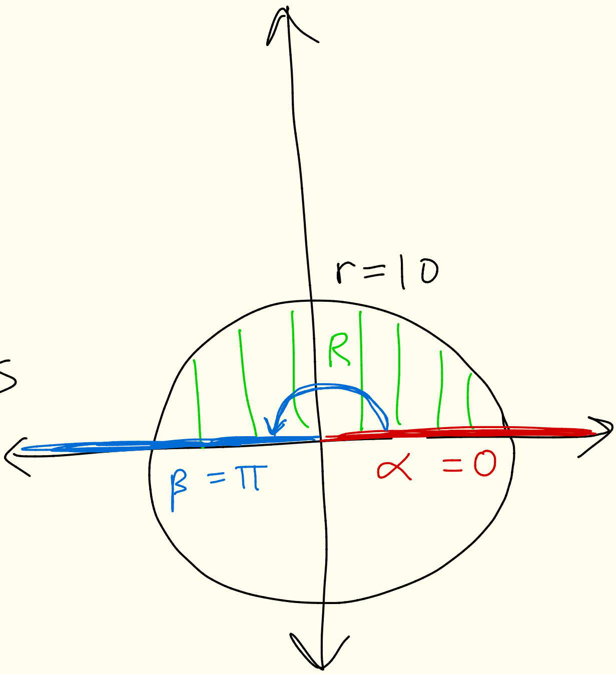
Ex: Find the area of the top half of a circle of radius 10.

$r = 10 = f(\theta)$

$\alpha = 0$

$\beta = \pi$



So the area is

$$\int_0^{\pi} \frac{1}{2}[10]^2 \, d\theta$$

$$= \frac{10^2}{2} \int_0^{\pi} d\theta = \frac{10^2}{2} \theta \Big|_0^{\pi} = \frac{10^2}{2}(\pi - 0)$$

$$= \frac{10^2}{2}\pi = 50\pi$$