Math 5402 - Spring 2020 - Final

Directions:

1. Pick a consecutive 2.5-hour window to take this exam, such as 2pm-4:30pm. You may only use 2.5 hours of consecutive time. Do not split the time (like 12-1:30 and then 5-6).
2. You can only use your mind to take this exam and your one page of notes with theorems and definitions. No help from any sources or people. No books, no notes, no online, etc.
3. No calculators.
4. Use blank paper (like printer paper) if you have it please.
5. On the first page of your exam, before any of your solutions, put these two things:
   (a) Your name.
   (b) Copy this statement and then sign your signature after it:

   "Everything on this test is my own work. I did not use any sources or talk to anyone about this exam." **_____your signature_____**
6. After your name and the above statement with signature, start putting your solutions to the problems. Please put them in order. That is, first problem 1, then problem 2, etc. You can put each one on its own page.
7. Scan and email to me by Thursday the 14th at 12pm.

The problems are on the next page.

Pick 5 out of the 7 problems.

Only turn in 5 problems. If you turn in more than 5, I will grade the first 5 in your list.
(Note that problem 7 is on the next page.)

1. (a) Use the polynomial $x^2 + 1$ to construct a finite field $\mathbb{F}_9$ of size 9. List all of the elements of $\mathbb{F}_9$.

   (b) Now find two elements $\alpha_1, \alpha_2 \in \mathbb{F}_9$ where $\alpha_1^2 + 1 = 0$ and $\alpha_2^2 + 1 = 0$.

2. Let E be the splitting field of the polynomial $x^6 - 5$ over the rationals $\mathbb{Q}$.

   (a) Find E and $[E : \mathbb{Q}]$. Explain with all the details.

   (b) List out the elements of $\text{Gal}(E/\mathbb{Q})$.
   (*Make sure to explain what the elements do to the elements of E, ie describe the elements of Gal$(E/\mathbb{Q})$ like we did in class.*)

3. Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$. Then R is a ring under regular addition and multiplication in the real numbers. Let

$$S = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

   Then S is a ring under regular matrix addition and multiplication.

   Now let $\phi : R \to S$ be defined by $\phi(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$. Is ϕ a homomorphism? Why or why not. Is ϕ an isomorphism? Why or why not.

4. Let R be an integral domain.

   (a) Given $x \in R$ prove that $(x) = \{xr \mid r \in R\}$ is an ideal of R.
   (*Prove this directly by checking the ideal properties. That is, verify it like how we verified that subsets were ideals in class.*)

   (b) Let $a, b \in R$. Prove that $(a) = (b)$ if and only if $a = bu$ where $u \in R$ is a unit.

5. Let K be a finite extension of a field F with $[K : F] = 47$. Let $a \in K$ with $a \notin F$. Prove that $F(a) = K$.

6. Show that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.

   (*You can use facts such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{\frac{3}{2}}$, $\sqrt{\frac{2}{3}}$ are not rational. Though you don't necessarily need these facts.*)

7. Let R be a principal ideal domain. Let I be an ideal with $I \neq \{0\}$ and $I \neq R$. Prove that there exists a maximal ideal M of R with $I \subseteq M$.

   (For this problem, you may assume the following fact: Since R is a PID, every non-zero element of R that isn't a unit can be written as the product of irreducible elements of R. That is, if $x \in R$, then $x = a_1 \cdot a_2 \cdots a_n$ where $a_i \in R$ and each a_i is irreducible in R.)