Math 4570
8/23/21

- I made an email list Using your calstatela email. I'll use that to mass email the class. If you use a different email, let me know and I'll add it to the list.

- Testing will be done through canvas. There will be no class on test day. The test will appear on canvas at say 6am and will stay there until noon on the following day. [Ex: Mon 6am - Tues noon] Within that you pick the time window [prob 2hr or 2.5 hrs] you take the test. Canvas will time you. Upload a scan of solutions.

Def: A field F is a set with two binary operations denoted by + and $\cdot$, such that the following are true.

(F1) For every $a, b \in F$, there exist unique elements $a+b$ and $a \cdot b$ in F.

(F2) For every $a, b, c \in F$ we have

$a+b=b+a$ $a \cdot b = b \cdot a$ (commutative properties)	$a+(b+c)=(a+b)+c$ $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ (associative properties)	$a\cdot(b+c)$ $=a\cdot b+a\cdot c$ $(b+c)\cdot a =$ $b\cdot a + c\cdot a$ (distributive properties)

(F3) There exists elements 0 and 1 in F where $a+0=0+a=a$ and $a\cdot 1 = 1\cdot a = a$ for all a in F.

(F4) For every $a \in F$ there exists $d \in F$ where $a+d=d+a=0$.

(F5) For every $a \in F$ with $a \neq 0$, there exists $f \in F$ where

$$a \cdot f = f \cdot a = 1$$

HW: 0, 1, d, f from F3 / F4 / F5 are unique.

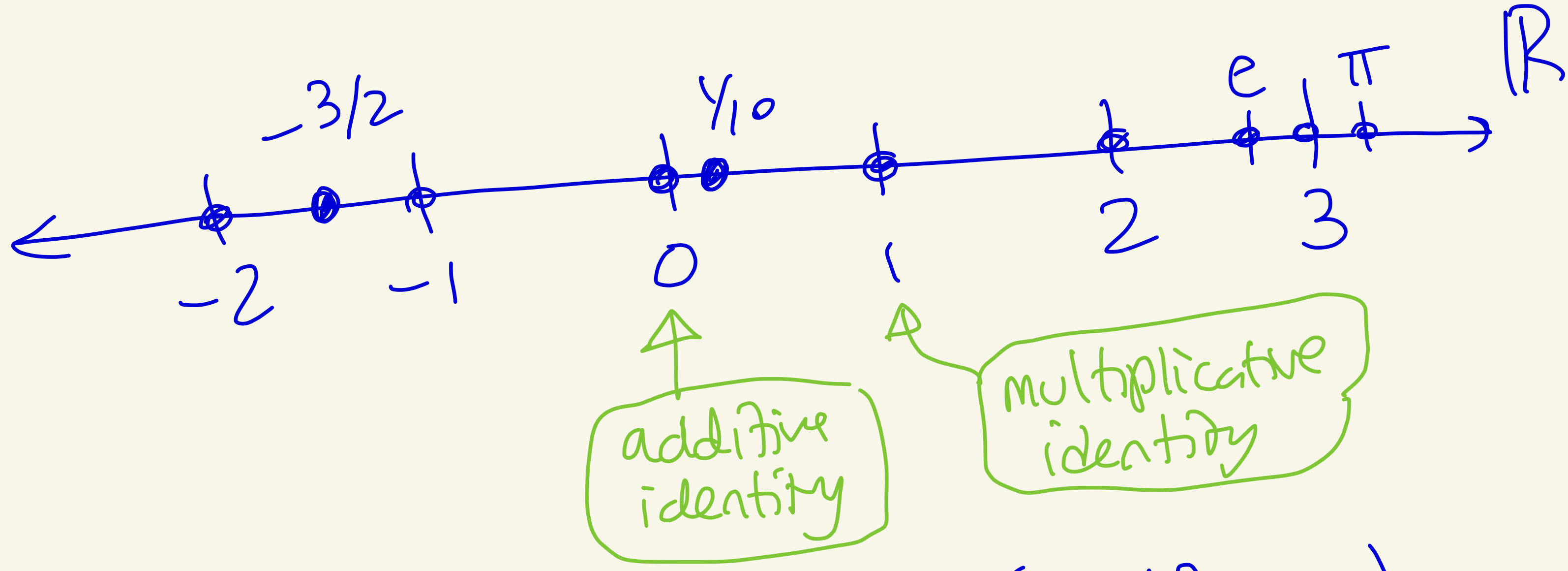
We call 0 the additive identity of F.

We call 1 the multiplicative identity of F.

We denote d in F4 as $-a$ and call it the additive inverse of a.

We denote f in F5 as a^{-1} and call it the multiplicative inverse of a.

Ex: $F = \mathbb{R}$ the set of real numbers is a field.



$a = \frac{1}{2}$, $-a = -\frac{1}{2}$ (additive inverse)

$a = \pi$, $a^{-1} = \frac{1}{\pi}$ (multiplicative inverse)

Ex: $F = \mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$

$= \left\{ -1, 0, 1, 10, \frac{1}{2}, \frac{-3}{7}, \ldots \right\}$

[rational numbers]

is a field.

$a = \frac{-3}{7}$, $-a = \frac{3}{7}$, $a^{-1} = \frac{-7}{3}$

Ex:

$$F = \mathbb{C} = \{ x+iy \mid x,y \in \mathbb{R} \}$$
$$= \{ 1 = 1+0i,\ 0 = 0+0i,\ \tfrac{1}{2} = \tfrac{1}{2}+0i,\ 1+i, \ldots \}$$

[$i^2 = -1$]

$\mathbb{C}$ is a field.

[3450, 4550, 4460]

Ex: If p is a prime, then

$$\mathbb{Z}_p = \{ \bar{0}, \bar{1}, \bar{2}, \ldots, \overline{p-1} \}$$

is a field. [$\mathbb{Z}_p$ is called the integers modulo p.]

[We won't use $\mathbb{Z}_p$ in this class]

Def: Let F be a field.
A vector space over F is a set V with two operations. The first operation is addition which takes two elements $v_1, v_2 \in V$ and produces a unique element $v_1 + v_2 \in V$.
The second operation is called scalar multiplication, which takes one element $a \in F$ and one element $v \in V$ and produces a unique element $av \in V$.

could write $a \cdot v$

The set V is sometimes called the set of "vectors" and F is sometimes called the "scalars"

The following properties must hold:

(V1) For all $v_1, v_2 \in V$ we have $v_1 + v_2 = v_2 + v_1$.

commutative property

(V2) For every $v_1, v_2, v_3 \in V$ we have $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

[associative property]

(V3) There exists an element $\vec{0}$ in V where $\vec{0} + w = w + \vec{0} = w$ for all $w \in V$.

(V4) For every $w \in V$ there exists $z \in V$ with

$$w + z = z + w = \vec{0}$$

(V5) For each $w \in V$ we have

$1w = w$ [Here 1 is from F]

(V6) For every $a, b \in F$ and $w \in V$ we have

$$(ab)w = a(bw)$$

V7 For all $a \in F$ and $v_1, v_2 \in V$ we have
$$a(v_1 + v_2) = av_1 + av_2$$

V8 For all $a, b \in F$ and $w \in V$ we have $(a+b)w = aw + bw$

<u>Note:</u> Later we will show that $\vec{0}$ from V3 and the z from V4 are unique.

$\vec{0}$ is called the <u>zero vector</u> in V

z is called the <u>additive inverse of w</u> and will be written $z = -w$.

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<u>Ex:</u> $F = \mathbb{R}$,

$$V = \mathbb{R}^2 = \{ (x,y) \mid x, y \in \mathbb{R} \}$$

Then $V = \mathbb{R}^2$ is a vector space over $F = \mathbb{R}$.

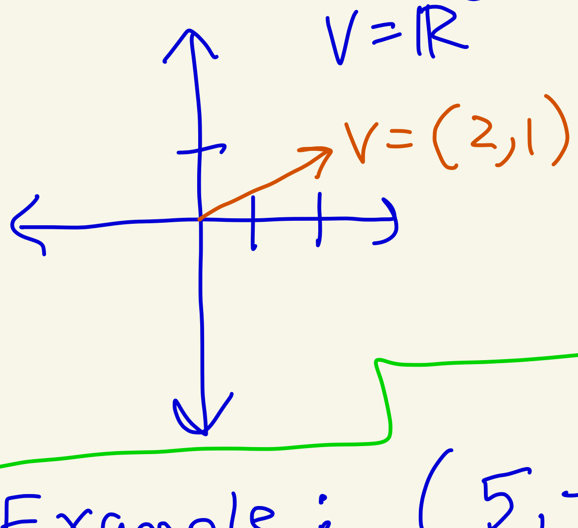
Where

$$(a,b) + (x,y) = (a+x, b+y)$$ ← vector addition

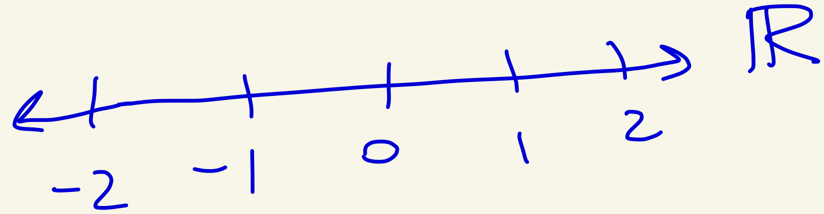
$$\alpha (x,y) = (\alpha x, \alpha y)$$ ← scalar multiplication

α = alpha

<u>Vectors</u>



<u>scalars/field</u>



Example: $(5,-1) + (2,7) = (7,6)$

$$3(5,-1) = (15,-3)$$

<u>Ex:</u> Let F be a field.

Let

$$V = F^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in F\}$$

where $n \geq 1$.

Then $V = F^n$ is a vector space over F using the following operations.

Let $\alpha \in F$ and

$$v = (a_1, a_2, \dots, a_n)$$

$$w = (b_1, b_2, \dots, b_n)$$

define vector addition as

$$v + w = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and scalar multiplication as

$$\alpha v = (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$$

proof: Let $\alpha, \beta \in F$ and $v, w, z \in V = F^n$ where $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$ and $z = (z_1, z_2, \ldots, z_n)$.

V1 We have that

$$
\begin{aligned}
v + w &= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \\
&= (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \\
&= (w_1 + v_1, w_2 + v_2, \ldots, w_n + v_n) \\
&= (w_1, w_2, \ldots, w_n) + (v_1, v_2, \ldots, v_n) \\
&= w + v
\end{aligned}
$$

Since F is a field $a+b=b+a$ $\forall a, b \in F$ F2 prop

Pg 4

V2 We have that

$$v+(w+z) = (v_1, v_2, \dots, v_n) + \left[(w_1, w_2, \dots, w_n) + (z_1, z_2, \dots, z_n)\right]$$

$$= (v_1, v_2, \dots, v_n) + (w_1+z_1, w_2+z_2, \dots, w_n+z_n)$$

$$= (v_1+(w_1+z_1), v_2+(w_2+z_2), \dots, v_n+(w_n+z_n))$$

$$= ((v_1+w_1)+z_1, (v_2+w_2)+z_2, \dots, (v_n+w_n)+z_n)$$

$$= (v_1+w_1, v_2+w_2, \dots, v_n+w_n) + (z_1, z_2, \dots, z_n)$$

$$= \left[(v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n)\right] + (z_1, z_2, \dots, z_n)$$

$$= [v+w]+z$$

F2 prop
$a+(b+c)=(a+b)+c$
$\forall a,b,c \in F$

V3 Define

$$\vec{0} = (0, 0, \ldots, 0)$$

where 0 is the zero element of F.

Then,

$$z + \vec{0} = (z_1, z_2, \ldots, z_n) + (0, 0, \ldots, 0)$$

$$= (z_1 + 0, z_2 + 0, \ldots, z_n + 0)$$

$$= (z_1, z_2, \ldots, z_n)$$

$$= z$$

F3 prop
$a + 0 = 0 + a = a$
$\forall a \in F$

and

$$\vec{0} + z = (0, 0, \ldots, 0) + (z_1, z_2, \ldots, z_n)$$

$$= (0 + z_1, 0 + z_2, \ldots, 0 + z_n)$$

$$= (z_1, z_2, \ldots, z_n)$$

$$= z$$

V4 Given $v = (v_1, v_2, \ldots, v_n)$

consider $-v = (-v_1, -v_2, \ldots, -v_n)$

where $-v_i$ is the additive inverse of v_i in F.

Using F4

Then,

$$v + (-v) = (v_1 - v_1, v_2 - v_2, v_n - v_n)$$
$$= (0, 0, \ldots, 0) = \vec{0}$$

and

$$(-v) + v = (-v_1 + v_1, -v_2 + v_2, \ldots, -v_n + v_n)$$
$$= (0, 0, \ldots, 0) = \vec{0}$$

(V5) Let 1 be the multiplicative identity of F.

Then,

$$1 \cdot v = 1 \cdot (v_1, v_2, \ldots, v_n)$$

$$= (1 v_1, 1 v_2, \ldots, 1 v_n)$$

F3

$$\overset{\triangledown}{=} (v_1, v_2, \ldots, v_n) = v$$

(V6) We have that

$$(\alpha\beta) w = (\alpha\beta)(w_1, w_2, \ldots, w_n)$$

$$= ((\alpha\beta) w_1, (\alpha\beta) w_2, \ldots, (\alpha\beta) w_n)$$

F2

$a(bc) = (ab)c$ $\forall a,b,c \in F$

$$\overset{\triangledown}{=} (\alpha(\beta w_1), \alpha(\beta w_2), \ldots, \alpha(\beta w_n))$$

$$= \alpha(\beta w_1, \beta w_2, \ldots, \beta w_n)$$

$$= \alpha[\beta(w_1, w_2, \ldots, w_n)]$$

$$= \alpha[\beta w]$$

V7 We have that

$$\alpha(v+w)$$

$$= \alpha\left[(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)\right]$$

$$= \alpha(v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

$$= (\alpha(v_1 + w_1), \alpha(v_2 + w_2), \ldots, \alpha(v_n + w_n))$$

$$= (\alpha v_1 + \alpha w_1, \alpha v_2 + \alpha w_2, \ldots, \alpha v_n + \alpha w_n)$$

$$= (\alpha v_1, \alpha v_2, \ldots, \alpha v_n) + (\alpha w_1, \alpha w_2, \ldots, \alpha w_n)$$

$$= \alpha(v_1, v_2, \ldots, v_n) + \alpha(w_1, w_2, \ldots, w_n)$$

$$= \alpha v + \alpha w$$

F2
$a(b+c) = ab + ac$
$\forall a, b, c \in F$

V8 We have that

$$(\alpha+\beta)w = (\alpha+\beta)(w_1, w_2, \ldots, w_n)$$

$$= ((\alpha+\beta)w_1, (\alpha+\beta)w_2, \ldots, (\alpha+\beta)w_n)$$

$$= (\alpha w_1 + \beta w_1, \alpha w_2 + \beta w_2, \ldots, \alpha w_n + \beta w_n)$$

F2
$(a+b)c = ac+bc$
$\forall a,b,c \in F$

$$= (\alpha w_1, \alpha w_2, \ldots, \alpha w_n) + (\beta w_1, \beta w_2, \ldots, \beta w_n)$$

$$= \alpha(w_1, w_2, \ldots, w_n) + \beta(w_1, w_2, \ldots, w_n)$$

$$= \alpha w + \beta w$$

Since V1 - V8 are true, $V = F^n$ is a vector space over F. $\blacksquare$

Ex:

$V = \mathbb{R}^5$ is a vector space over $F = \mathbb{R}$

$V = \mathbb{Q}^{10,000,000}$ is a vector space over $F = \mathbb{Q}$

<u>Ex:</u> Let F be a field. Let $V = M_{m,n}(F)$ be the set of all $m \times n$ matrices with entries from F. Then one can show that V is a vector space over F where vector addition is defined as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

[more on next page]

and scalar multiplication is defined as

$$\alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

Where

$$\vec{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

<u>Proof:</u> Similar to last example. $\blacksquare$

Ex: $F = \mathbb{R}$

$$V = M_{2,3}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; a,b,c,d,e,f \in \mathbb{R} \right\}$$

$$\vec{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example of computation is

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & \pi \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 3 & 0 \\ 5 & 2 & 1+\pi \end{pmatrix}$$

and

$$\frac{1}{2}\begin{pmatrix} 3 & 0 & 1 \\ \pi & \sqrt{2} & 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 & 1/2 \\ \frac{\pi}{2} & \frac{\sqrt{2}}{2} & \frac{5}{2} \end{pmatrix}$$

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Office hours

Monday 12:30 – 1:30

Tuesday 12:30 – 2:00

Zoom link is on canvas under "Office hours" page.

Ex: Let $F = \mathbb{R}$ or $F = \mathbb{C}$.

Let $n \geq 0$ be an integer.

Define

$$P_n(F) = \{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \mid a_i \in F \}$$

So, $P_n(F)$ are all polynomials of degree $\leq n$ with coefficients from the field F.

One can show that $V = P_n(F)$ is a vector space over F where vector addition is given by:

$$(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$$

and scalar multiplication is given by

$$\alpha(a_0 + a_1 x + \dots + a_n x^n)$$
$$= (\alpha a_0) + (\alpha a_1) x + \dots + (\alpha a_n) x^n$$

Note: In $P_n(F)$, the zero vector is $\vec{0} = 0 + 0x + \dots + 0x^n$.

Equality:

We define equality as follows:

Let $f = a_0 + a_1 x + \dots + a_n x^n$

and $g = b_0 + b_1 x + \dots + b_n x^n$.

We define $f = g$ iff

$$a_0 = b_0, \; a_1 = b_1, \dots, \; a_n = b_n$$

Ex: Let $F = \mathbb{R}$.

Consider

$$V = P_4(\mathbb{R})$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_i \in \mathbb{R} \}$$

$$= \{ 0, \ 5, \ \pi + 3x^2 - x^4, \ x^4, \ \ldots \}$$

$5 = 5 + 0x + 0x^2 + 0x^3 + 0x^4$

$0 = 0 + 0x + 0x^2 + 0x^3 + 0x^4$

example of adding:

$$(\pi + 3x^2 - x^4) + (1 - x^2 + x^3)$$

$$= (\pi + 1) + 2x^2 + x^3 - x^4$$

example of scaling:

$$\frac{1}{2}(1 - 6x^2 + x^4) = \frac{1}{2} - 3x^2 + \frac{1}{2}x^4$$

$P_4(\mathbb{R})$ is like $\mathbb{R}^5$

[pg 5]

$$1 + x - x^2 + 5x^3 - 7x^4$$ ← in $P_4(\mathbb{R})$

$$\updownarrow$$

$$(1, 1, -1, 5, -7)$$ ← in $\mathbb{R}^5$

<u>Theorem:</u> Let V be a vector space over a field F.

(1) The element $\vec{0}$ from (V3) is unique.

[That is, there is only one vector $\vec{0}$ in V that satisfies $\vec{0}+w=w+\vec{0}=w$ for all $w \in V$.]

(2) Given $w \in V$, the element z from (V4) where $w+z=z+w=\vec{0}$ is unique.

[Recall we write z as $-w$]

<u>proof:</u>

(1) Suppose $\vec{0}_1, \vec{0}_2 \in V$ where

$$\vec{0}_1+w=w+\vec{0}_1=w$$

and $\vec{0}_2+w=w+\vec{0}_2=w$

for all $w \in V$.

So, $\vec{0}_1$ and $\vec{0}_2$ are both zero vectors.

Then,

$$\vec{0}_1 = \vec{0}_1 + \vec{0}_2 = \vec{0}_2$$

$\boxed{w = w + \vec{0}_2}$ $\boxed{\vec{0}_1 + w = w}$

Thus, $\vec{0}_1 = \vec{0}_2$.

So there can be only one zero vector.

(2) Let $w \in V$.

Suppose $z_1, z_2 \in V$ where

$$w + z_1 = z_1 + w = \vec{0}$$

and $w + z_2 = z_2 + w = \vec{0}$.

So, z_1, z_2 are both additive inverses for w

We have $w + z_1 = \vec{0}$.

Add z_2 to both sides to get

$$z_2 + (w + z_1) = z_2 + \vec{0}$$

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Thus, using associativity we have

$$\underbrace{(z_2 + w)}_{\vec{0}} + z_1 = z_2$$

Thus, $\vec{0} + z_1 = z_2$.

So, $z_1 = z_2$.

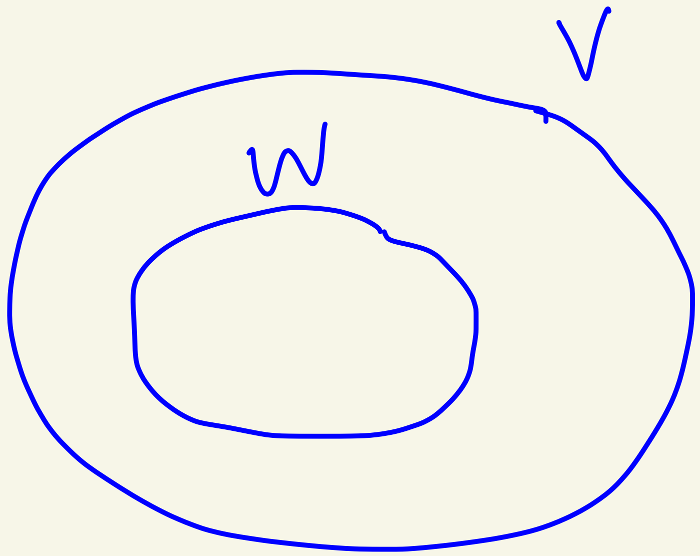
Ergo, there is only one additive inverse for w.

$\blacksquare$

<u>Def:</u> Let V be a vector space over a field F.

Let $W \subseteq V$.

We say that W is a <u>subspace</u> of V if W is a vector space over F using the same vector addition and scalar multiplication as in V

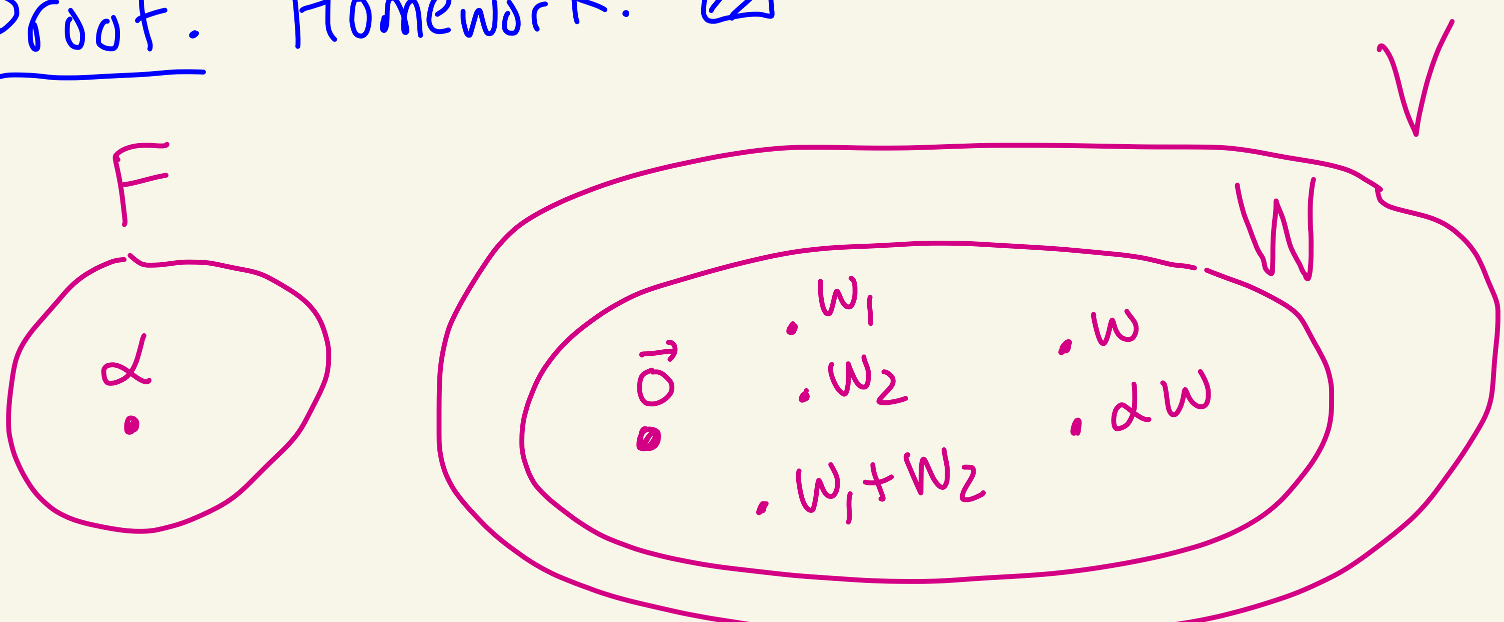


Theorem: Let V be a vector space over a field F. Let W be a subset of V.

W is a subspace of V if and only if the following three conditions hold:

1. $\vec{0} \in W$ — you can actually just show $W \neq \emptyset$
2. If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$. — W is closed under $+$
3. If $\alpha \in F$ and $w \in W$, then $\alpha w \in W$ — W is closed under scaling

Proof: Homework. $\boxed{}$



PICTURE OF 1, 2, 3

Ex: Let $V = \mathbb{R}^3$, $F = \mathbb{R}$.

Let

$$W = \{ (0, b, c) \mid b, c \in \mathbb{R} \}$$
$$= \{ (0, 1, \pi), (0, -1, \sqrt{2}), \ldots \}$$

Is W a subspace of V?

It is!

Let's prove it.

① Setting $b = 0$, $c = 0$ gives

$(0, b, c) = (0, 0, 0)$ is in W.

So, $\vec{0} \in W$.

② Let $w_1, w_2 \in W$.

Then, $w_1 = (0, b_1, c_1)$ and

$w_2 = (0, b_2, c_2)$ where b_1, c_1, b_2, c_2 are in $\mathbb{R}$.

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Then,

$w_1 + w_2 = (0, b_1 + b_2, c_1 + c_2)$

which is in W, since $b_1 + b_2, c_1 + c_2 \in \mathbb{R}$

3 Let $\alpha \in \mathbb{R}$ and $w \in W$.

Then, $w = (0, b, c)$ where $b, c \in \mathbb{R}$.

And $\alpha w = (0, \alpha b, \alpha c)$ which is still in W, since $\alpha b, \alpha c \in \mathbb{R}$.

By 1, 2, and 3

W is a subspace of $V = \mathbb{R}^3$.

$\blacksquare$

Ex: Let

$V = P_2(\mathbb{R})$ and $F = \mathbb{R}$.

Let

$$W = \{ 1 + bx \mid b \in \mathbb{R} \}$$
$$= \{ 1+2x, 1-3x, \ldots \}$$

Is W a subspace of $P_2(\mathbb{R})$?

No. For example

$1+2x, 1-3x \in W$

but

$$(1+2x) + (1-3x) = 2 - x \notin W$$

not 1

Note: Let V be a vector space over F.

V has at least these subspaces:

$$W = \{ \vec{0} \}$$ trivial subspace of V

$$W = V$$

$\boxed{\text{HW 2}}$ — Bases of vector spaces | pg 15

Def: Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$.

① The span of $v_1, v_2, \ldots, v_n$ is defined to be

$$\text{span}(\{v_1, v_2, \ldots, v_n\}) = \left\{ \underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}_{\text{is called a linear combination of } v_1, v_2, \ldots, v_n} \;\middle|\; \alpha_1, \ldots, \alpha_n \in F \right\}$$

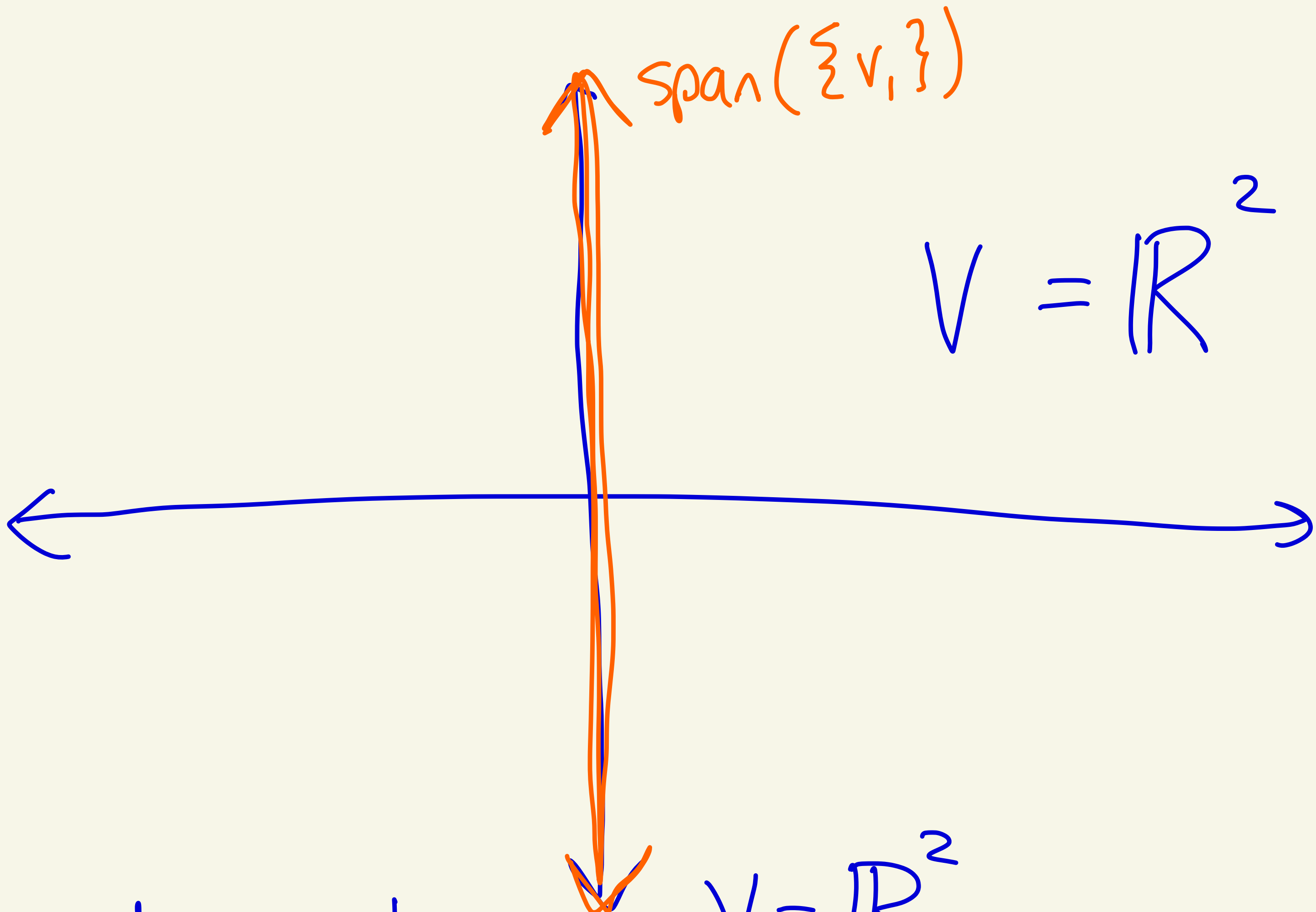
② If $V = \text{span}(\{v_1, v_2, \ldots, v_n\})$ then we say that $v_1, v_2, \ldots, v_n$ span V or we say that $v_1, v_2, \ldots, v_n$ is a spanning set for V.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

Let $v_1 = (0,1)$.

Then,

$$\begin{aligned} \text{span}(\{v_1\}) &= \{\alpha_1 v_1 \mid \alpha_1 \in \mathbb{R}\} \\ &= \{\alpha_1 (0,1) \mid \alpha_1 \in \mathbb{R}\} \\ &= \{(0,\alpha_1) \mid \alpha_1 \in \mathbb{R}\} \end{aligned}$$



v_1 does not span $V = \mathbb{R}^2$.

<u>Ex:</u> Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

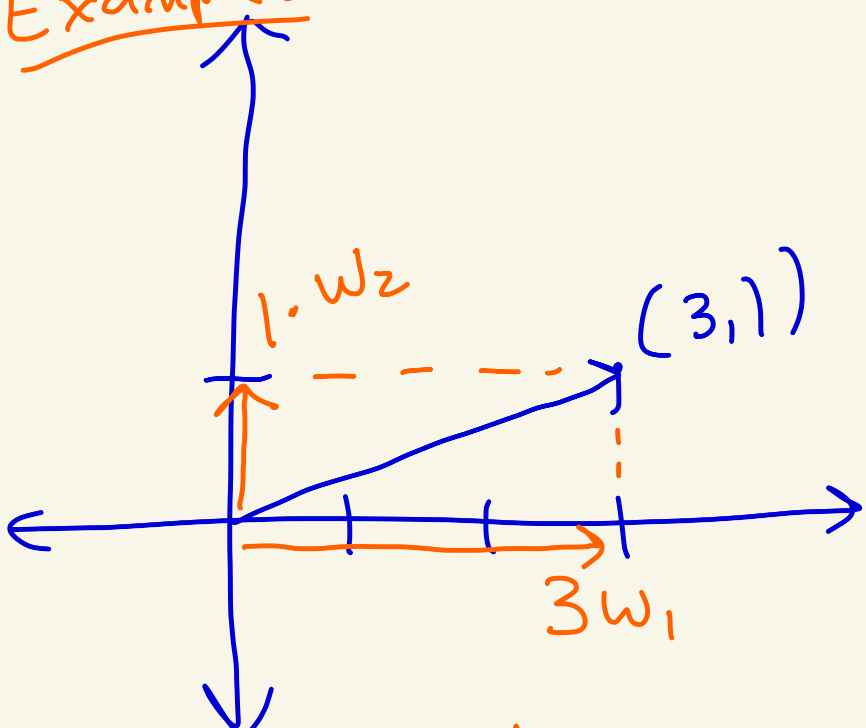
Let $w_1 = (1,0)$, $w_2 = (0,1)$.

Then,

$$\begin{aligned}
\operatorname{span}(\{w_1, w_2\}) &= \{\alpha_1 w_1 + \alpha_2 w_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\
&= \{\alpha_1 (1,0) + \alpha_2 (0,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\
&= \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} \\
&= \mathbb{R}^2
\end{aligned}$$

So, $\mathbb{R}^2$ is spanned by w_1 and w_2.

<u>Example:</u>



$(3,1) = 3w_1 + 1 \cdot w_2$

Math 4570

9/1/21

Last time we talked about spanning.

We showed $v_1 = (1,0)$, $v_2 = (0,1)$ span $V = \mathbb{R}^2$.

Why? Because given any $(a,b) \in \mathbb{R}^2$ then

$$(a,b) = a(1,0) + b(0,1)$$

That is, every vector in $\mathbb{R}^2$ is a linear combo. of v_1, v_2.

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $v_1 = (2,1)$, $v_2 = (-1,1)$.

Do v_1, v_2 span $\mathbb{R}^2$?

Let $(a,b) \in \mathbb{R}^2$.

The question is: Can we always solve the following equation for c_1, c_2 no matter what (a,b) is?

$$(a,b) = c_1 \underbrace{(2,1)}_{v_1} + c_2 \underbrace{(-1,1)}_{v_2}$$

The above equation is equivalent to

$$(a,b) = (2c_1 - c_2,\ c_1 + c_2)$$

This is equivulent to

$$\boxed{\begin{aligned} 2c_1 - c_2 &= a \\ c_1 + c_2 &= b \end{aligned}}$$

3 operations for Gaussian elimination

1. interchange two rows
2. multiply a row by a non-zero constant
3. Add a multiple of one row to another row.

2550

We had $\boxed{\begin{aligned} 2c_1 - c_2 &= a \\ c_1 + c_2 &= b \end{aligned}}$

$$\left(\begin{array}{cc|c} 2 & -1 & a \\ 1 & 1 & b \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 2 & -1 & a \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & -3 & -2b+a \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & b \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{3}b \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} c_1 + c_2 &= b \\ c_2 &= -\tfrac{1}{3}a + \tfrac{2}{3}b \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

(2) gives $c_2 = -\frac{1}{3}a + \frac{2}{3}b$.

Sub into (1) to get

$$c_1 = b - c_2 = b - \left(-\frac{1}{3}a + \frac{2}{3}b\right)$$

$$= \frac{1}{3}a + \frac{1}{3}b.$$

Thus, given any $(a,b) \in \mathbb{R}^2$ we can write

$$(a,b) = \underbrace{\left(\frac{1}{3}a + \frac{1}{3}b\right)}_{c_1} \cdot \underbrace{(2,1)}_{v_1} + \underbrace{\left(-\frac{1}{3} + \frac{2}{3}b\right)}_{c_2} \cdot \underbrace{(-1,1)}_{v_2}$$

for example,

$$(1,1) = \frac{2}{3}(2,1) + \frac{1}{3}(-1,1)$$

We showed that
$$\mathbb{R}^2 = \text{span}\{(2,1), (-1,1)\}$$

<u>Lemma:</u> (Hw 1 #4a)
Let V be a vector space over a field F. Let $\vec{0}$ be the zero vector of V and let 0 be the zero element of F.
Then, $0w = \vec{0}$ for all $w \in V$.

<u>proof:</u> We have that
$$0w \overset{F3}{=} (0+0)w \overset{V8}{=} 0w + 0w$$
We know $-(0w)$ exists in V by V4.

pg 6

Thus,

$$\underbrace{-(0w)+0w}_{\vec{0}} = \underbrace{-(0w)+0w}_{\vec{0}}+0w$$

So, $\vec{0} = \underbrace{\vec{0}+0w}_{0w}$

Thus, $\vec{0} = 0w$. $\blacksquare$

Theorem: Let V be a vector space over a field F.
Let $v_1, v_2, \ldots, v_n \in V$.

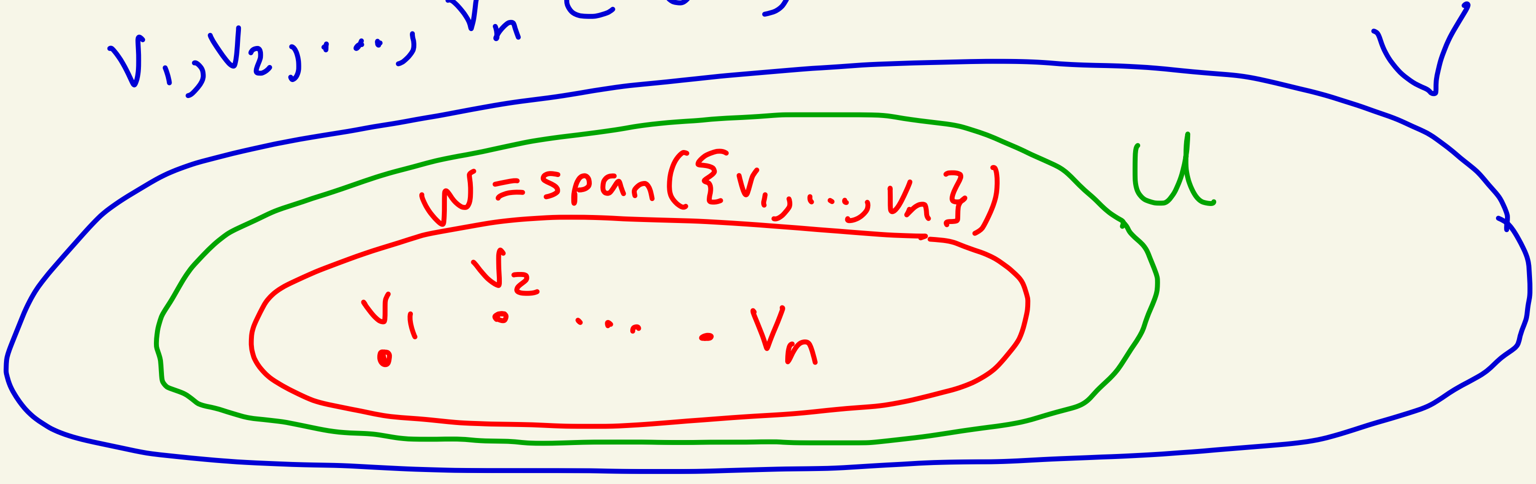
Let

$$W = \text{span}(\{v_1, v_2, \ldots, v_n\}) = \{c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \mid c_1, \ldots, c_n \in F\}$$

Then:

① W is a subspace of V.

② W is the "smallest" subspace that contains $v_1, v_2, \ldots, v_n$. That is, if U is any subspace with $v_1, v_2, \ldots, v_n \in U$, then $W \subseteq U$.



proof:

① Let's show W is a subspace of V.

(i) If we set $c_1 = c_2 = \dots = c_n = 0$ then we have that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n =$$

$$= 0 v_1 + 0 v_2 + \dots + 0 v_n$$

lemma

$$= \vec{0} + \vec{0} + \dots + \vec{0}$$

$$= \vec{0}.$$

Thus, $\vec{0} \in W$.

(ii) Let's show W is closed under $+$.

Let $w_1, w_2 \in W$.

Then, $w_1 = s_1 v_1 + s_2 v_2 + \dots + s_n v_n$

and $w_2 = t_1 v_1 + t_2 v_2 + \dots + t_n v_n$

where $s_1, s_2, \dots, s_n, t_1, t_2, \dots, t_n \in F$.

Then,

$$w_1 + w_2 = s_1 v_1 + s_2 v_2 + \dots + s_n v_n + t_1 v_1 + t_2 v_2 + \dots + t_n v_n$$

$$= \underbrace{(s_1 + t_1)}_{\text{in } F} v_1 + \underbrace{(s_2 + t_2)}_{\text{in } F} v_2 + \dots + \underbrace{(s_n + t_n)}_{\text{in } F} v_n$$

V8

$av + bv = (a+b)v$

Thus, $w_1 + w_2 \in W$, since $s_1 + t_1, s_2 + t_2, \dots, s_n + t_n \in F$.

(iii) Let's show W is closed under scalar multiplication.

Let $z \in W$ and $\alpha \in F$.

We need to show that $\alpha z \in W$.

Since $z \in W$ we know that

$$z = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

for some $c_1, c_2, \dots, c_n \in F$.

Then,

$$\alpha z = \alpha(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$$\overset{V7}{=} \alpha(c_1 v_1) + \alpha(c_2 v_2) + \cdots + \alpha(c_n v_n)$$

$$\overset{V6}{=} \underbrace{(\alpha c_1)}_{\text{in } F \text{ (F1)}} v_1 + \underbrace{(\alpha c_2)}_{\text{in } F \text{ (F1)}} v_2 + \cdots + \underbrace{(\alpha c_n)}_{\text{in } F \text{ (F1)}} v_n$$

V7: $a(v_1 + v_2) = av_1 + av_2$

V6: $(ab)w = a(bw)$

Thus, $\alpha z \in W$, because $\alpha c_1, \alpha c_2, \ldots, \alpha c_n \in F$.

By (i), (ii), (iii),

W is a subspace of V. $\boxed{1}$

2 Let $W = \text{span}(\{v_1, v_2, \dots, v_n\})$

Let U be a subspace of V

where $v_1, v_2, \dots, v_n \in U$

We want to show that $W \subseteq U$.

Let $x \in W$.

Then, $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

where $c_1, c_2, \dots, c_n \in F$.

Since $v_1, v_2, \dots, v_n \in U$ and U is a subspace of V we know that $c_1 v_1, c_2 v_2, \dots, c_n v_n \in U$. U is closed under scalar mult.

Since $c_1 v_1, c_2 v_2, \dots, c_n v_n \in U$ and V is a subspace of V we know that $c_1 v_1 + c_2 v_2 + \dots + c_n v_n \in U$. U is closed under +

Thus, $x \in U$.

So, $W \subseteq V$. $\boxed{2}$

Math 4570

9/8/21

<u>Def:</u> Let V be a vector space over a field F.
Let $v_1, v_2, \dots, v_n \in V$.
We say that $v_1, v_2, \dots, v_n$ are <u>linearly dependent</u> if there exists $c_1, c_2, \dots, c_n \in F$, that are not all zero, such that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$$

If there are no such $c_1, c_2, \dots, c_n$ then we say that $v_1, v_2, \dots, v_n$ are <u>linearly independent</u>.

Ex: Let $V = \mathbb{R}^3$ and $F = \mathbb{R}$.

Let $v_1 = (1, 0, 1)$
$v_2 = (-1, 2, 1)$
$v_3 = (0, 2, 2)$

Are v_1, v_2, v_3 linearly dependent or linearly independent?

We want to see what the solutions are to

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

which is

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 \\ 0 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ 2c_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2c_3 \\ 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 - c_2 \\ 2c_2 + 2c_3 \\ c_1 + c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} c_1 - c_2 \qquad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}}$$

Let's solve the system:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

leading variables c_1, c_2
free variable c_3

We get:

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \quad (1) \\ c_2 + c_3 &= 0 \quad (2) \end{aligned}}$$

solve for leading variables

$$\boxed{\begin{aligned} c_1 &= c_2 \quad (1) \\ c_2 &= -c_3 \quad (2) \end{aligned}}$$

Give free variables new name.

Let $\boxed{c_3 = t}$.

Solve (1) & (2) by back substitution.

(2) gives $\boxed{c_2 = -c_3 = -t}$

(1) gives $\boxed{c_1 = c_2 = -t}$

Thus, the solutions to

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

are:

$$\boxed{\begin{aligned} c_1 &= -t \\ c_2 &= -t \\ c_3 &= t \end{aligned} \quad \text{where } t \text{ is any real number}}$$

$F = \mathbb{R}$

Thus,

$$-t v_1 - t v_2 + t v_3 = \vec{0}$$

for any $t \in \mathbb{R}$.

For example if $t = 1$, then

$$\boxed{-v_1 - v_2 + v_3 = \vec{0}}$$

dependency equation for v_1, v_2, v_3

$v_3 = v_1 + v_2$

Thus, v_1, v_2, v_3 are linearly dependent.

Ex: Let $V = P_2(\mathbb{R})$ and $F = \mathbb{R}$.

Let
$$w_1 = -3 + 4x^2$$
$$w_2 = 5 - x + 2x^2$$
$$w_3 = 1 + x + 3x^2$$

Are w_1, w_2, w_3 linearly dependent or linearly independent?

Consider the equation
$$c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}$$

This becomes
$$c_1(-3+4x^2) + c_2(5 - x + 2x^2) + c_3(1 + x + 3x^2) = 0 + 0x + 0x^2$$

This is equivalent to
$$(-3c_1 + 5c_2 + c_3) + (-c_2 + c_3)x + (4c_1 + 2c_2 + 3c_3)x^2 = 0 + 0x + 0x^2$$

Thus, we get

$$\boxed{\begin{aligned} -3c_1 + 5c_2 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \\ 4c_1 + 2c_2 + 3c_3 &= 0 \end{aligned}}$$

Solving we get

$$\left(\begin{array}{ccc|c} -3 & 5 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & -\frac{5}{3} & \frac{-1}{3} & 0 \\ 0 & -1 & 1 & 0 \\ 4 & 2 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -\frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & -1 & 1 & 0 \\ 0 & \frac{26}{3} & \frac{13}{3} & 0 \end{array}\right)$$

$$\xrightarrow[3R_3 \to R_3]{-R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 26 & 13 & 0 \end{array}\right)$$

$$\xrightarrow{-26R_2 + R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 39 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{39}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -5/3 & -1/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} c_1 - \tfrac{5}{3}c_2 - \tfrac{1}{3}c_3 &= 0 \\ c_2 - c_3 &= 0 \\ c_3 &= 0 \end{aligned}}$$

leading variables c_1, c_2, c_3

no free variables

Solve for leading variables:

$$\boxed{\begin{aligned} c_1 &= \tfrac{5}{3}c_2 + \tfrac{1}{3}c_3 && (1) \\ c_2 &= c_3 && (2) \\ c_3 &= 0 && (3) \end{aligned}}$$

Back substitute.

(3) gives $\boxed{c_3 = 0}$

(2) gives $\boxed{c_2 = c_3 = 0}$

(1) gives $\boxed{c_1 = \frac{5}{3}c_2 + \frac{1}{3}c_3 = \frac{5}{3}(0) + \frac{1}{3}(0) = 0}$

Thus the only solution to

$$c_1 w_1 + c_2 w_2 + c_3 w_3 = \vec{0}$$

is $c_1 = 0,\ c_2 = 0,\ c_3 = 0.$

Thus, w_1, w_2, w_3 are linearly independent.

Summary:

You can always write

$$0 \cdot v_1 + 0 v_2 + \dots + 0 v_n = \vec{0}$$

If this is the only solution to

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$$

then $v_1, v_2, \dots, v_n$ are linearly independent.

If there are more solutions then just the zero solution above then $v_1, v_2, \dots, v_n$ are linearly dependent.

Def: Let V be a vector space over a field F.

Let $v_1, v_2, \ldots, v_n \in V$.

We say that $v_1, v_2, \ldots, v_n$ form a **basis** for V if

(1) $\text{span}(\{v_1, v_2, \ldots, v_n\}) = V$

and (2) $v_1, v_2, \ldots, v_n$ are linearly independent.

Ex: Let $V=\mathbb{R}^2$ and $F=\mathbb{R}$.

Let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Claim: v_1, v_2 is a basis for $V=\mathbb{R}^2$

proof:

① Last class we showed that $\text{Span}(\{v_1, v_2\}) = \mathbb{R}^2$.

② Let's show that v_1, v_2 are linearly independent.

Suppose $c_1 v_1 + c_2 v_2 = \vec{0}$

That is, $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So, $c_1 = 0$, $c_2 = 0$ is the only solution to $c_1 v_1 + c_2 v_2 = \vec{0}$.

Thus, v_1, v_2 are lin. ind.

By ① and ②, v_1, v_2 are a basis for $V=\mathbb{R}^2$ over $F=\mathbb{R}$. ∎

Ex: Let

$V = M_{2,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\}$

and $F = \mathbb{R}$.

Let

$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Let $\beta = \{v_1, v_2, v_3, v_4\}$

Claim: β is a basis for $M_{2,2}(\mathbb{R})$

---

proof of claim:

① Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{R})$.

Then,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{span}(\beta)$. So, β spans $M_{2,2}(\mathbb{R})$

(2) Suppose

$$c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 = \vec{0}$$

This becomes

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Which becomes

$$\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Which gives

$$c_1 = 0, \ c_2 = 0, \ c_3 = 0, \ c_4 = 0.$$

Thus, v_1, v_2, v_3, v_4 are lin. ind.

By (1) and (2), $\beta = \{v_1, v_2, v_3, v_4\}$ form a basis for $V = M_{2,2}(\mathbb{R})$ over $F = \mathbb{R}$. $\blacksquare$

Math 4570
9/13/21

Last time we talked about what a basis is.

The next two classes we will prove some theorems about bases.

Theorem: Let V be a vector space over a field F.
Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a subset of V.
Then β is a basis for V if and only if every vector $x \in V$ can be expressed uniquely in the form
$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$
where $c_1, c_2, \ldots, c_n \in F$.

Proof:

$(\Leftarrow)$ Suppose every vector $x \in V$ can be written uniquely in the form $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$, where $c_i \in F$.

We want to show that β is a basis for V.

Since every $x \in V$ is of the form $x = c_1 v_1 + \dots + c_n v_n$ we know that

$$V = \text{span}(\{v_1, \dots, v_n\}) = \text{span}(\beta).$$

We now need to show that $v_1, v_2, \dots, v_n$ are lin. ind.

$x = \vec{0}$

Suppose we want to solve

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}.$$

We know we have

$$0v_1 + 0v_2 + \dots + 0v_n = \vec{0}$$

By our initial assumption with $x = \vec{0}$ this must be the only solution to $c_1v_1 + c_2v_2 + \dots + c_nv_n = \vec{0}$.

Thus, $v_1, v_2, \dots, v_n$ are linearly independent.

So, $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis.

($\Rightarrow$) Let β be a basis for V.

Pick some $x \in V$.

Since β is a basis for V, β spans V.

Thus, there exist $c_1, c_2, \dots, c_n \in F$ where $x = c_1v_1 + c_2v_2 + \dots + c_nv_n$. (*)

Let's show this expression is unique.

Suppose we also had

$$x = c_1' v_1 + c_2' v_2 + \dots + c_n' v_n \qquad (**)$$

for some $c_1', c_2', \dots, c_n' \in F$.

Computing $(*) - (**)$ we get

$$\vec{0} = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \dots + (c_n - c_n')v_n$$

Since $v_1, v_2, \dots, v_n$ are lin. ind. we have $c_1 - c_1' = 0$, $c_2 - c_2' = 0$, $\dots$, $c_n - c_n' = 0$.

Thus, $c_1 = c_1'$, $c_2 = c_2'$, $\dots$, $c_n = c_n'$.

So, x can be written uniquely in the form $x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$.

$\blacksquare$

Notation for the next Theorem

Consider the system

$$\boxed{\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}} \quad (*)$$

Let

$$\boxed{\begin{aligned} A_1 &= (10, -3, \tfrac{1}{3}) \\ A_2 &= (0, 5, -1) \\ A_3 &= (-1, 1, 0) \\ X &= (x_1, x_2, x_3) \end{aligned}}$$

Then $(*)$ can be rewritten as

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ A_3 \cdot X &= 0 \end{aligned}} \quad \text{Same as } (*)$$

Adding $\frac{1}{10} * (\text{row } 1)$ to (row 3)

$$\boxed{\begin{aligned} 10x_1 - 3x_2 + \tfrac{1}{3}x_3 &= 0 \\ 5x_2 - x_3 &= 0 \\ \tfrac{7}{10}x_2 + \tfrac{1}{30}x_3 &= 0 \end{aligned}}$$

which can be represented by

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ \left(\tfrac{1}{10}A_1 + A_3\right) \cdot X &= 0 \end{aligned}}$$

<u>Theorem:</u> Let

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} (*)$$

be a system of m equations and n unknowns where $a_{ij} \in F$ where F is a field.

If $n > m$, then $(*)$ has a non-trivial solution.

[That is, there is a solution $(x_1, x_2, \ldots, x_n) \in F^n$ to $(*)$ with $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$]

Proof: We induct on m [the # of equations] $\left[\begin{matrix}\text{pg}\\8\end{matrix}\right.$

base case: Suppose $m=1$.
We also assume $n > m = 1$. So, $n \geq 2$.
So, $(*)$ becomes

$$\boxed{a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0} \quad (*)$$

If $a_{11} = a_{12} = \dots = a_{1n} = 0$, then an example of a non-trivial solution would be $x_1 = x_2 = \dots = x_n = 1$.
Suppose one of the constants isn't 0.
Without loss of generality, assume $a_{11} \neq 0$.

means: the same proof will work in other situations.

Then $(*)$ becomes

$$\boxed{x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n)}$$

Set $x_2 = x_3 = \dots = x_n = 1$ and

$$x_1 = -a_{11}^{-1}(a_{12} + \dots + a_{1n}).$$

This gives a non-trivial solution to $(*)$.

Note we definitely used $n \geq 2$ to get the non-trivial solution.

So, the base case $m=1$ is true.

$$\boxed{\begin{array}{l}\underline{\text{Induction hypothesis}} \\ \text{Now assume the theorem is true} \\ \text{for any linear system of } m-1 \\ \text{equations with more than} \\ m-1 \text{ unknowns}\end{array}}$$

Suppose we have a system (*) of m equations and n unknowns with $n > m > 1$.

If all the $a_{ij} = 0$, then set $x_1 = x_2 = \dots = x_n = 1$ and we get a non-trivial solution.

Now suppose some coefficient $a_{ij} \neq 0$. By renumbering the equations and variables we may assume $a_{11} \neq 0$.

$a_{11} \neq 0$

set

$$
\begin{aligned}
A_1 &= (a_{11}, a_{12}, \dots, a_{1n}) \\
A_2 &= (a_{21}, a_{22}, \dots, a_{2n}) \\
&\vdots \\
A_m &= (a_{m1}, a_{m2}, \dots, a_{mn}) \\
X &= (x_1, x_2, \dots, x_n)
\end{aligned}
$$

Then (*) becomes

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ A_2 \cdot X &= 0 \\ &\vdots \\ A_m \cdot X &= 0 \end{aligned}} \quad (**)$$

By subtracting a multiple of the first row and adding it to the rows below it we can eliminate x_1 in rows 2 through m. We get that (**) becomes

$$\boxed{\begin{aligned} A_1 \cdot X &= 0 \\ (A_2 - a_{21} a_{11}^{-1} A_1) \cdot X &= 0 \\ &\vdots \\ (A_m - a_{m1} a_{11}^{-1} A_1) \cdot X &= 0 \end{aligned}}$$

no x_1 in these rows

The last equations

$$\boxed{\begin{array}{c} (A_2 - a_{21} a_{11}^{-1} A_1) \cdot X = 0 \\ \vdots \qquad\qquad \vdots \\ (A_m - a_{m1} a_{11}^{-1} A_1) \cdot X = 0 \end{array}} \quad (***)$$

are a system of $m-1$ equations with $n-1 > m-1$ unknowns.

Thus, by the induction hypothesis we can find a solution

$$(x_2, x_3, \ldots, x_n) \neq (0, 0, \ldots, 0)$$

to $(***)$.

Now using this solution $(x_2, \dots, x_n)$ to (***) we can also solve $A_1 \cdot X = 0$ by setting

Pg 13

$$x_1 = -a_{11}^{-1}(a_{12}x_2 + \dots + a_{1n}x_n)$$

[because $A_1 \cdot X = 0$ is $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$ and $a_{11} \neq 0$]

Set $X = (x_1, x_2, \dots, x_n)$.

We have $A_1 \cdot X = 0$.

We also have that $i \geq 2$ then

$$A_i \cdot X \overset{(***)}{=} a_{i1} a_{11}^{-1} \underbrace{A_1 \cdot X}_{0} = 0$$

Thus we have solved

$$A_1 \cdot X = 0$$
$$A_2 \cdot X = 0$$
$$\vdots$$
$$A_m \cdot X = 0.$$

with a non-trivial solution.

$\blacksquare$

Math 4570

9/15/21

Theorem: Let V be a vector space over a field F.

Let $v_1, v_2, \ldots, v_m \in V$ where $V = \text{span}(\{v_1, v_2, \ldots, v_m\})$.

Let $w_1, w_2, \ldots, w_n \in V$.

If $n > m$, then $w_1, w_2, \ldots, w_n$ are linearly dependent.

---

proof: Since $v_1, v_2, \ldots, v_m$ span V we can write

$$\begin{aligned}
w_1 &= a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m \\
w_2 &= a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m \\
&\ \ \vdots \\
w_n &= a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m
\end{aligned}$$

where $a_{ij} \in F$.

pg 2

For any $c_1, c_2, \ldots, c_n \in F$ we have that

$$
\begin{aligned}
c_1 w_1 + c_2 w_2 + \cdots + c_n w_n &= \\
&= c_1 (a_{11} V_1 + a_{21} V_2 + \cdots + a_{m1} V_m) \\
&\quad + c_2 (a_{12} V_1 + a_{22} V_2 + \cdots + a_{m2} V_m) \\
&\quad\ \ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\quad + c_n (a_{1n} V_1 + a_{2n} V_2 + \cdots + a_{mn} V_m) \\
&= (c_1 a_{11} + c_2 a_{12} + \cdots + c_n a_{1n}) V_1 \\
&\quad + (c_1 a_{21} + c_2 a_{22} + \cdots + c_n a_{2n}) V_2 \\
&\quad\ \ \vdots \qquad\qquad\qquad \vdots \\
&\quad + (c_1 a_{m1} + c_2 a_{m2} + \cdots + c_n a_{mn}) V_m
\end{aligned}
$$

From the theorem from Monday, since $n > m$ we know that

$$\boxed{\begin{aligned} c_1 a_{11} + c_2 a_{12} + \dots + c_n a_{1n} &= 0 \\ c_1 a_{21} + c_2 a_{22} + \dots + c_n a_{2n} &= 0 \\ &\vdots \\ c_1 a_{m1} + c_2 a_{m2} + \dots + c_n a_{mn} &= 0 \end{aligned}}$$

has a non-trivial solution $(\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n) \neq (0, 0, \dots, 0)$.

Plugging this solution into the previous page we will get

$$\hat{c}_1 w_1 + \hat{c}_2 w_2 + \dots + \hat{c}_n w_n$$
$$= 0 v_1 + 0 v_2 + \dots + 0 v_m = \vec{0}$$

Thus, $w_1, w_2, \dots, w_n$ are lin. dep. $\blacksquare$

<u>Corollary:</u> Let V be a vector space over a field F. Suppose $\beta_1 = \{v_1, v_2, \ldots, v_a\}$ and $\beta_2 = \{w_1, w_2, \ldots, w_b\}$ are both bases for V. Then $a = b$.

<u>proof:</u>

Since β_1 is a basis for V we know that β_1 spans V.

If $b > a$, then by the previous theorem, β_2 would be a linearly dependent set of vectors.

But β_2 is a basis, so the β_2 is a set of linearly independent vectors.

Thus, $b \leq a$.

Now we show $a \le b$.

Since β_2 is a basis for V we know that β_2 spans V.

If $a > b$, then by the previous theorem, β_1 would be a linearly dependent set of vectors.

But β_1 is a basis, so the β_1 is a set of linearly independent vectors.

Thus, $\boxed{a \le b}$.

Since $b \le a$ and $a \le b$ we know that $a = b$. ∎

The previous Corollary allows us to make the following definition.

Def: Let V be a vector space over a field F. We say that V is finite dimensional if it has a basis consisting of a finite number of elements.

If V has a basis with n elements then we say that V has dimension n and write $\underbrace{\dim(V) = n}_{\text{some write}}$

$\dim_F(V) = n$

$V = \{\vec{0}\}$ is called the <u>trivial</u> vector space.

A special case is when $V = \{\vec{0}\}$.
This vector space has no basis.
We define $V = \{\vec{0}\}$ to have dimension zero,
that is $\dim(\{\vec{0}\}) = 0$.

<u>Ex:</u> Let F be a field and $V = F^n$ where $n \geq 1$.

Recall $V = F^n$ is a vector space over F.

We now show that $\dim(F^n) = n$

<u>proof:</u> We will construct what is called the standard basis.

Let v_i be the vector with a 1 in the i-th spot and 0's everywhere else.

That is,

$$v_1 = (1, 0, 0, \ldots, 0)$$
$$v_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots \qquad \vdots$$
$$v_n = (0, 0, 0, \ldots, 1)$$

Let $\beta = \{v_1, v_2, \ldots, v_n\}$

We will now show that β is a basis for $V = F^n$ which will give us that $\dim(F^n) = n$.

① β spans $V = F^n$:

Let $x \in F^n$

Then, $x = (f_1, f_2, \ldots, f_n)$ where $f_1, f_2, \ldots, f_n \in F$.

So,

$$\begin{aligned} x &= (f_1, f_2, \ldots, f_n) \\ &= (f_1, 0, \ldots, 0) + (0, f_2, \ldots, 0) \\ &\quad + \cdots + (0, 0, \ldots, f_n) \\ &= f_1(1, 0, \ldots, 0) + f_2(0, 1, \ldots, 0) \\ &\quad + \cdots + f_n(0, 0, \ldots, 1) \\ &= f_1 v_1 + f_2 v_2 + \cdots + f_n v_n \end{aligned}$$

Thus, $x \in \text{span}(\beta)$.

Therefore, β spans $V = F^n$.

(2) β is linearly independent:

Suppose

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \vec{0}$$

where $c_1, c_2, \dots, c_n \in F$.

Then,

$$c_1(1,0,\dots,0) + c_2(0,1,\dots,0) + \dots + c_n(0,0,\dots,1) = (0,0,\dots,0)$$

So, $(c_1,0,\dots,0) + (0,c_2,\dots,0) + \dots + (0,0,\dots,c_n) = (0,0,\dots,0)$.

Ergo, $(c_1, c_2, \dots, c_n) = (0,0,\dots,0)$.

So, $c_1 = 0, c_2 = 0, \dots, c_n = 0$.

Thence, $v_1, v_2, \dots, v_n$ are lin. independent. $\blacksquare$

Ex: Let $F = \mathbb{R}$ or $F = \mathbb{C}$.

Let

$$V = P_n(F) = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_i \in F\}$$

One can show that

$$\left.\begin{aligned} v_0 &= 1 \\ v_1 &= x \\ v_2 &= x^2 \\ &\vdots \\ v_n &= x^n \end{aligned}\right\} \text{n+1 vectors}$$

is a basis for $P_n(F)$ over F.

So, $\dim(P_n(F)) = n+1$

<u>Ex:</u> Let F be a field and $V = M_{m,n}(F)$ be the set of $m \times n$ matrices with entries from F.

One can show that

$$\dim(M_{m,n}(F)) = m \cdot n$$

For example,

$$M_{3,2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \middle| \, a,b,c,d,e,f \in \mathbb{R} \right\}$$

A basis for $M_{3,2}(\mathbb{R})$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So, $\dim(M_{3,2}(\mathbb{R})) = 3 \cdot 2 = 6$

Math 4570

9/20/21

<u>Theorem</u>: Let V be a vector space over a field F.
Suppose $\dim(V) = n > 0$.
Then the following are true:

1. Let $v_1, v_2, \ldots, v_m \in V$.

   (a) If $m > n$, then $v_1, v_2, \ldots, v_m$ are linearly dependent.

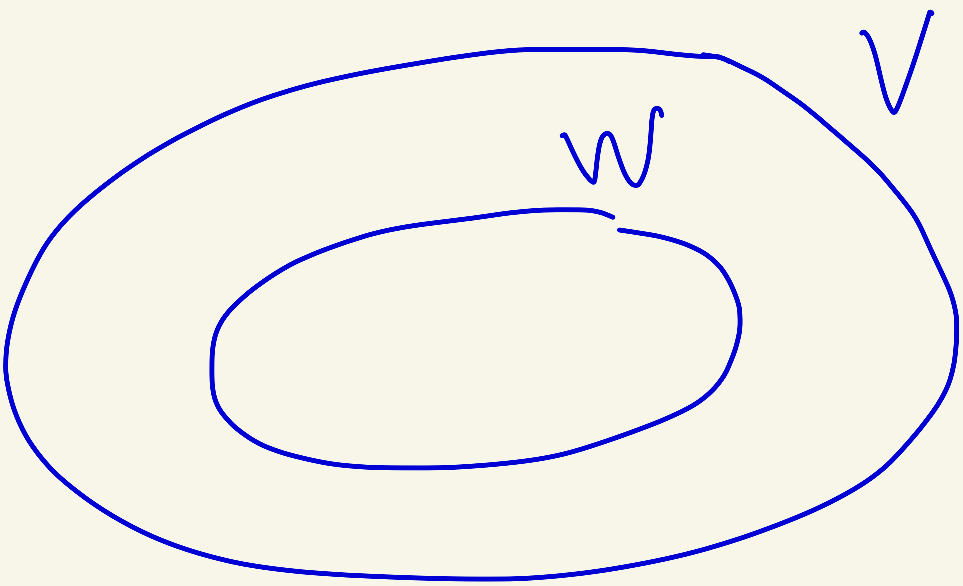
   (b) If $m < n$, then $v_1, v_2, \ldots, v_m$ do not span V.

   (c) If $m = n$ and $v_1, v_2, \ldots, v_m$ span V, then $v_1, v_2, \ldots, v_m$ are also linearly independent and hence form a basis for V.

   (d) If $m = n$ and $v_1, v_2, \ldots, v_m$ are linearly independent, then $v_1, v_2, \ldots, v_m$ span V and hence form a basis for V.

2 Let W be a subspace of V.
Then W is finite-dimensional
and $\dim(W) \leq \underbrace{n}_{\dim(V)}$

Moreover, $W = V$ if and only
if $\dim(W) = \dim(V)$.



proof: We have that $\dim(V) = n$. pg 3

① Let $v_1, v_2, \ldots, v_m \in V$.

(a) Suppose that $m > n$.

Since $\dim(V) = n$ we know that V has a basis with n vectors.

So, V is spanned by n vectors.

From a previous theorem, since $m > n$ we know that $v_1, v_2, \ldots, v_m$ are linearly dependent.

(b) Suppose $m < n$.

Let's show that $v_1, v_2, \ldots, v_m$ do not span V.

Suppose instead that $v_1, v_2, \ldots, v_m$ did span V.

Then from our previous results, since $m<n$, and $v_1, v_2, \ldots, v_m$ span V, we would have that any set of n vectors must be linearly dependent.

But since $\dim(V)=n$ there must be a basis for V of size n.

So, there is a set of n vectors in V that are linearly independent.

Contradiction.

So, $v_1, v_2, \ldots, v_m$ do not span V.

(c) Suppose $m = n$ and $v_1, v_2, \ldots, v_m$ span V

We want to show that $v_1, v_2, \ldots, v_m$ are linearly independent.

HW 2 - #7b

Suppose $V \neq \{\vec{0}\}$ is spanned by some finite set S of vectors. Prove that some subset of S is a basis for V

Let $S = \{v_1, v_2, \ldots, v_m\}$.

By this HW problem, there is a subset S' of S that is a basis for V.

Since $\dim(V) = n$, every basis for V has n vectors in it.

So, S' has $m = n$ vectors.

Thus, $S' = S$. Thus, $S = \{v_1, v_2, \ldots, v_m\}$ is a basis for V and is thus linearly independent.

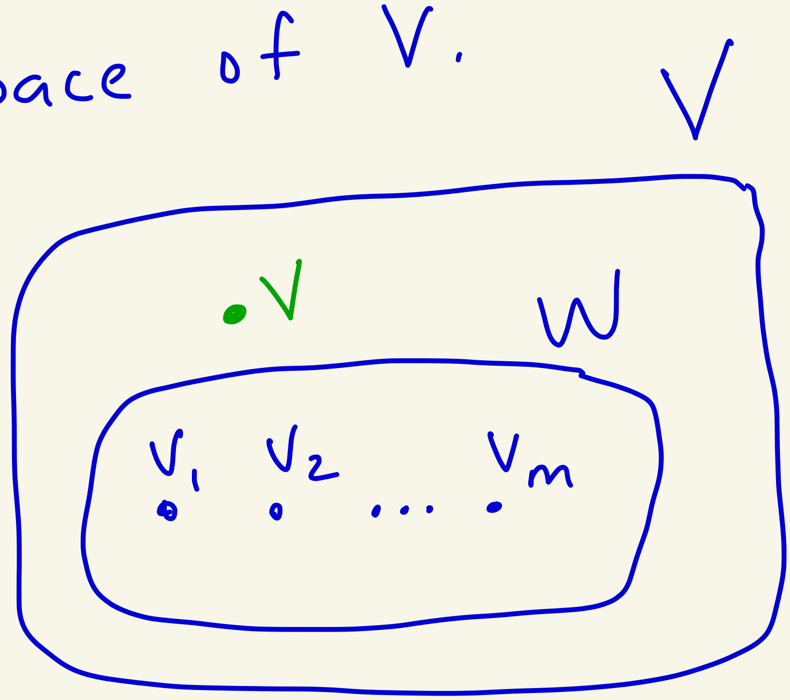
(d) Suppose $m = n = \dim(V)$ and $v_1, v_2, \ldots, v_m$ are linearly independent.

We want to show that $v_1, v_2, \ldots, v_m$ span V and hence are a basis for V.

Let $W = \text{span}(\{v_1, v_2, \ldots, v_m\})$.

So W is a subspace of V.

We will now show that $W = V$.



We know $W \subseteq V$.

We need to show that $V \subseteq W$.

Let $v \in V$.

Since $\dim(V) = n = m$ we know that the $n+1 = m+1$ vectors $v_1, v_2, \ldots, v_m, v$ are linearly dependent from part (a).

Thus, there exist

$c_1, c_2, \ldots, c_m, c_{m+1} \in F$,

not all equal to zero, where

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + c_{m+1} v = \vec{0}$

If $c_{m+1} = 0$, then

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m = \vec{0}$

with not all $c_1, c_2, \ldots, c_m$ equalling zero.

But this would contradict the fact that $v_1, v_2, \ldots, v_m$ are linearly independent.

Thus, $c_{m+1} \neq 0$.

So, we can solve for v in

$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m + c_{m+1} v = \vec{0}$

and we get

$$v = \underbrace{c_{m+1}^{-1}}_{\substack{\text{exists} \\ \text{since } c_{m+1} \neq 0}} \left(-c_1 v_1 - c_2 v_2 - \dots - c_m v_m\right)$$

So,

$$v = \left(-c_{m+1}^{-1} c_1\right) v_1 + \left(-c_{m+1}^{-1} c_2\right) v_2 + \dots + \left(-c_{m+1}^{-1} c_m\right) v_m$$

Thus, $v \in \text{span}\left(\{v_1, v_2, \dots, v_m\}\right) = W$.

So, $V = W$ and $v_1, v_2, \dots, v_m$ span V and are thus a basis for V. $\boxed{1}$

Now for part 2.

(2)

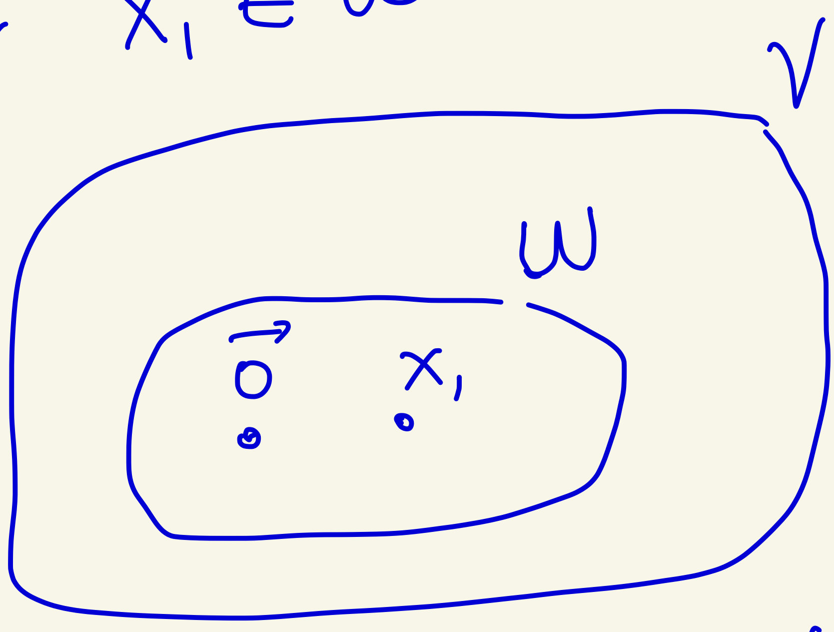
Let W be a subspace of V.

We first will show that W is finite-dimensional and $\dim(W) \leq n = \dim(V)$.

If $W = \{\vec{0}\}$, then W is finite-dimensional and $\dim(W) = 0 < n = \dim(V)$.

Now suppose $W \neq \{\vec{0}\}$.

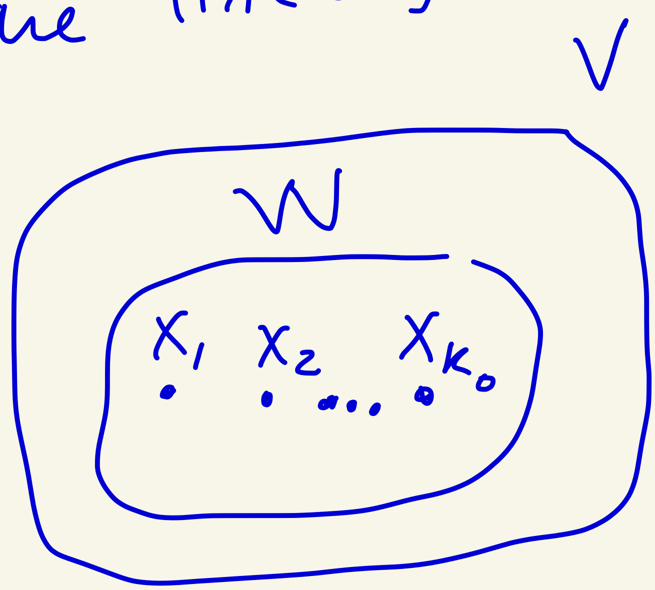
Then there exists $x_1 \in W$ with $x_1 \neq \vec{0}$.



Then, $\{x_1\}$ is a linearly independent set of vectors.

Because if $c_1 x_1 = \vec{0}$ then $c_1 = 0$ because $x_1 \neq \vec{0}$.

Continue to add vectors from W to this set such that at each stage k, the vectors $\{x_1, x_2, \ldots, x_k\}$ are linearly independent.



Since $W \subseteq V$ and $\dim(V) = n$, by part (a), there must reach a stage $k_0 \leq n$ where $S_0 = \{x_1, x_2, \ldots, x_{k_0}\}$ is linearly independent but adding any new vector from W to S_0 will yield a linearly dependent set.

HW 2 − 7(a)

Let S be a finite set of linearly independent vectors from V and let $x \in V$ with $x \notin S$.
Then $S \cup \{x\}$ is linearly dependent iff $x \in \text{span}(S)$

Let $x \in W$.
If $x \in S_0$, then $x \in \text{span}(S_0)$.
If $x \notin S_0$, then by the construction of S_0 we have that $S_0 \cup \{x\}$ is linearly dependent. So by HW 2, 7(a), $x \in \text{span}(S_0)$.

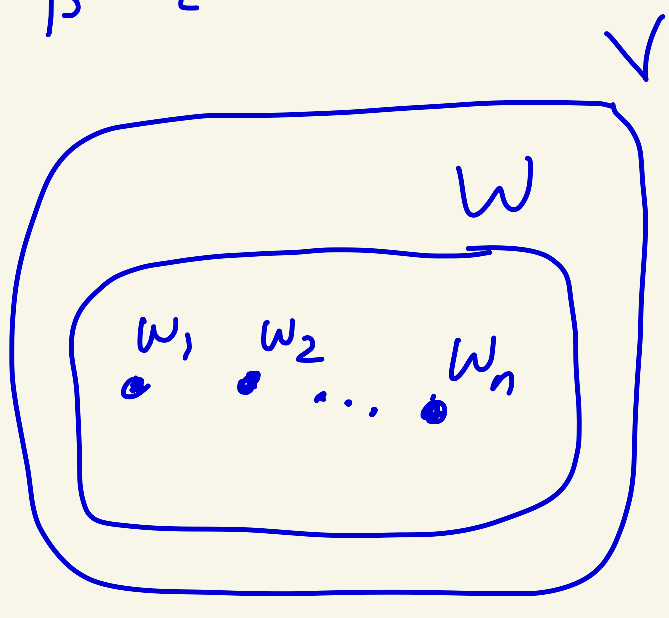
Thus, if $x \in W$, then $x \in \text{span}(S_0)$.
So, $W = \text{span}(S_0)$.
Since S_0 is a lin. ind. set, S_0 is a basis for W. Thus,
$\dim(W) = k_0 \leq n = \dim(V)$.

Now we show that $W = V$ iff $\dim(W) = \dim(V)$.

($\Rightarrow$) If $V = W$, then $\dim(V) = \dim(W)$.

($\Leftarrow$) Now suppose $\dim(W) = \dim(V)$.
Let's show that $W = V$.

Then W has a basis of $n = \dim(V)$ elements, call it $\beta = \{w_1, w_2, \dots, w_n\}$



So, $W = \text{span}(\beta)$.

By part 1(d), since β is a set of n vectors that are linearly independent and $n = \dim(V)$, they must span V also!

So, β is a basis for V.
Thus, $W = \text{span}(\beta) = V$. $\blacksquare$

Math 4570

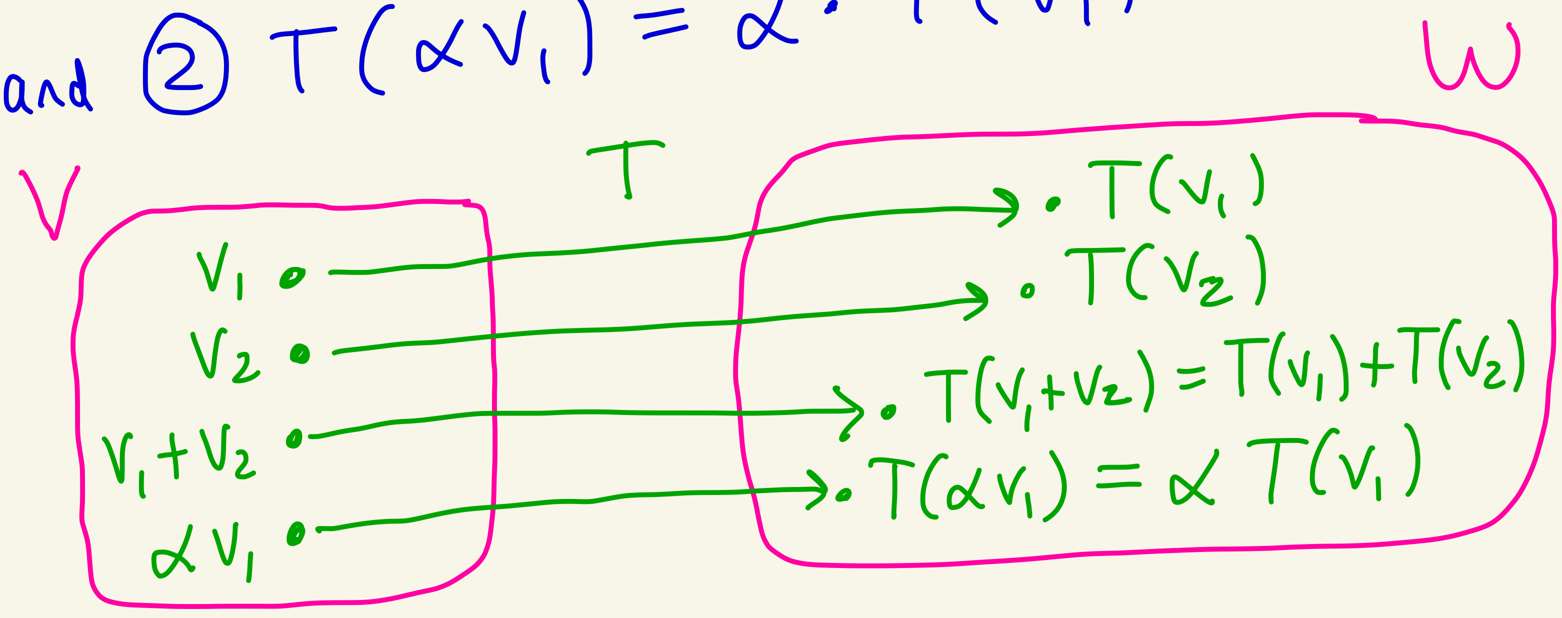
9/22/21

Linear Transformations - HW 3

<u>Def:</u> Let V and W be vector spaces over a field F. Let $T: V \to W$ be a function between them. We say that T is a <u>linear transformation</u> if for every $v_1, v_2 \in V$ and $\alpha \in F$ we have that

① $T(v_1 + v_2) = T(v_1) + T(v_2)$

and ② $T(\alpha v_1) = \alpha \cdot T(v_1)$



People sometimes say that T "preserves" vector addition and scalar multiplication

You can condense ① and ② into one condition:

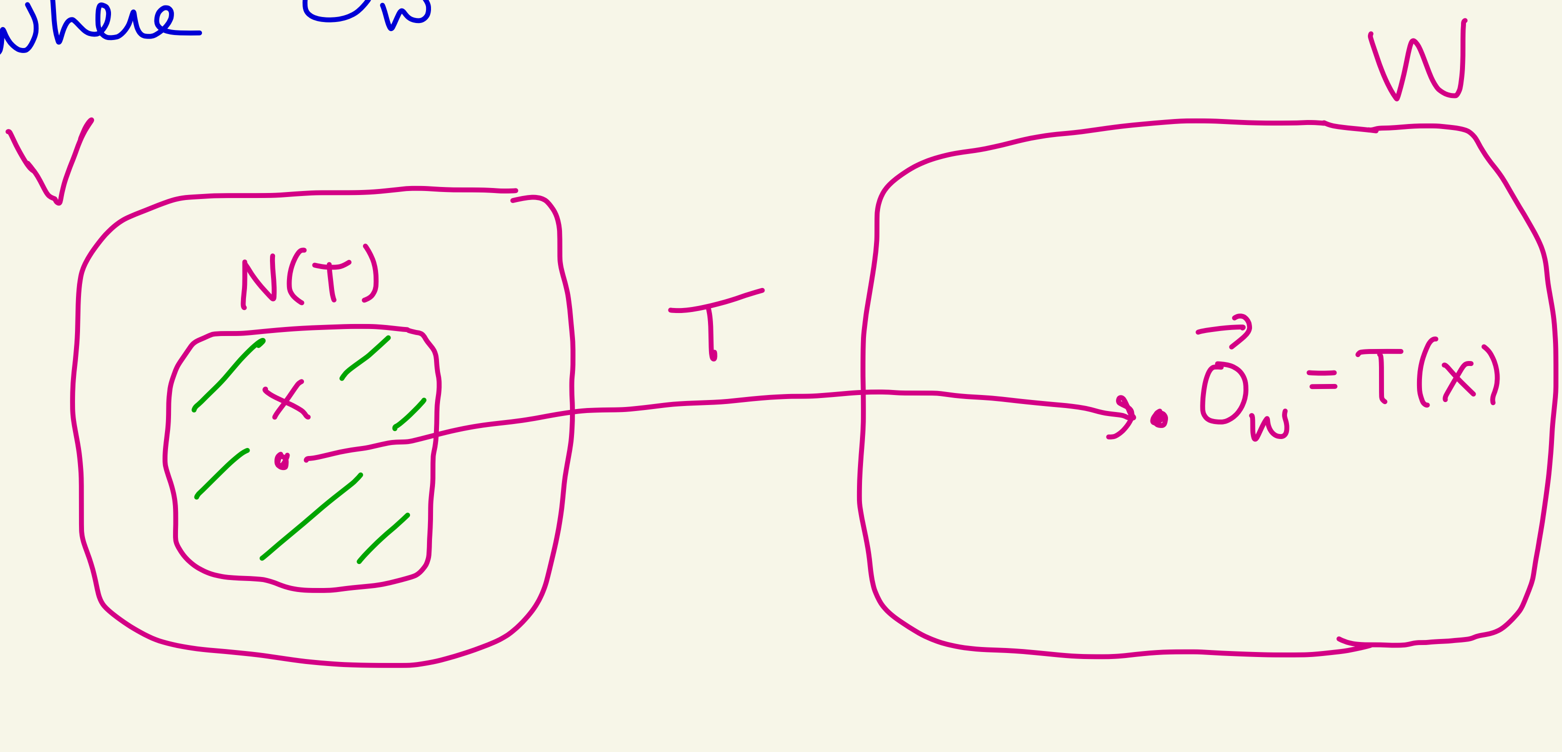
$$T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2)$$

for all $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in F$

We define the <u>nullspace</u> (or <u>kernel</u>) of T to be

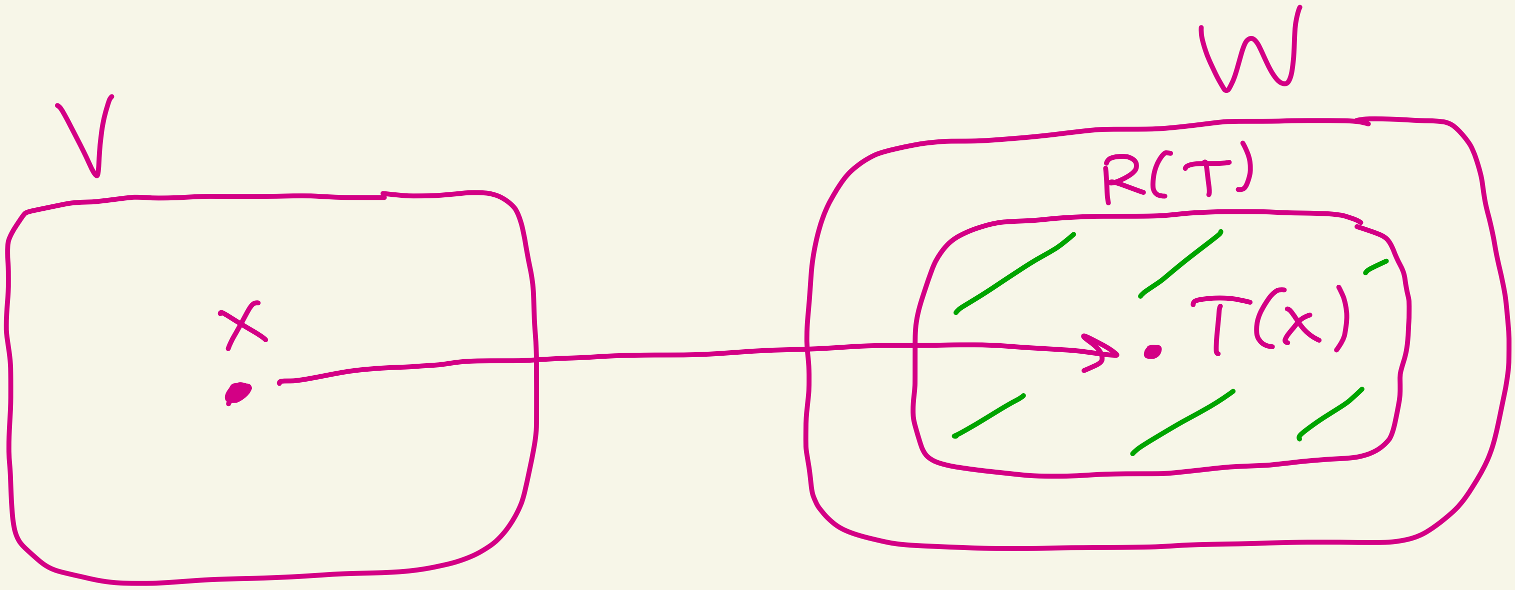
$$N(T) = \{ x \in V \mid T(x) = \vec{0}_W \}$$

where $\vec{0}_W$ is the zero vector of W.



We define the <u>range</u> (or <u>image</u>) of T to be

$$R(T) = \{ T(x) \mid x \in V \}$$



<u>Comment:</u> We will show later that $N(T)$ is a subspace of V and $R(T)$ is a subspace of W

If $N(T)$ is finite-dimensional then we call the dimension of $N(T)$ the <u>nullity of T</u> and write

$$\text{nullity}(T) = \dim(N(T))$$

If $R(T)$ is finite-dimensional then we call the dimension of $R(T)$ the <u>rank of T</u> and write

$$\text{rank}(T) = \dim(R(T))$$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x,y,z) = (x,y)$

Here $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $F = \mathbb{R}$.

For example,

$T(1,\pi,10) = (1,\pi)$

$T(-1,\frac{1}{2},3) = (-1,\frac{1}{2})$

T is a linear transformation:

proof: Let $v_1, v_2 \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

Then, $v_1 = (x_1,y_1,z_1)$ and $v_2 = (x_2,y_2,z_2)$ where $x_1,y_1,z_1,x_2,y_2,z_2 \in \mathbb{R}$.

① Then,

$$T(v_1+v_2)$$

$$= T((x_1,y_1,z_1)+(x_2,y_2,z_2))$$

$$= T(x_1+x_2, y_1+y_2, z_1+z_2)$$

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$$= (x_1 + x_2, y_1 + y_2)$$

$$= (x_1, y_1) + (x_2, y_2)$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

$$= T(v_1) + T(v_2)$$

② We also have that

$$T(\alpha v_1) = T(\alpha(x_1, y_1, z_1))$$

$$= T(\alpha x_1, \alpha y_1, \alpha z_1)$$

$$= (\alpha x_1, \alpha y_1)$$

$$= \alpha \cdot (x_1, y_1)$$

$$= \alpha \cdot T(x_1, y_1, z_1)$$

$$= \alpha \cdot T(v_1)$$

$\blacksquare$

Nullspace of T:

$$N(T) = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0)\}$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{(x,y) = (0,0)}_{x=0 \text{ and } y=0}\}$$

$$= \boxed{\{(0,0,z) \mid z \in \mathbb{R}\}} \leftarrow \boxed{N(T)}$$

$$= \{z \cdot (0,0,1) \mid z \in \mathbb{R}\}$$

$$= \operatorname{span}(\{(0,0,1)\})$$

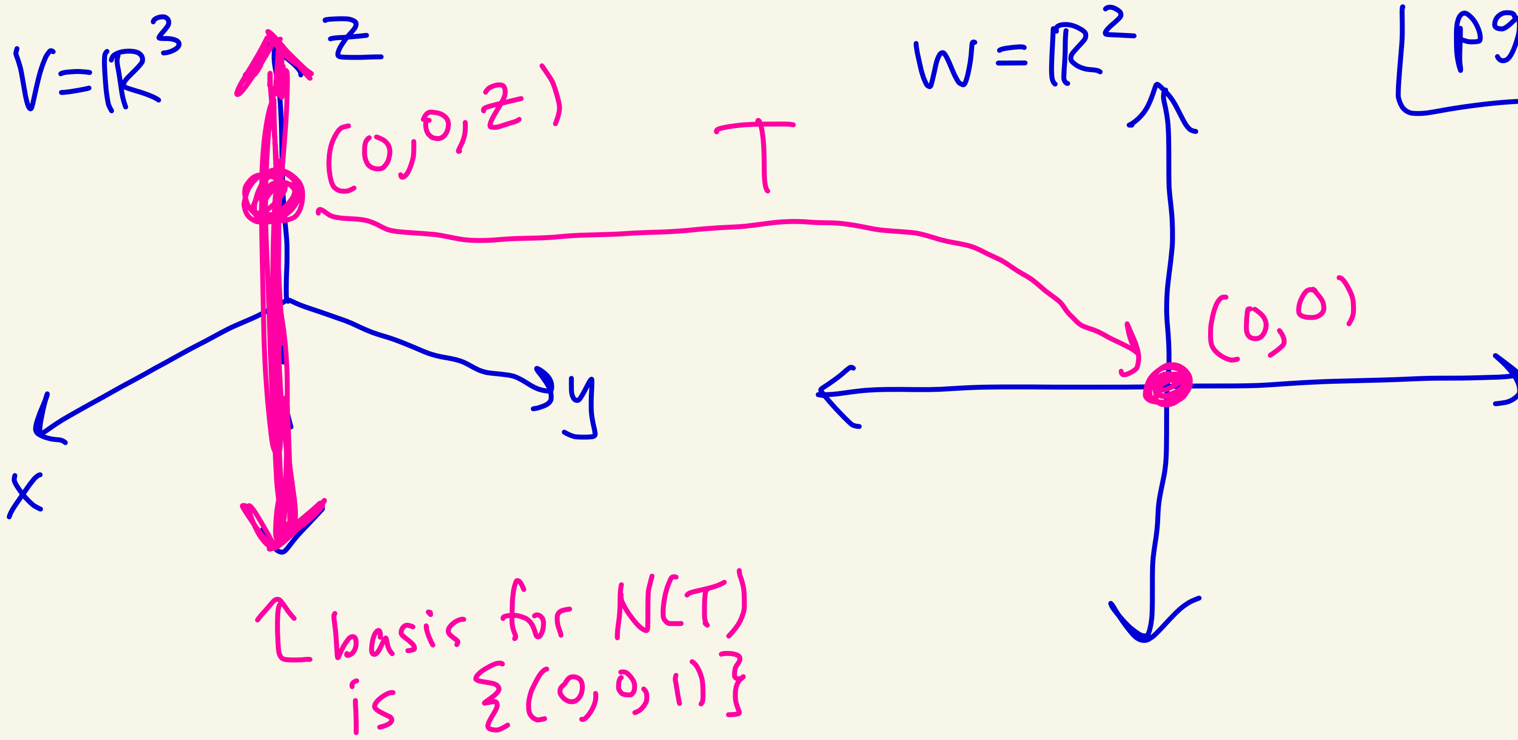
Let $\beta = \{(0,0,1)\}$.

Then β spans $N(T)$.

By HW 2 #6 since β consists of one non-zero vector, β is a linearly independent set

So, β is a basis for $N(T)$ and

$$\text{nullity}(T) = \dim(N(T)) = \left(\#\text{ elements in basis } \beta\right) = 1$$



$\boxed{\text{Range of } T}$

$$R(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$$

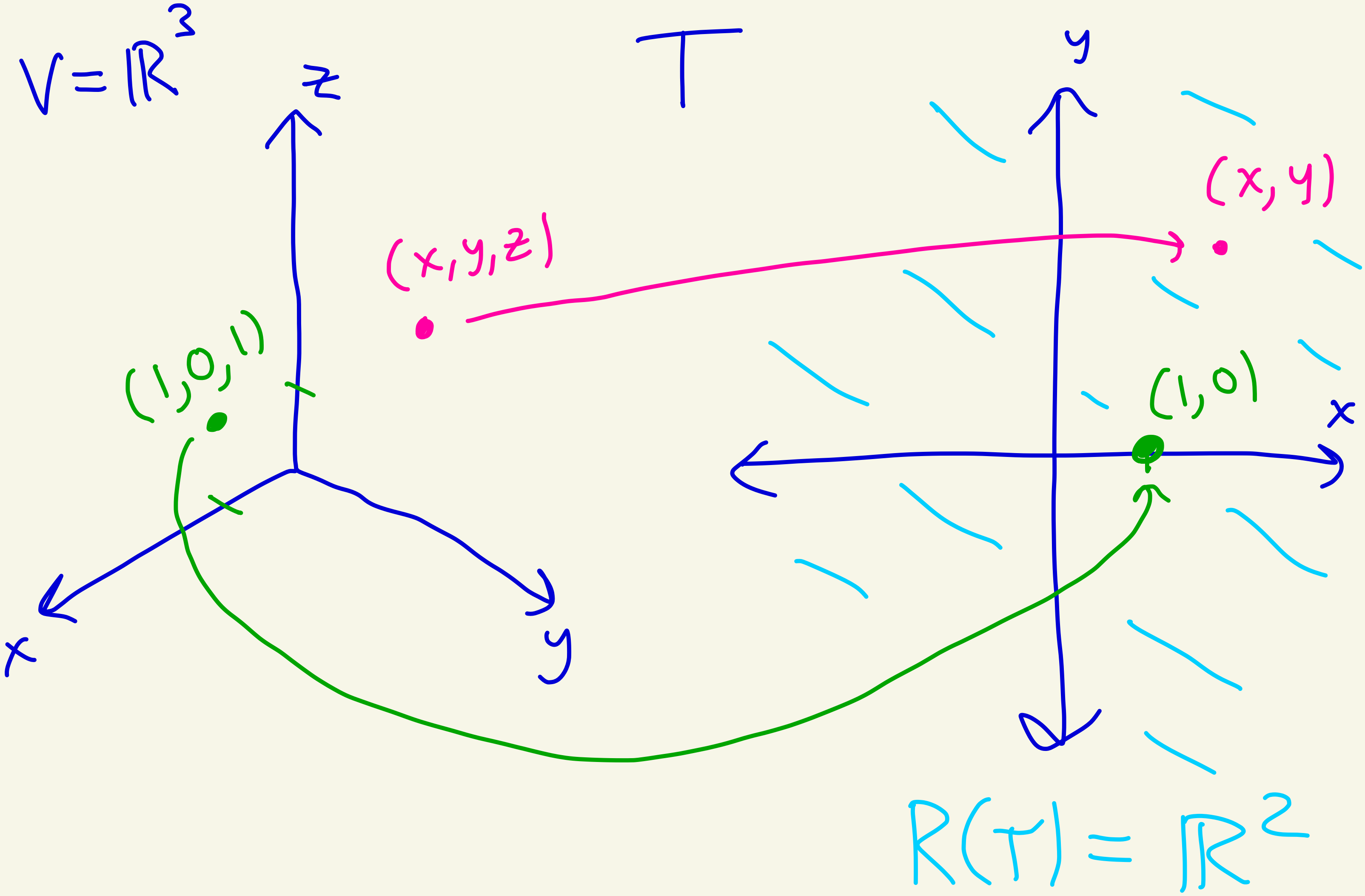
$$= \{(x,y) \mid (x,y,z) \in \mathbb{R}^3\}$$

$$= \{(x,y) \mid x,y \in \mathbb{R}\}$$

$$= \mathbb{R}^2$$

Thus, $\text{rank}(T) = \dim(R(T))$

$$= \dim(\mathbb{R}^2) = 2$$



Note:

$$\dim(V) = \text{nullity}(T) + \text{rank}(T)$$

$$3 = 1 + 2$$

Ex: Let $n \geq 1$ be fixed and

$$T : \underbrace{P_n(\mathbb{R})}_{\substack{\text{polys} \\ \text{of degree} \\ \leq n}} \longrightarrow \underbrace{P_{n-1}(\mathbb{R})}_{\substack{\text{polys} \\ \text{of degree} \\ \leq n-1}}$$

where $T(f) = f'$.
Here f' is the derivative of the polynomial f.

T is a linear transformation:

Let $f_1, f_2 \in P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$.
Then,

$$T(f_1 + f_2) = (f_1 + f_2)' = f_1' + f_2' = T(f_1) + T(f_2)$$

property of derivatives

and

$$T(\alpha f_1) = (\alpha f_1)' = \alpha f_1' = \alpha T(f_1)$$

$\blacksquare$

Nullspace of T:

$N(T) =$

$= \{a_0 + a_1 x + \dots + a_n x^n \mid T(a_0 + a_1 x + \dots + a_n x^n) = \vec{0}\}$

$= \{a_0 + a_1 x + \dots + a_n x^n \mid \underbrace{a_1}_{0} + \underbrace{2a_2}_{0} x + \dots + \underbrace{n a_n}_{0} x^{n-1} = \vec{0}\}$

$= \{a_0 + a_1 x + \dots + a_n x^n \mid a_1 = a_2 = \dots = a_n = 0\}$

$= \{a_0 \mid a_0 \in \mathbb{R}\}$ ← constant polynomials

$= \{a_0 \cdot 1 \mid a_0 \in \mathbb{R}\}$

$= \operatorname{span}(\{1\})$

Let $\beta = \{1\}$. Then β spans $N(T)$.

Since β consists of one non-zero vector by HW 2 #6, β is a lin. ind. set.

Thus, β is a basis for $N(T)$.

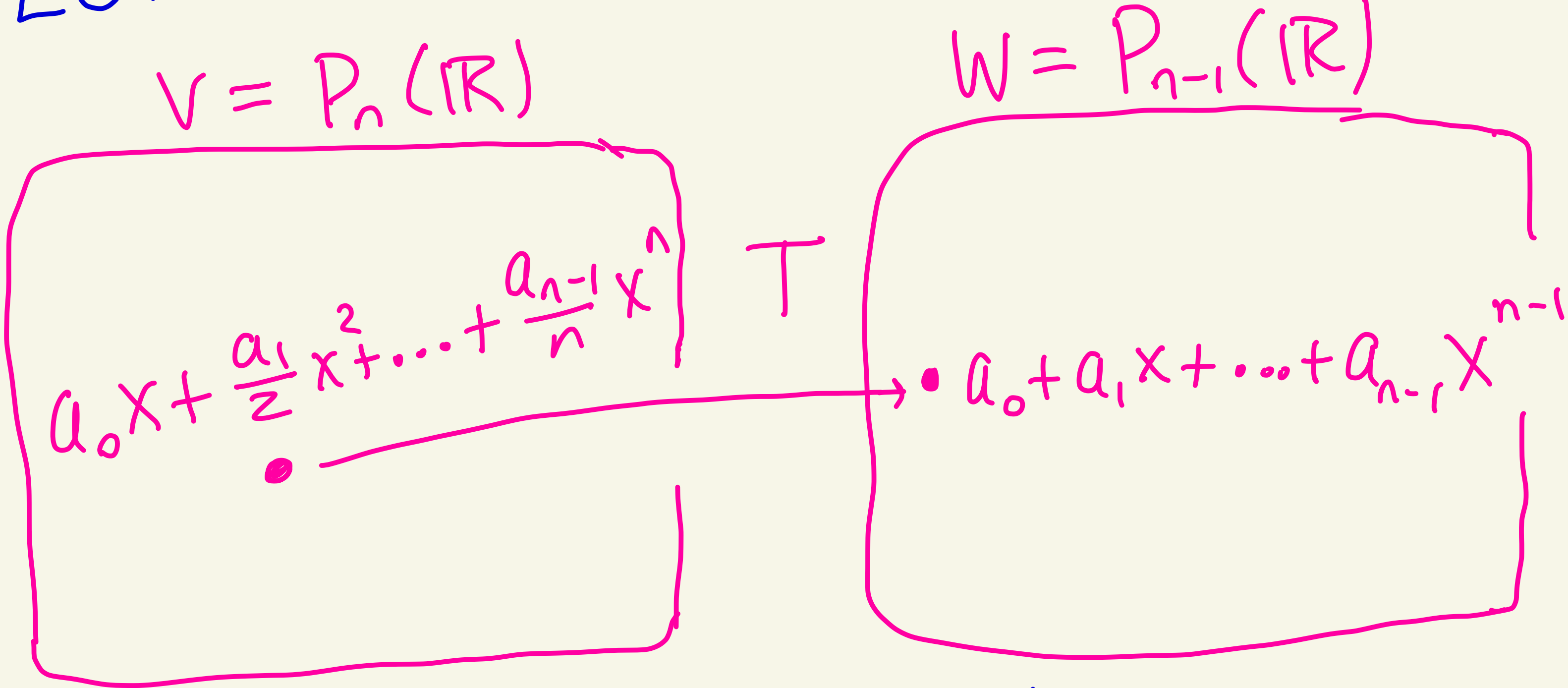
So, nullity$(T) = \left(\begin{array}{l}\#\text{ elements in} \\ \text{basis } \beta\end{array}\right) = 1$.

Range of T:

I claim that T is onto.

That is, $R(T) = P_{n-1}(\mathbb{R})$.

Let $a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in P_{n-1}(\mathbb{R})$.



Integrate and notice that

$$a_0 x + \frac{a_1}{2} x^2 + \dots + \frac{a_{n-1}}{n} x^n \in P_n(\mathbb{R})$$

and

$$T\left(a_0 x + \frac{a_1}{2} x^2 + \dots + \frac{a_{n-1}}{n} x^n\right)$$

$$= a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$$

Thus, T is onto $P_{n-1}(\mathbb{R}) = W$.

Thus,

$R(T) = P_{n-1}(\mathbb{R})$.

So, $\operatorname{rank}(T) = \dim(P_{n-1}(\mathbb{R}))$

$= (n-1) + 1 = n$.

Note:

$$\dim(\underbrace{P_n(\mathbb{R})}_{V}) = \operatorname{nullity}(T) + \operatorname{rank}(T)$$

$$n+1 \quad = \quad 1 \quad + \quad n$$

Math 4570

9/27/21

Test 1 is on Monday Oct 18 Test 1 covers HW 1 and HW 2

No class on Test day.
Test is done on canvas.
Test will appear at 5am on Monday 10/18 and dissapear at 12pm noon on Tuesday 10/19. During that time period you pick a 2.5 hour time window to take the test, scan, and upload your answers [2 hrs for test, 30 min to scan]. Canvas will time you once you open the test.

I put a "practice taking a test" Module in case you haven't taken a test on canvas before to see what its like to download an exam and upload your solutions.

Try it out if needed.

(HW 3 continued...)

Another way to make a linear transformation is by matrix multiplication

---

<u>Def:</u> Let F be a field.
Let A be an $m \times n$ matrix with coefficients from F.
We can construct a linear transformation

$$L_A : F^n \longrightarrow F^m$$

where $L_A(x) = Ax$ for any $x \in F^n$.
[Here Ax is matrix multiplication]

L_A is called the <u>left-multiplication by A transformation</u>.

$$\underbrace{A}_{m \times n} \underbrace{x}_{n \times 1}$$

result is $m \times 1$

Note: L_A above is a linear transformation because if $x, y \in F^n$ and $\alpha, \beta \in F$ then

$$L_A(\alpha x + \beta y) = A(\alpha x + \beta y)$$
$$= A(\alpha x) + A(\beta y)$$
$$= \alpha A x + \beta A y$$
$$= \alpha L_A(x) + \beta L_A(y)$$

property of matrix multiplication

$\blacksquare$

Ex: Let $F = \mathbb{C}$.

Let
$$A = \begin{pmatrix} i & 1+i & -3-5i \\ 0 & 1 & -1-i \end{pmatrix}$$

be in $M_{2\times 3}(\mathbb{C})$.

$\boxed{i^2 = -1}$

Then,
$$L_A : \mathbb{C}^3 \longrightarrow \mathbb{C}^2$$

where
$$L_A\begin{pmatrix} a \\ b \\ c \end{pmatrix} = A \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$
$$= \begin{pmatrix} i & 1+i & -3-5i \\ 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

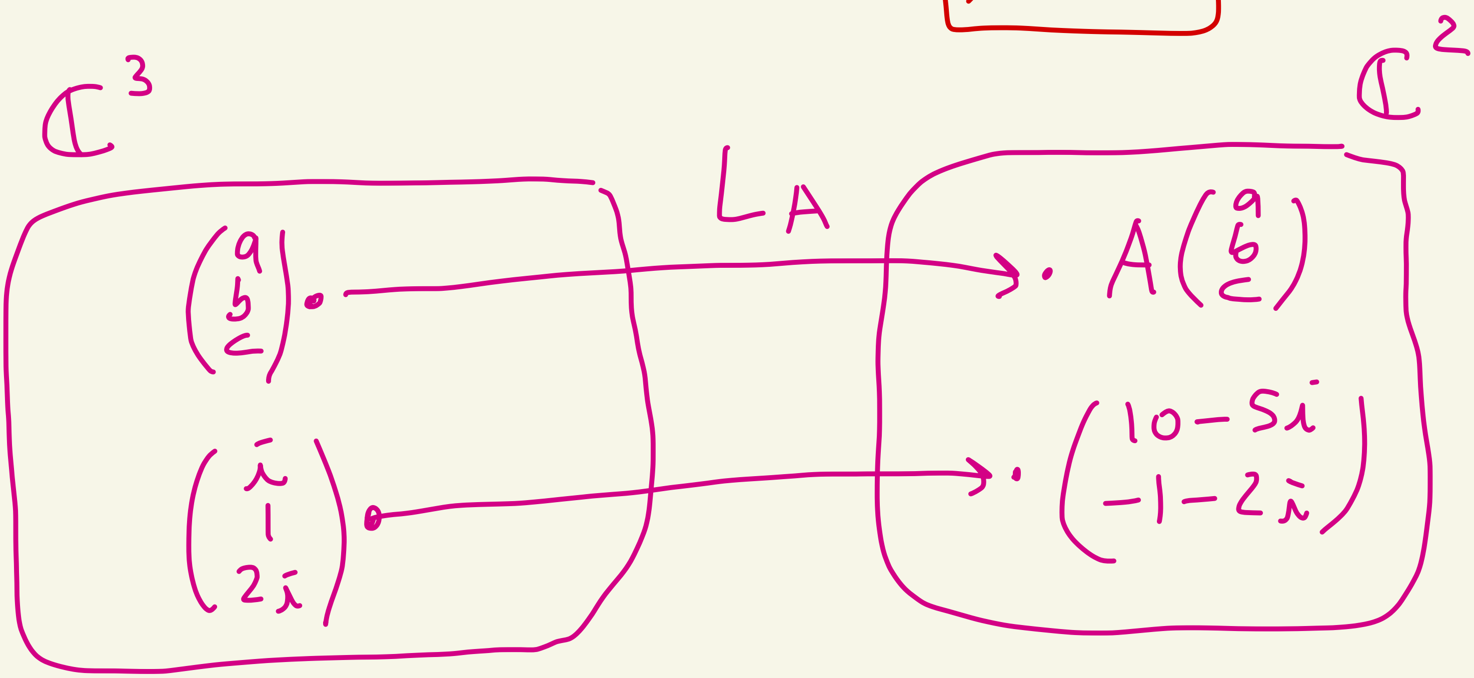
For example,

$$L_A\begin{pmatrix} i \\ 1 \\ 2i \end{pmatrix} = \begin{pmatrix} i & 1+i & -3-5i \\ 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} i \\ 1 \\ 2i \end{pmatrix}$$

$$= \begin{pmatrix} (i)(i)+(1+i)(1)+(-3-5i)(2i) \\ (0)(i)+(1)(1)+(-1-i)(2i) \end{pmatrix}$$

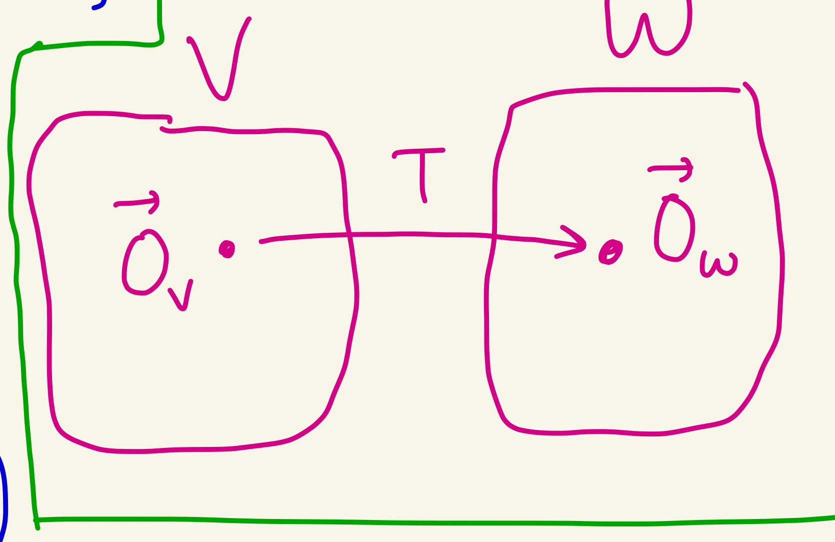
$$= \begin{pmatrix} i^2+1+i-6i-10i^2 \\ 0+1-2i+2i^2 \end{pmatrix} = \begin{pmatrix} 10-5i \\ -1-2i \end{pmatrix}$$

$i^2 = -1$



Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W respectively. Then, $T(\vec{0}_V) = \vec{0}_W$.

PICTURE



proof: (HW problem)

We have that

$$T(\vec{0}_V) = T(\vec{0}_V + \vec{0}_V)$$

$$= T(\vec{0}_V) + T(\vec{0}_V)$$

T is linear

Thus, $T(\vec{0}_V) = T(\vec{0}_V) + T(\vec{0}_V)$ in W.

Add the additive inverse $-T(\vec{0}_V)$ to both sides to get that

$$\underbrace{-T(\vec{O}_v) + T(\vec{O}_v)}_{\vec{O}_w} = -\underbrace{T(\vec{O}_v) + T(\vec{O}_v)}_{\vec{O}_w} + T(\vec{O}_v)$$

So, $\vec{O}_w = \underbrace{\vec{O}_w + T(\vec{O}_v)}_{T(\vec{O}_v)}$

Thus, $T(\vec{O}_v) = \vec{O}_w$. $\blacksquare$

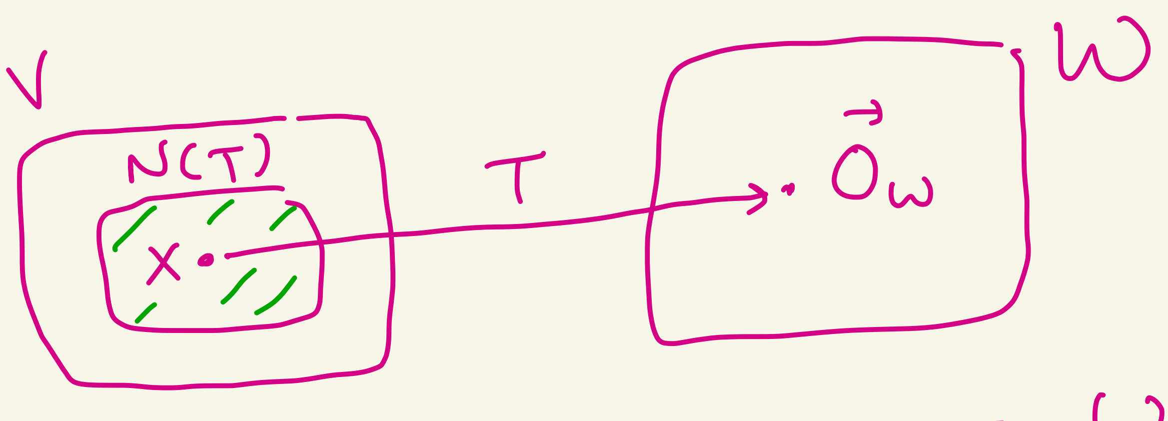
Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

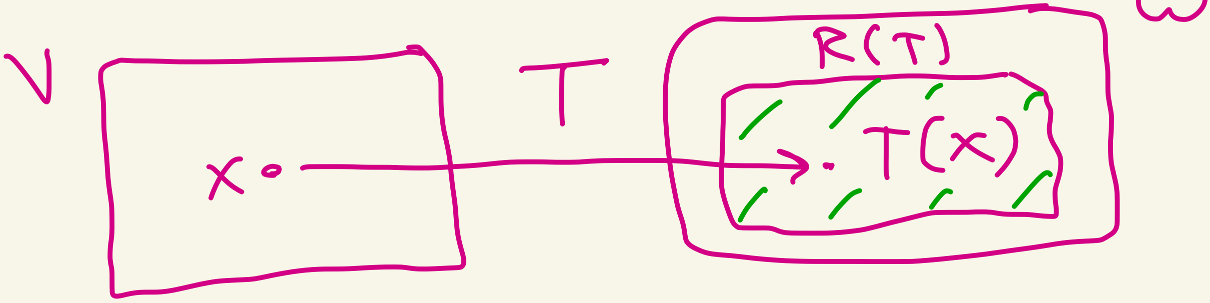
Then:

① $N(T) = \{ x \in V \mid T(x) = \vec{0}_W \}$ is a subspace of V

and

② $R(T) = \{ T(x) \mid x \in V \}$ is a subspace of W.





<u>proof:</u> Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors of V and W.

① Let's show that $N(T)$ is a subspace of V.

(i) By the previous theorem today we know that $T(\vec{0}_V) = \vec{0}_W$.
This tells us that $\vec{0}_V \in N(T)$.

(ii) Let's show $N(T)$ is closed under $+$.

Let $x, y \in N(T)$.

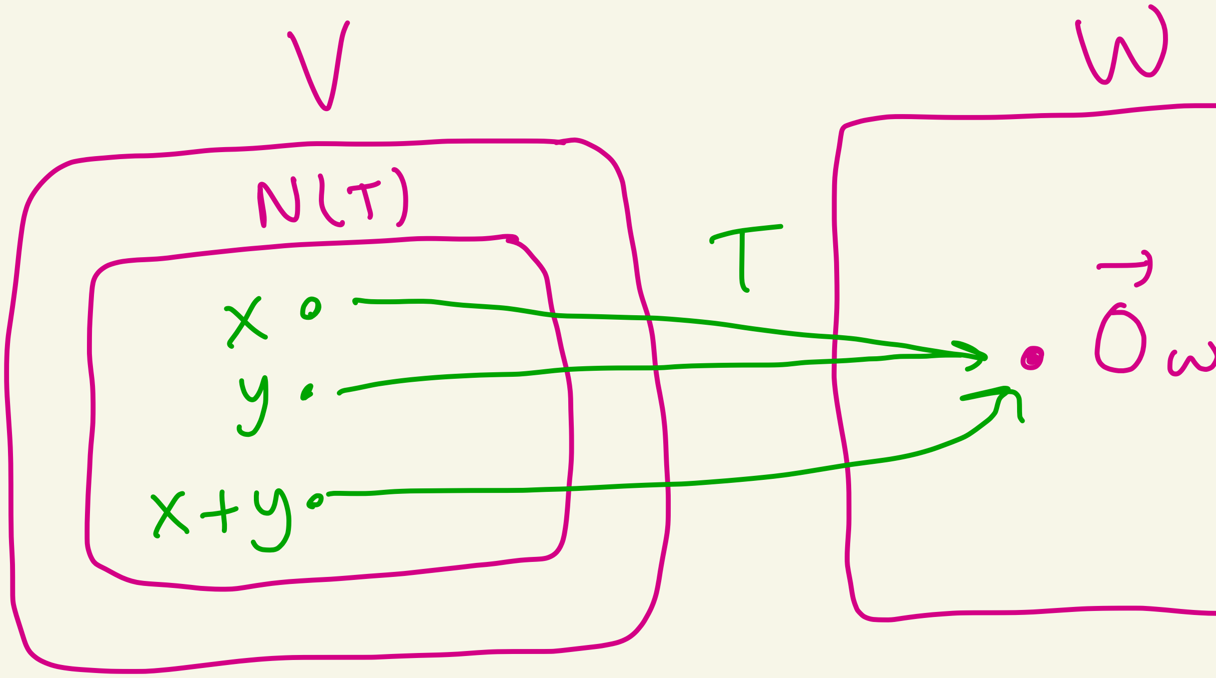
Then, $T(x) = \vec{0}_W$ and $T(y) = \vec{0}_W$

So, $T(x+y) = T(x) + T(y)$

$= \vec{0}_W + \vec{0}_W = \vec{0}_W$

Since T is linear

Thus, $T(x+y) = \vec{0}_W$.
So, $x+y \in N(T)$.



(iii) Let's show $N(T)$ is closed under scalar mult.

Let $z \in N(T)$ and $\alpha \in F$.

Since $z \in N(T)$, we know that $T(z) = \vec{0}_w$.

Thus,

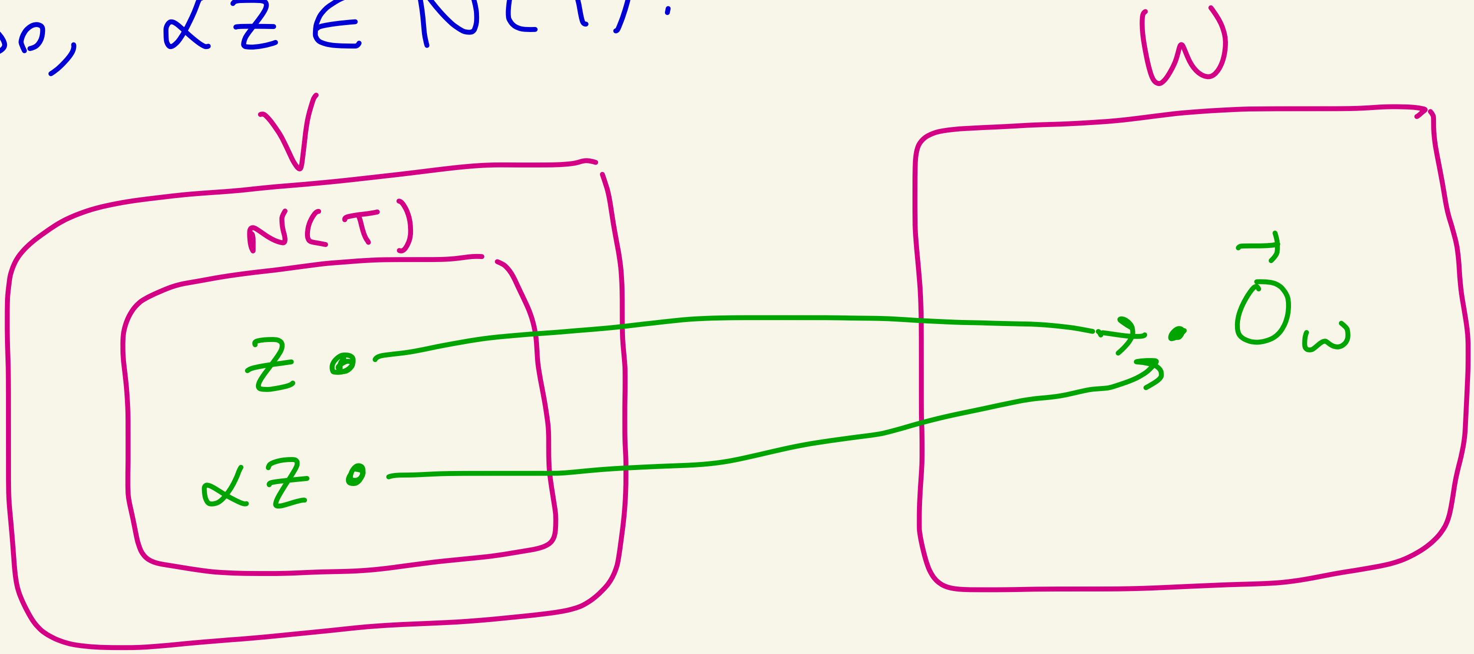
$$T(\alpha z) = \alpha T(z)$$

since T is linear

$$= \alpha \cdot \vec{0}_w = \vec{0}_w$$

Thus, $T(\alpha z) = \vec{0}_W$.

So, $\alpha z \in N(T)$.



By (i), (ii), and (iii) we have that $N(T)$ is a subspace of V.

Pg 14

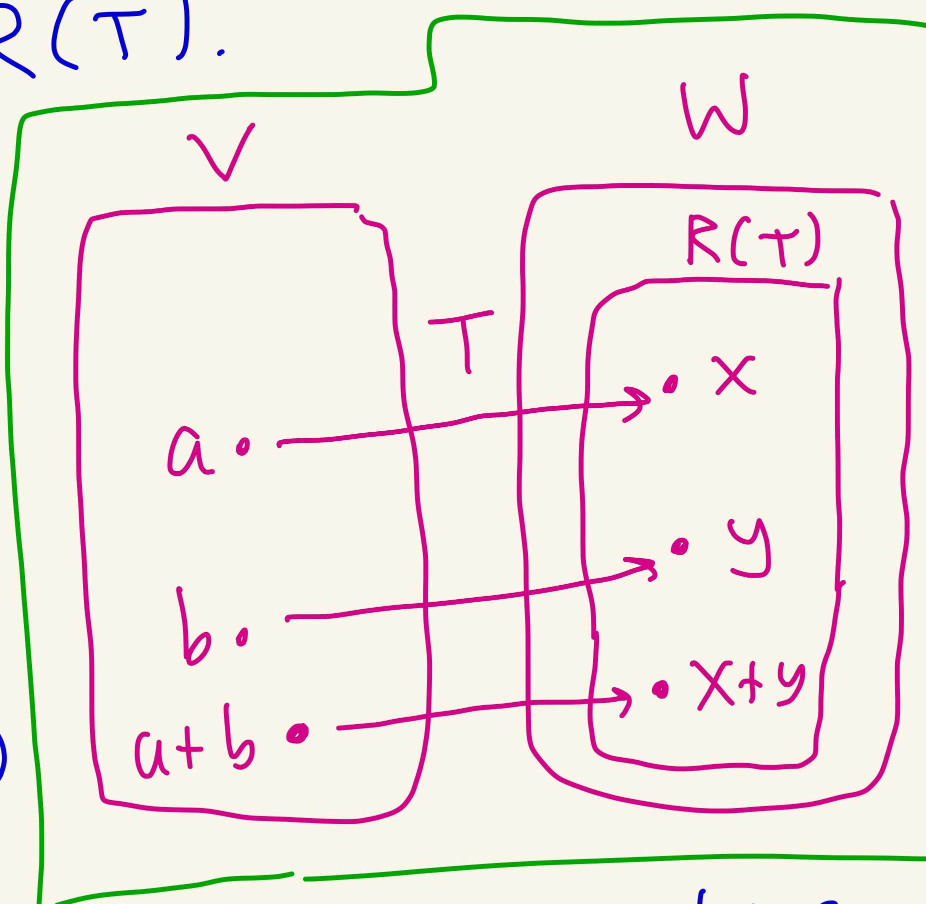
② Let's show $R(T)$ is a subspace of W.

Recall $R(T) = \{ T(x) \mid x \in V \}$

(i) Because $\vec{0}_W = T(\vec{0}_V)$ and $\vec{0}_V \in V$ we know that $\vec{0}_W \in R(T)$.

(ii) Let's show $R(T)$ is closed under $+$.

Let $x, y \in R(T)$.



Then there exist $a, b \in V$ with $T(a) = x$ and $T(b) = y$.

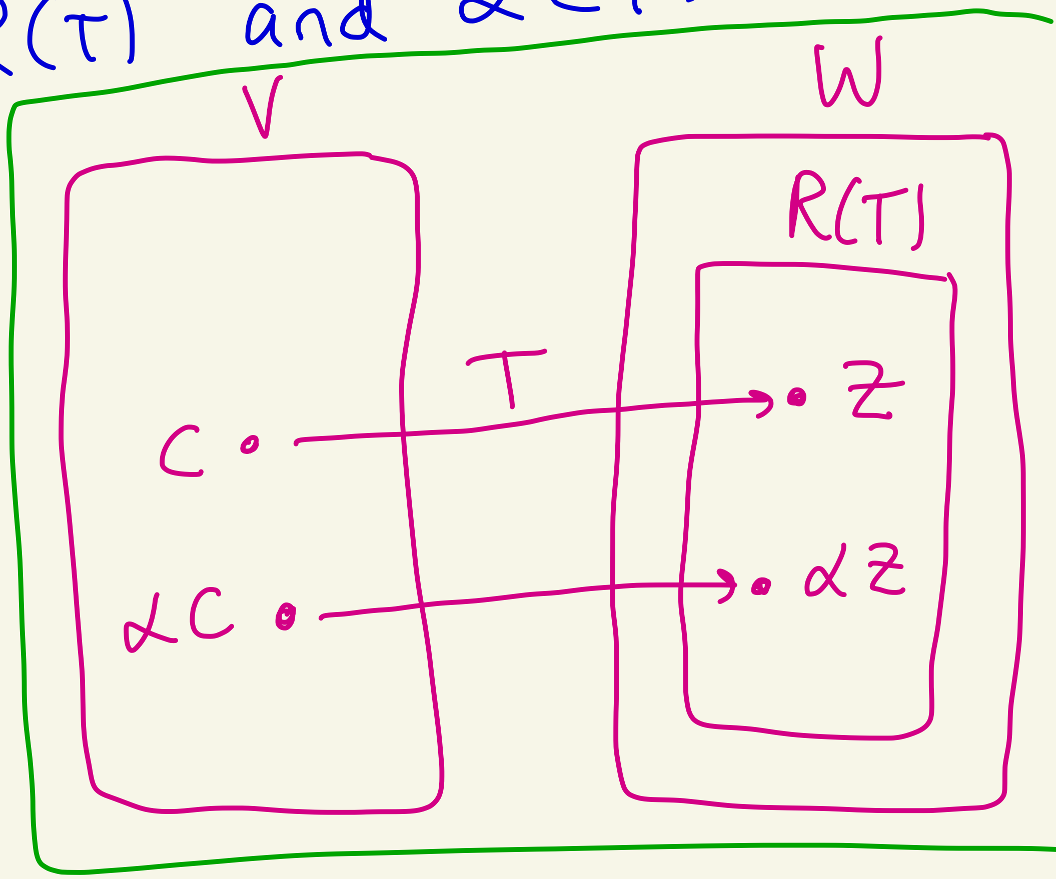
Thus,

$$x + y = T(a) + T(b) = T(a+b)$$

Since $x+y = T(a+b)$ and $a+b \in V$ we have that $x+y \in R(T)$.

(iii) Let's show $R(T)$ is closed under scalar mult.

Let $z \in R(T)$ and $\alpha \in F$.



Thus,

$$z = T(c)$$

where $c \in V$.

Then,

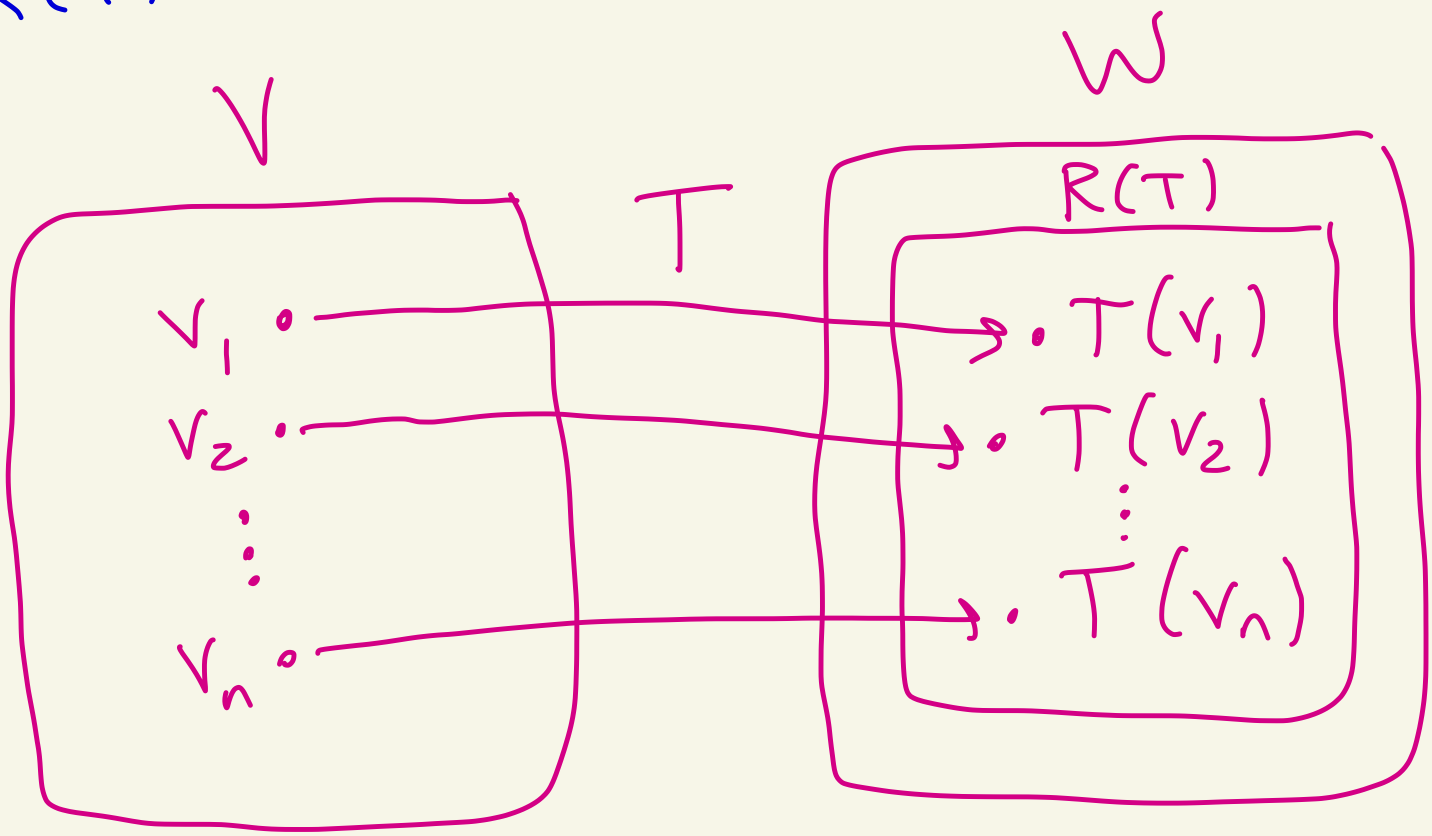
$$\alpha z = \alpha T(c) = T(\alpha c).$$

Since $\alpha z = T(\alpha c)$ where $\alpha c \in V$ we know that $\alpha z \in R(T)$.

By (i), (ii), (iii), $R(T)$ is a subspace of W. $\blacksquare$

Lemma: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

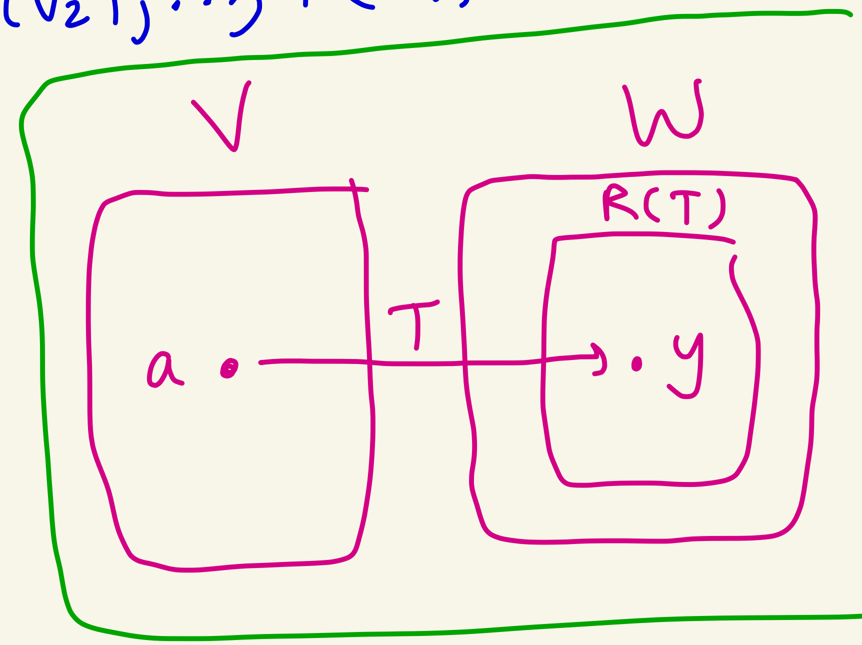
If $v_1, v_2, \ldots, v_n \in V$ and
$$V = \text{span}(\{v_1, v_2, \ldots, v_n\}),$$
then
$$R(T) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$$



pg 17

<u>proof:</u> Suppose $v_1, v_2, \ldots, v_n \in V$ and $v_1, v_2, \ldots, v_n$ span V.

Lets show $T(v_1), T(v_2), \ldots, T(v_n)$ spans $R(T)$.



Let $y \in R(T)$. Then there exists $a \in V$ where $y = T(a)$.

Because $a \in V$ and $v_1, v_2, \ldots, v_n$ span V, we know that

$$a = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Thus,

$$y = T(a) = T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_n T(v_n)$$

HW 3 problem since T is linear

So, $y \in \operatorname{span}(\{T(v_1), \ldots, T(v_n)\})$

So, $T(v_1), \ldots, T(v_n)$ span $R(T)$. ▨

Math 4570

9/29/21

See Monday notes about test 1

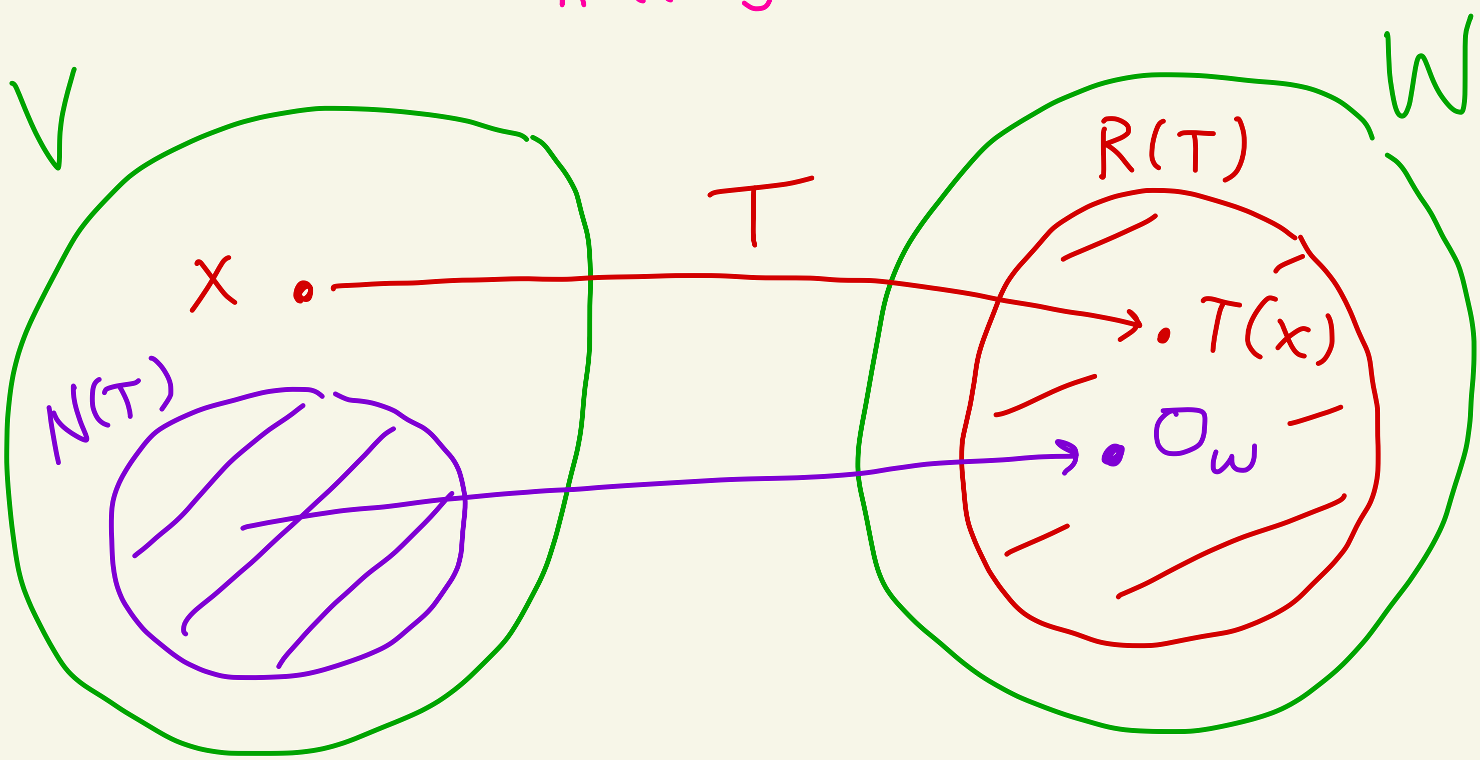
Rank-Nullity Theorem

Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear transformation.

If V is finite dimensional, then

1. $N(T)$ is finite dimensional
2. $R(T)$ is finite dimensional

and 3. $\dim(V) = \underbrace{\dim(N(T))}_{\text{nullity}(T)} + \underbrace{\dim(R(T))}_{\text{rank}(T)}$



Proof: Let $n = \dim(V)$.

By Monday's theorem, $N(T)$ is a subspace of V.

Thus, since V is finite dimensional, $N(T)$ is finite dimensional [Thm from class]

Also, if we set $k = \dim(N(T))$ then $k \leq n$. [Thm from class]

Thus, there exists a basis $\{v_1, v_2, \ldots, v_k\}$ for $N(T)$.

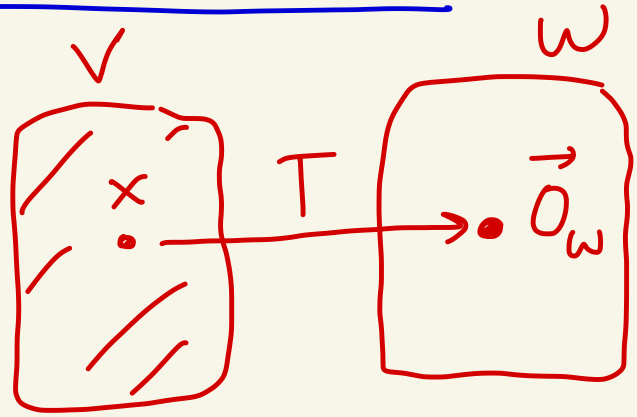
Let $\vec{0}_V$ and $\vec{0}_W$ be the zero vectors for V and W.

Note that $T(\vec{0}_V) = \vec{0}_W$ and so $\vec{0}_W \in R(T)$.

Let's now break the proof into two cases.

Case 1: Suppose $R(T) = \{\vec{0}_W\}$

Then, $T(x) = \vec{0}_W$ for every $x \in V$.



Then, $N(T) = V$.

So, $\dim(R(T)) = 0$ and thus $R(T)$ is finite-dimensional

And,

$$\dim(V) = \dim(V) + 0$$
$$= \dim(N(T)) + \dim(R(T)).$$

$$\boxed{\begin{aligned} V &= N(T) \\ 0 &= \dim(R(T)) \end{aligned}}$$

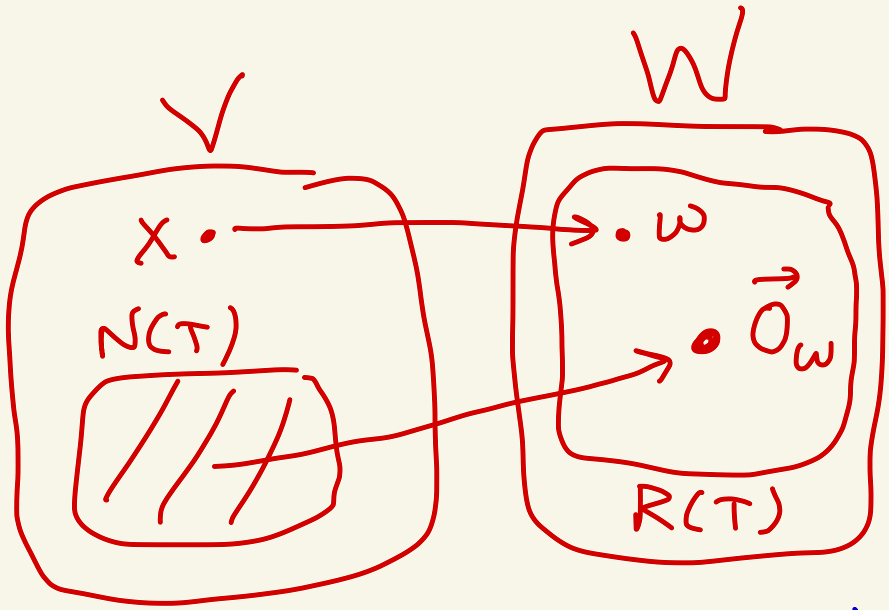
Case 2: Suppose $R(T) \neq \{\vec{0}_W\}$

Then in this case $R(T)$ contains at least one non-zero vector $w \neq \vec{0}_W$

So, there exists $x \in V$ where $T(x) = w \neq \vec{0}_W$

Thus,

$N(T) \neq V$.



By HW 2 #9 we can extend the basis for $N(T)$ to all of V.

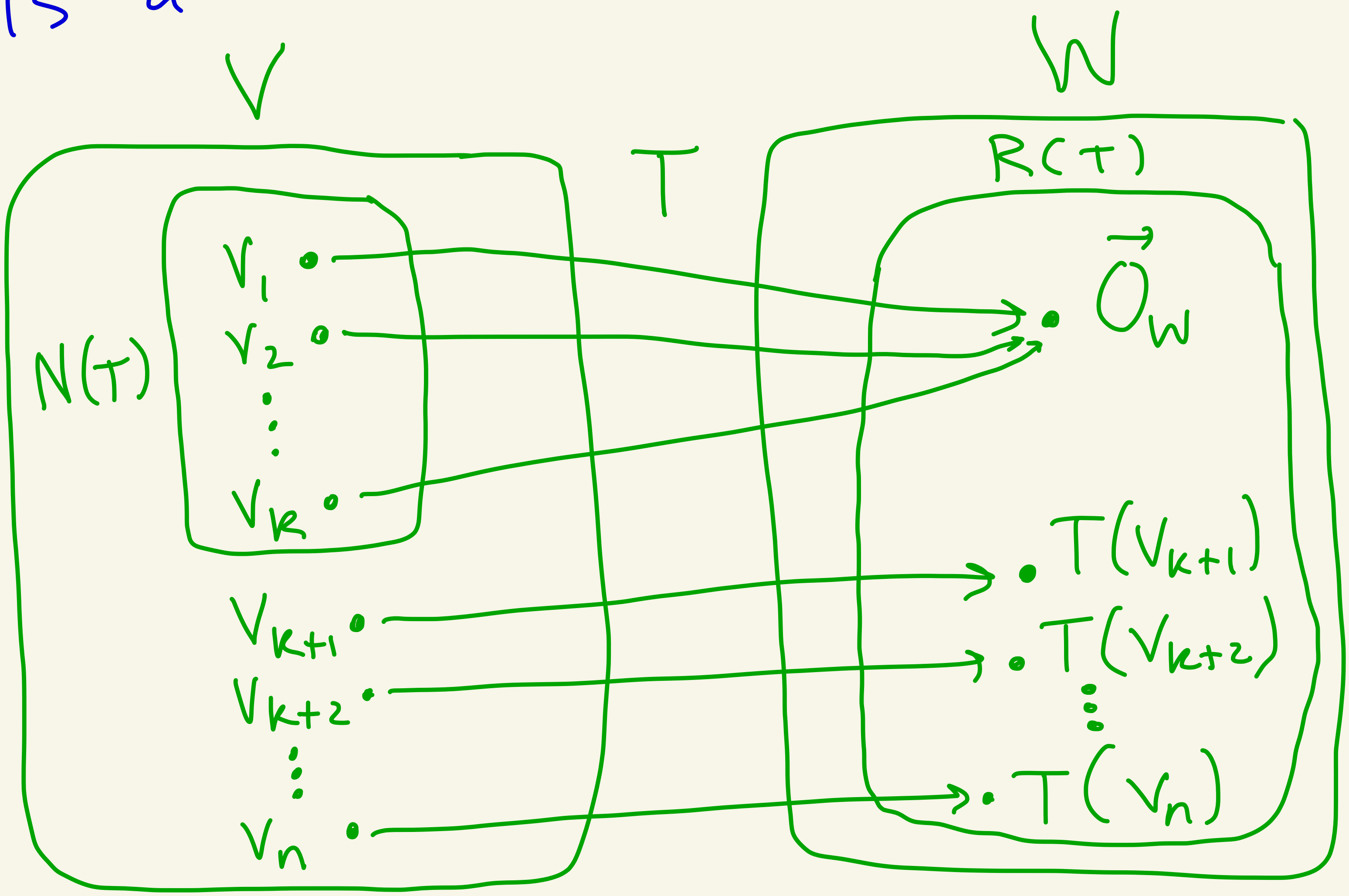
That is there exist

$v_{k+1}, v_{k+2}, \ldots, v_n \in \underbrace{V - N(T)}_{\text{in } V \text{ but not in } N(T)}$

where

$$\beta = \{ \underbrace{v_1, v_2, \ldots, v_k}_{\text{basis for } N(T)}, \underbrace{v_{k+1}, v_{k+2}, \ldots, v_n}_{\text{not in } N(T)} \}$$

is a basis for V.



We will show that

$$\beta' = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$$

is a basis for $R(T)$.

Note once we've done this, then we will have finished the proof of the theorem because then $R(T)$ will be finite dimensional and

$$\begin{aligned} \dim(V) &= n \\ &= k + (n-k) \\ &= \dim(N(T)) + \left(\begin{matrix}\#\text{ elements} \\ \text{in } \beta'\end{matrix}\right) \\ &= \dim(N(T)) + \dim(R(T)). \end{aligned}$$

So, let's now show that β' is a basis for $R(T)$.

By a theorem from Monday, since $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ spans V, we know that

$$R(T) = \operatorname{span}\left(\{T(v_1), T(v_2), \ldots, T(v_k), T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}\right)$$

$$= \operatorname{span}\left(\{\vec{0}_W, \vec{0}_W, \ldots, \vec{0}_W, T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}\right)$$

$$= \operatorname{span}\left(\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}\right)$$

Thus, β' spans $R(T)$.

Let's now show β' is a linearly independent set.

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Suppose

$$c_{k+1}T(v_{k+1}) + c_{k+2}T(v_{k+2}) + \dots + c_n T(v_n) = \vec{0}_W$$

Since T is linear we have

$$T(c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \dots + c_n v_n) = \vec{0}_W$$

Thus, $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \dots + c_n v_n$ is in $N(T)$.

Since $N(T)$ has $\{v_1, v_2, \dots, v_k\}$ as a basis we must have that

$$c_{k+1}v_{k+1} + \dots + c_n v_n = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

for some $c_1, c_2, \dots, c_k \in F$.

Thus,

$$-c_1v_1 - \dots - c_kv_k + c_{k+1}v_{k+1} + \dots + c_nv_n = \vec{0}_V$$

But $\beta = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ is a basis for V and hence is linearly independent.

So the above equation implies that

$$-c_1 = -c_2 = \dots = -c_k = c_{k+1} = \dots = c_n = 0$$

In particular,

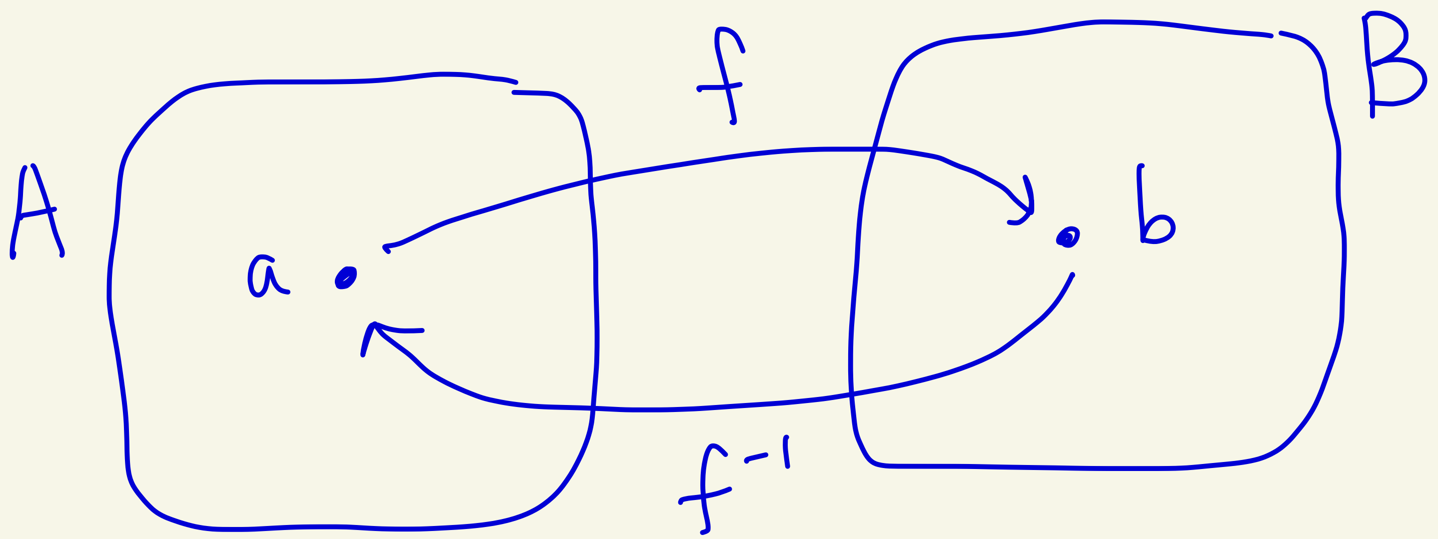
$$c_{k+1} = c_{k+2} = \dots = c_n = 0.$$

Thus, $\beta' = \{T(v_{k+1}), \dots, T(v_n)\}$ is linearly independent.

So, β' is a basis for $R(T)$. $\blacksquare$

Recall: Suppose $f: A \to B$ is 1-1 and onto where A and B are sets. Then $f^{-1}: B \to A$ is defined by

$f^{-1}(b) = a$ iff $f(a) = b$.



Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

Then you can check that T is a linear transformation and its 1-1 and onto.

Let's find $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$.

$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

iff $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

iff $\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

iff $\boxed{\begin{aligned} a+b &= c \\ a-b &= d \end{aligned}}$

Let's solve this system.

$$\left(\begin{array}{cc|c} 1 & 1 & c \\ 1 & -1 & d \end{array}\right)$$

$$\xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & -2 & d-c \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & 1 & c \\ 0 & 1 & -\frac{d}{2}+\frac{c}{2} \end{array}\right)$$

row echelon form

Thus,
$$\boxed{\begin{aligned} a+b &= c \\ b &= -\frac{d}{2}+\frac{c}{2} \end{aligned}} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

(2) gives $b = -\frac{d}{2}+\frac{c}{2}$.

(1) gives $a = c-b = c-\left(-\frac{d}{2}+\frac{c}{2}\right)$
$= \frac{1}{2}c+\frac{1}{2}d$

pg 14

Thus,

$$T^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c + \frac{1}{2}d \\ \frac{1}{2}c - \frac{1}{2}d \end{pmatrix}$$

You can check that T^{-1} is linear by checking that

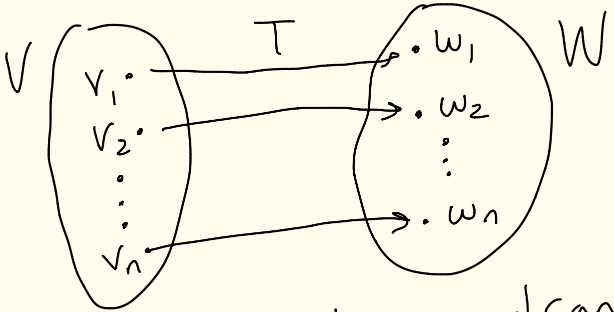
$$T^{-1}(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T^{-1}(v_1) + \alpha_2 T^{-1}(v_2)$$

for all $v_1, v_2 \in \mathbb{R}^2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Theorem: Let V and W be vector spaces over a field F. Suppose that V is finite-dimensional and $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

$\boxed{\text{part 1}}$ Let $w_1, w_2, \ldots, w_n \in W$.

(1) There exists a unique linear transformation $T: V \to W$ where $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$



this unique linear transformation is given by the formula

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \quad (*)$$

(2) T given above is an isomorphism iff $\beta' = \{w_1, w_2, \ldots, w_n\}$ is a basis for W.

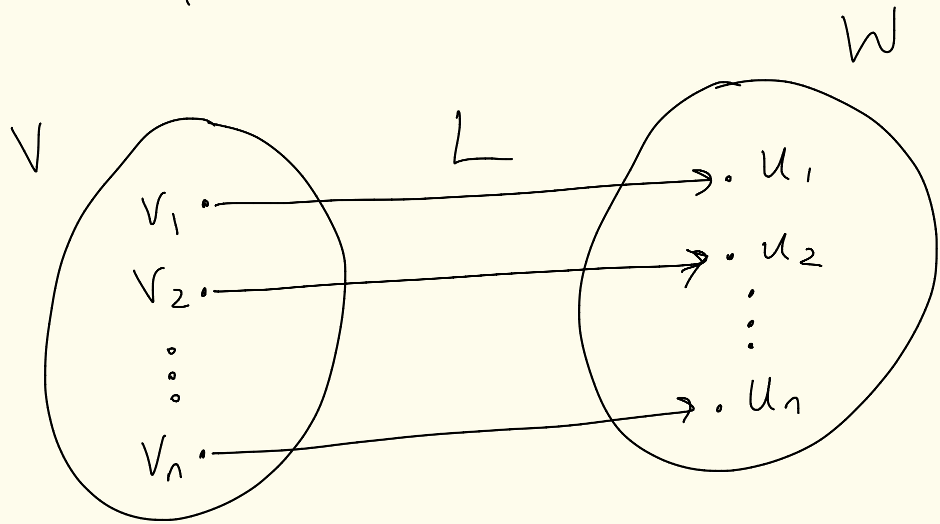
part 2

All linear transformations between V and W are constructed as in ① above. That is, if $L : V \to W$ is a linear transformation, set

$$u_i = L(v_i) \quad \text{for } i = 1, 2, \ldots, n$$

and then the formula for L is

$$L(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$
$$= c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$



proof: part 1

① Let T be defined by (*).
That is,

$$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

for any $c_i \in F$.

Let's show T is a linear transformation and $T(v_i) = w_i$ for all i.

Why is T linear?

Let $x, y \in V$ and $\alpha, \delta \in F$.
Since β is a basis for V, we can write $x = e_1 v_1 + \dots + e_n v_n$ and $y = d_1 v_1 + \dots + d_n v_n$ where $e_i, d_i \in F$. Then,

$$T(\alpha x + \delta y)$$
$$= T\big(\alpha(e_1 v_1 + \dots + e_n v_n) + \delta(d_1 v_1 + \dots + d_n v_n)\big) =$$
$$= T\big((\alpha e_1 + \delta d_1)v_1 + \dots + (\alpha e_n + \delta d_n)v_n\big) =$$

$= T\big((\alpha e_1 + \delta d_1)v_1 + \dots + (\alpha e_n + \delta d_n)v_n\big)$

$\overset{(*)}{=} (\alpha e_1 + \delta d_1)w_1 + \dots + (\alpha e_n + \delta d_n)w_n$

$= \alpha e_1 w_1 + \dots + \alpha e_n w_n$
$+ \delta d_1 w_1 + \dots + \delta d_n w_n$

$= \alpha(e_1 w_1 + \dots + e_n w_n)$
$+ \delta(d_1 w_1 + \dots + d_n w_n)$

$\overset{(*)}{=} \alpha T(e_1 v_1 + \dots + e_n v_n)$
$+ \delta T(d_1 v_1 + \dots + d_n v_n)$

$= \alpha T(x) + \delta T(y)$.

So, T is linear.

Also,

$T(v_1) = T(1 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n) = 1 \cdot w_1 = w_1$

$\vdots$

$T(v_n) = T(0 \cdot v_1 + 0 \cdot v_2 + \dots + 1 \cdot v_n) = 1 \cdot w_n = w_n$

So, $T(v_i) = w_i$ for all i.

Why is T unique?

Suppose $S : V \to W$ is another linear transformation with $S(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Let $x \in V$.

Then, since β is a basis for V,

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

And,

$$
\begin{aligned}
S(x) &= S(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\
&\overset{\text{S is linear}}{=} c_1 S(v_1) + c_2 S(v_2) + \cdots + c_n S(v_n) \\
&\overset{S(v_i) = w_i}{=} c_1 w_1 + c_2 v_2 + \cdots + c_n w_n \\
&= T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) \\
&\overset{\text{def of } T}{=} T(x)
\end{aligned}
$$

So, $S = T$ on V.

So, T is the unique linear transf. with $T(v_i) = w_i \ \forall i$

② $\boxed{T \text{ defined by } (*) \text{ is an isomorphism iff } \beta' = \{w_1, w_2, \ldots, w_n\} \text{ is a basis for } W.}$

($\Leftarrow$) Suppose β' is a basis for W.
Let's show that T defined by $(*)$ is 1-1 and onto, and hence an isomorphism

$\boxed{1\text{-}1}$: Suppose $T(x) = T(y)$ for some $x, y \in V$.
Since β is a basis for V,
$x = c_1 v_1 + \cdots + c_n v_n$ and $y = d_1 v_1 + \cdots + d_n v_n$
for $c_i, d_i \in F$.
Since $T(x) = T(y)$, by def of T, we have

$$\underbrace{c_1 w_1 + \cdots + c_n w_n}_{T(x)} = \underbrace{d_1 w_1 + \cdots + d_n w_n}_{T(y)}$$

So, $(c_1 - d_1) w_1 + \cdots + (c_n - d_n) w_n = \vec{0}$

By assumption, β' is a lin. ind. set, so
$0 = c_1 - d_1 = c_2 - d_2 = \cdots = c_n - d_n$

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So, $c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n$
and hence

$$x = c_1 v_1 + \cdots + c_n v_n = d_1 v_1 + \cdots + d_n v_n = y.$$

$\boxed{\text{onto}}$: We need to show $R(T) = W$.
By a previous thm, since $\beta = \{v_1, v_2, \ldots, v_n\}$
spans V, we know $R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$.
So,

$$R(T) = \text{span}(\{T(v_1), \ldots, T(v_n)\})$$

$$= \text{span}(\{w_1, \ldots, w_n\})$$

$$= W.$$

we are assuming $\beta' = \{w_1, \ldots, w_n\}$ is a basis for W

So, T is onto W.

Thus, T is an isomorphism.

($\Rightarrow$) Now suppose T is an isomorphism, ie 1-1 and onto.
Let's show β' is a basis for W.

Since T is onto, $R(T) = W$.

Therefore,

$$W = R(T) = \text{span}(\{T(v_1), \dots, T(v_n)\}) = \text{span}(\{w_1, \dots, w_n\})$$

So, β' spans W.

Is β' a lin. ind. set?

Suppose

$$d_1 w_1 + \dots + d_n w_n = \vec{0}_W$$

where $d_i \in F$.

Since T is 1-1 and onto, T^{-1} exists and is linear (from Monday) and $T^{-1}(w_i) = v_i$ for $i = 1, \dots, n$.

Since T^{-1} is linear, $T^{-1}(\vec{0}_w) = \vec{0}_v$.

So,

$$\vec{0}_v = T^{-1}(\vec{0}_w) = T^{-1}(d_1 w_1 + \dots + d_n w_n)$$

$$= d_1 T^{-1}(w_1) + \dots + d_n T^{-1}(w_n)$$

$$= d_1 v_1 + \dots + d_n v_n$$

Since $\beta = \{v_1, \dots, v_n\}$ is a basis and $\vec{0}_v = d_1 v_1 + \dots + d_n v_n$

we get $d_1 = d_2 = \dots = d_n = 0$.

Thus, β' is a lin. ind. set.

since if $d_1 w_1 + \dots + d_n w_n = \vec{0}_w$

then $d_1 = d_2 = \dots = d_n = 0$.

So, β' is a basis for W.

$\boxed{\text{part 2}}$

Suppose L is a linear transformation and $u_i = L(v_i)$ for $i = 1, 2, \ldots, n$.

Then,

$$L(c_1 v_1 + \cdots + c_n v_n)$$

$$= c_1 L(v_1) + \cdots + c_n L(v_n)$$

$$= c_1 u_1 + \cdots + c_n u_n.$$

L is linear

$\blacksquare$

Math 4570

10/4/21

HW 1 – 1(c)

There is an error in the solutions.
It is a vector space
I will fix this one.

SOME THINGS TO FOCUS ON FOR SURE FOR TEST 1

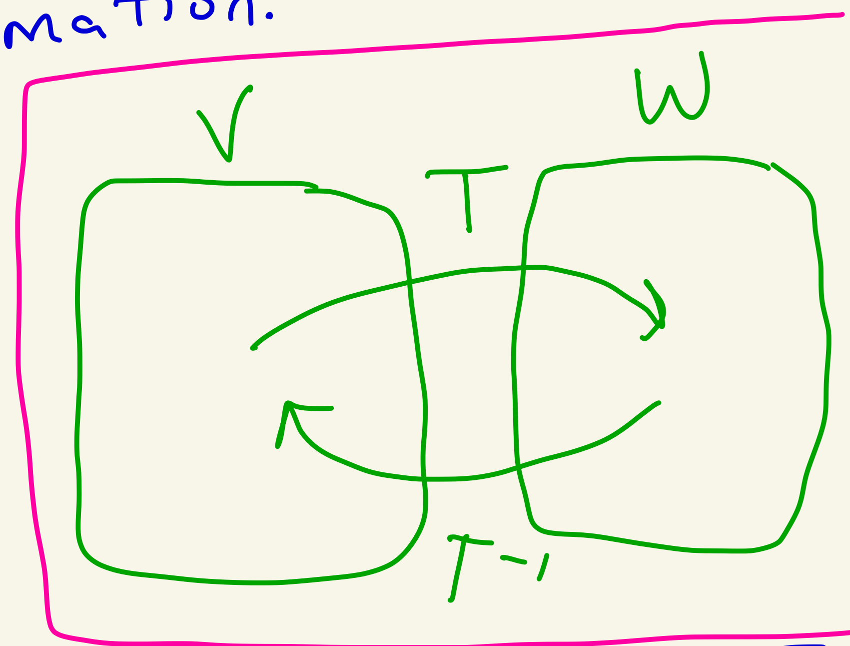
- Showing if some subset is a subspace.
- Finding a basis for V or W.
  Finding dimensions of V or W.
- Computationally see 2550 problems for HW 2.
  Does a set span?
  Is a set linearly dep/ind.?
  Is a set a basis?
  Is a

  Main vector spaces:
  $\mathbb{R}^n$, $P_n(\mathbb{R})$, $M_{m,n}(\mathbb{R})$

  F^n

  - Look at proofs in HW.

Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a 1-1 and onto linear transformation.
Then, $T^{-1}: W \to V$ is also a linear transformation.



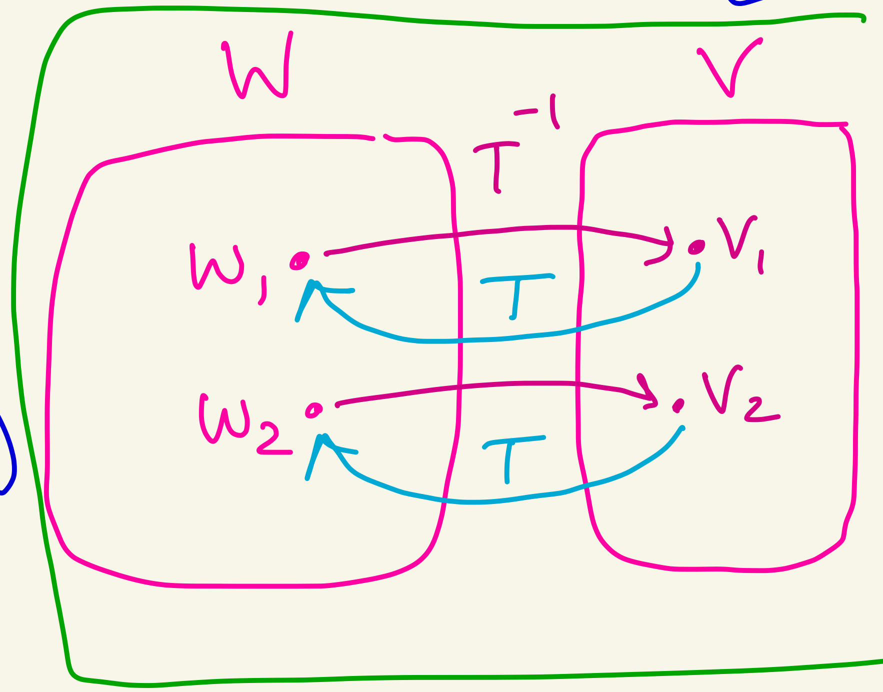
proof:

Because T is 1-1 and onto $T^{-1}: W \to V$ exists as a function. [MATH 3450]
We just need to show that T^{-1} is a linear transformation.

Let $\alpha_1, \alpha_2 \in F$ and $w_1, w_2 \in W$.

We will show that

$$T^{-1}(\alpha_1 w_1 + \alpha_2 w_2)$$
$$= \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2)$$



Then, there exist $v_1, v_2 \in V$ where $T^{-1}(w_1) = v_1$ and $T^{-1}(w_2) = v_2$.

By def of inverse, $T(v_1) = w_1$ and $T(v_2) = w_2$.

Thus,

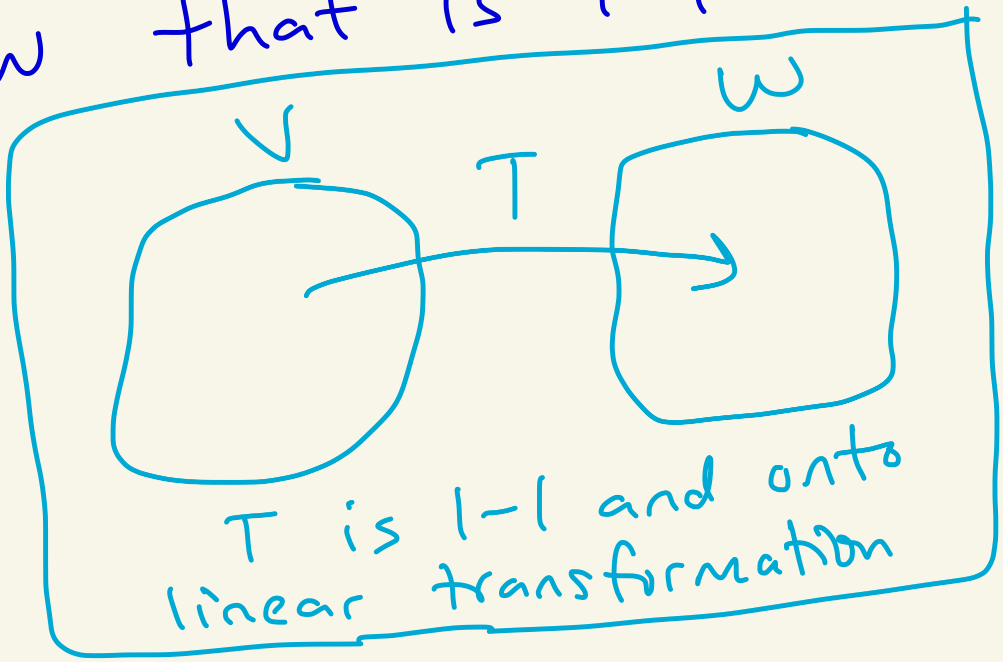
$$T^{-1}(\alpha_1 w_1 + \alpha_2 w_2) = T^{-1}(\alpha_1 T(v_1) + \alpha_2 T(v_2))$$
$$= T^{-1}(T(\alpha_1 v_1 + \alpha_2 v_2))$$
$$= \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 T^{-1}(w_1) + \alpha_2 T^{-1}(w_2) \quad \blacksquare$$

since T is linear

$T^{-1}(T(x)) = x$ for all $x \in V$, prop. of inverse

<u>Def:</u> Let V and W be vector spaces over a field F.

① An <u>isomorphism between V and W</u> is a linear transformation $T: V \to W$ that is 1-1 and onto.

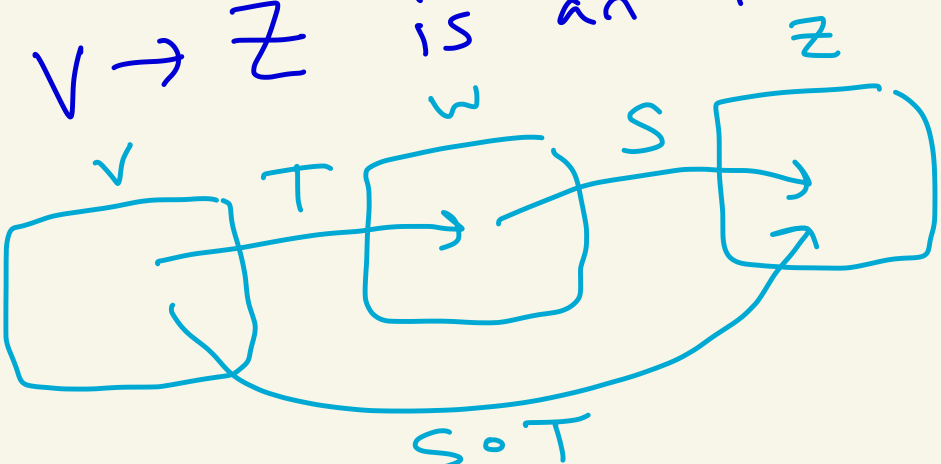


② We say that V and W are <u>isomorphic</u>, and write $V \cong W$, if there exists an isomorphism $T: V \to W$ between them.

<u>Note:</u> This def is well-defined by the following facts that one could show:

① If $T: V \to W$ is an isomorphism then $T^{-1}: W \to V$ is also an isomorphism.
Thus if $V \cong W$ then $W \cong V$.

② If $T: V \to W$ and $S: W \to Z$ are both isomorphisms, then $S \circ T: V \to Z$ is an isomorphism



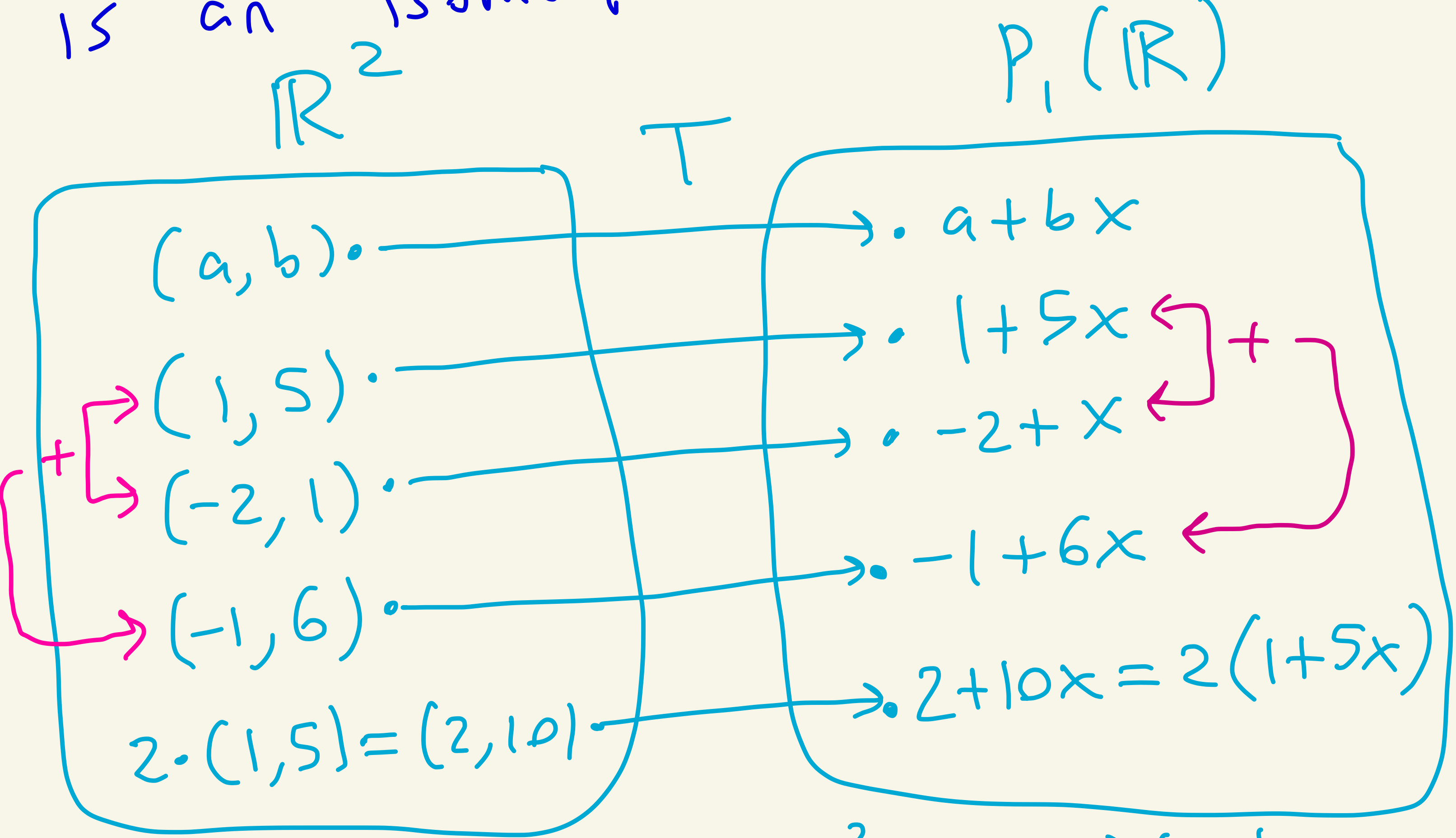
Thus if $V \cong W$ and $W \cong Z$ then $V \cong Z$.

$$\boxed{\begin{aligned}&(S \circ T)(x)\\ &= S(T(x))\end{aligned}}$$

<u>Ex:</u> Let $F = \mathbb{R}$. Let $V = \mathbb{R}^2$ and $W = P_1(\mathbb{R}) = \{a + bx \mid a, b \in \mathbb{R}\}$

Let $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ be defined by $T((a,b)) = a + bx$

We will show later that T is an isomorphism.



T is showing that $\mathbb{R}^2$ and $P_1(\mathbb{R})$ are structurally the same. The elements are just notated differently.

Theorem: (Constructing linear transformations $T: V \to W$ when V is finite-dimensional)

pg 7

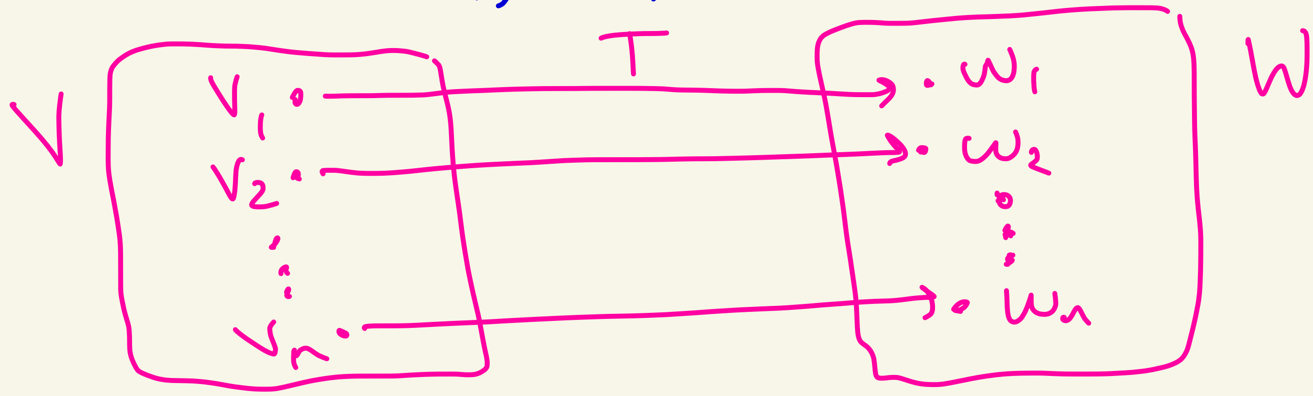
Let V and W be vector spaces over a field F. Suppose V is finite-dimensional and that $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V.

PART 1 Pick any $w_1, w_2, \ldots, w_n \in W$

① Then there exists a unique linear transformation $T: V \to W$ with $T(v_1) = w_1$, $T(v_2) = w_2, \ldots, T(v_n) = w_n$ given by the formula

$$\left.\begin{aligned} &T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) \\ &\quad = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \end{aligned}\right] \quad (*)$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.



(2) T given above in part 1 and $(*)$ is an isomorphism iff $\beta' = \{w_1, w_2, \dots, w_n\}$ is a basis for W.

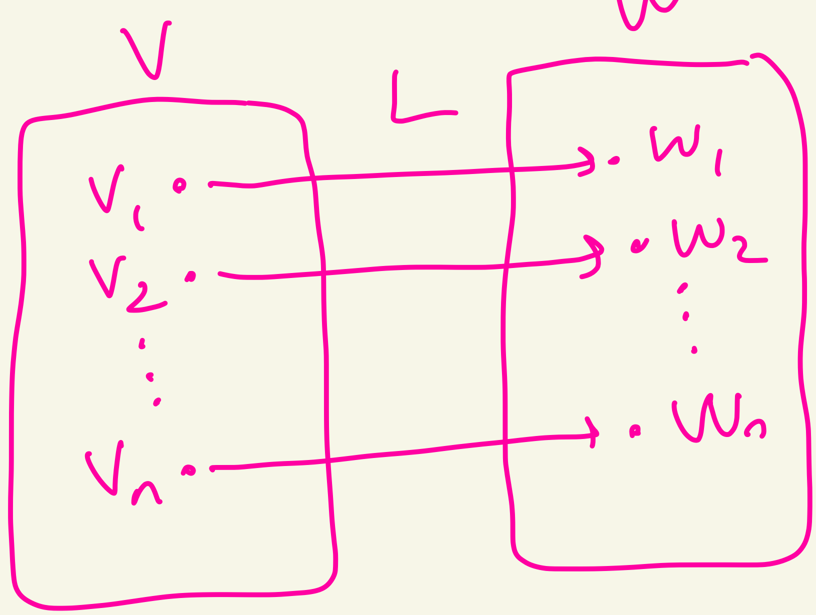
Part 2 All linear transformations between V and W are constructed as in part 1 above. That is, if $L: V \to W$ is a linear transformation, then pick

$$w_1 = L(v_1),\ w_2 = L(v_2),\ \dots,\ w_n = L(v_n)$$

and then the formula for L will be

$$L(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha_1 w_1 + \dots + \alpha_n w_n$$

as in $(*)$.



Proof:

We won't do this proof in class.

I will post it in on the website on the same day in the calendar. $\blacksquare$

Ex: Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let's make a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$.

Step 1: Pick a basis for $V = \mathbb{R}^3$.
Let's use the standard basis

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \{ v_1, v_2, v_3 \}$$

(from theorem)

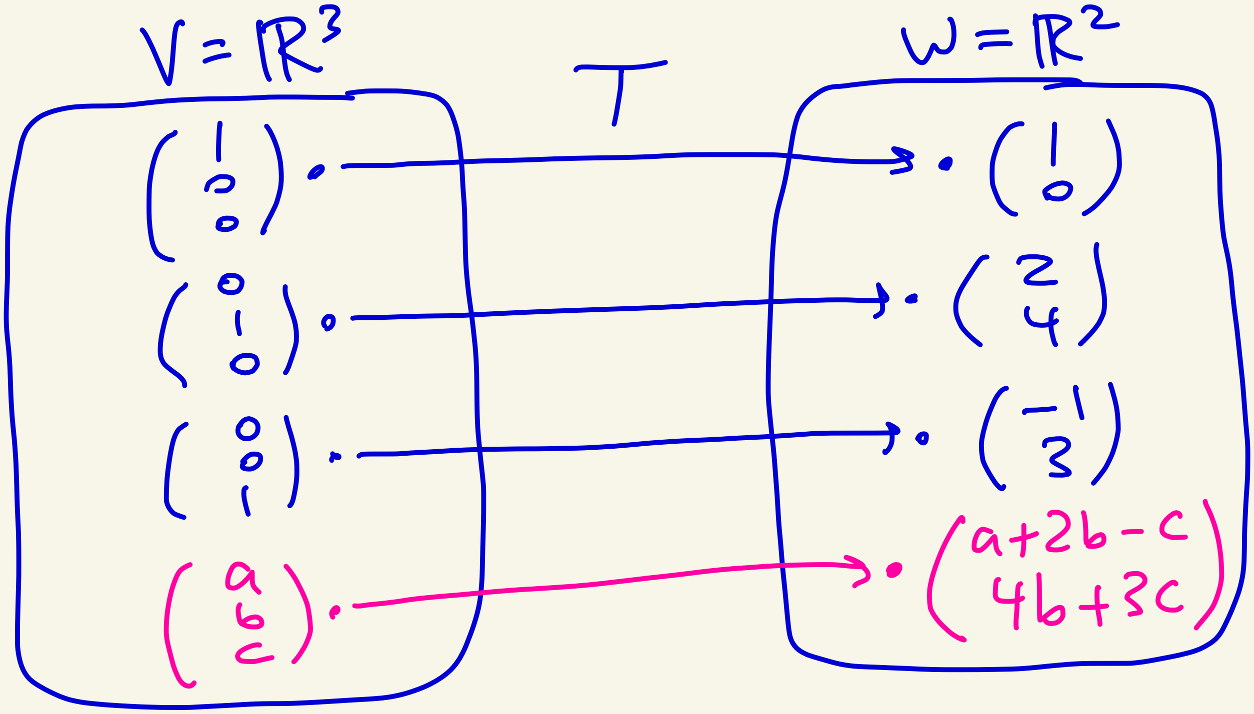
Step 2: Decide where β goes.

Pick:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_1$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = w_2$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 3 \end{pmatrix}}_{\text{can put any vectors here}} = w_3$$



Then in general for any $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ we have from the theorem that

$$T\begin{pmatrix}a\\b\\c\end{pmatrix} = T\left(a\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + b\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + c\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= a\cdot\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{w_1} + b\cdot\underbrace{\begin{pmatrix}2\\4\end{pmatrix}}_{w_2} + c\cdot\underbrace{\begin{pmatrix}-1\\3\end{pmatrix}}_{w_3}$$

$$= \begin{pmatrix}a+2b-c\\4b+3c\end{pmatrix}$$

T will be a linear transformation.

Math 4570

10/6/21

Ex: $V = \mathbb{R}^2$

$W = P_1(\mathbb{R}) = \{a + bx \mid a, b \in \mathbb{R}\}$

Let's build a linear transformation between these vector spaces.

Step 1: Pick a basis for $V = \mathbb{R}^2$.

Let's pick the standard basis $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

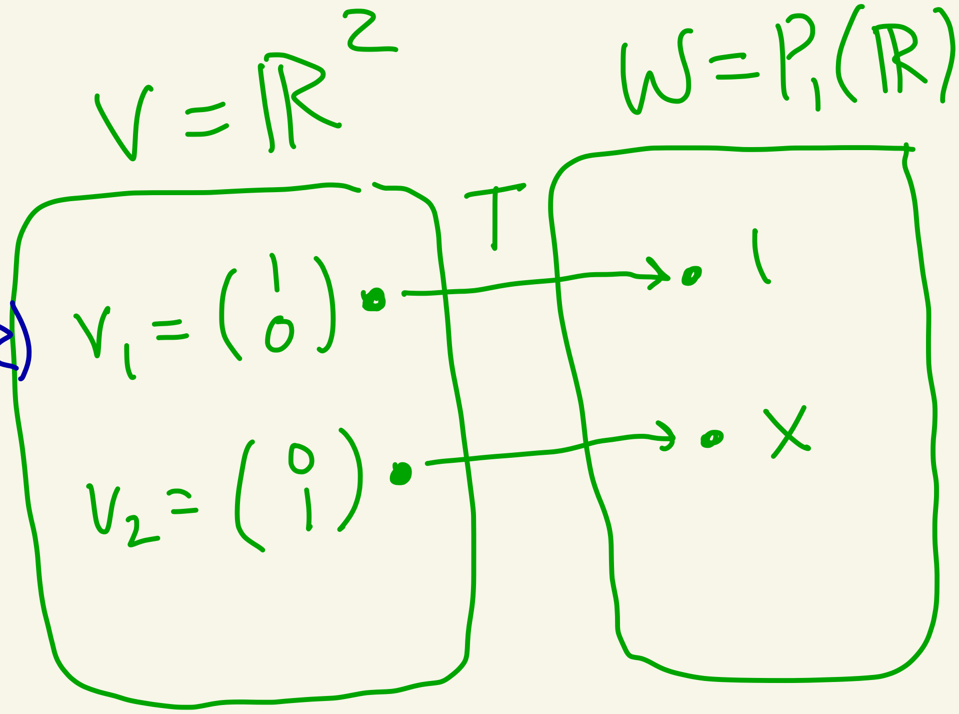
Step 2: Choose where each element of β goes.

You can send them anywhere.

Define $T : \mathbb{R}^2 \to P_1(\mathbb{R})$ where

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x$



There is only one way to make this linear transformation. And this is described in Mondays Theorem. The reason is as follows:

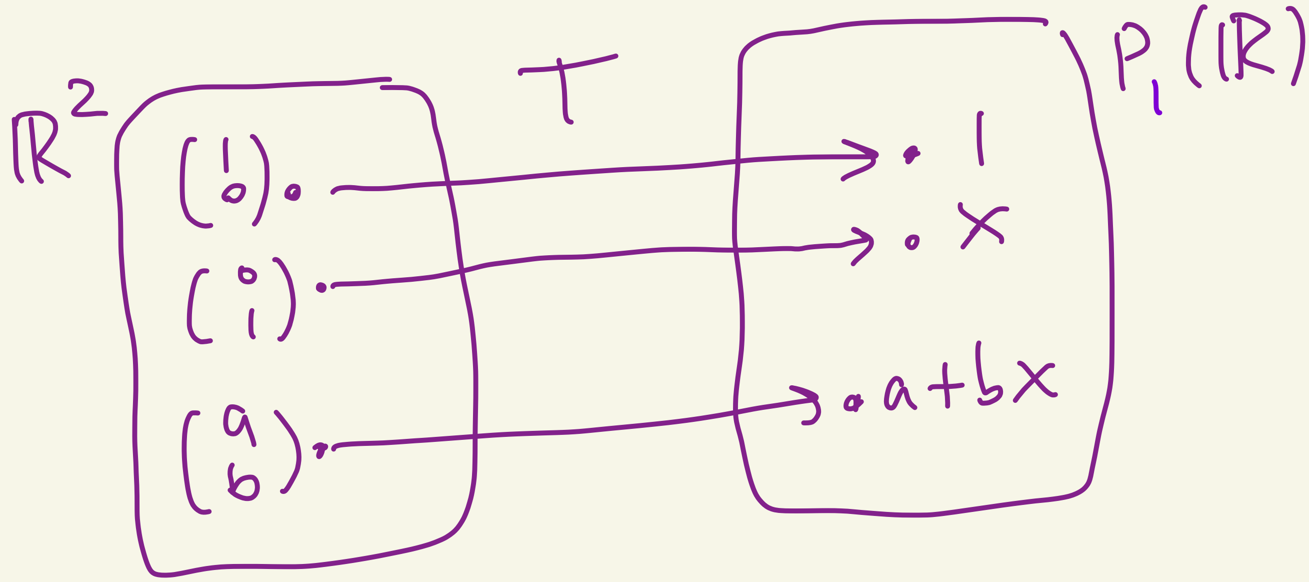
Suppose we have $v \in \mathbb{R}^2$.
Then $v = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a, b \in \mathbb{R}$.
So, to define T on v we need

$$T(v) = T\begin{pmatrix} a \\ b \end{pmatrix}$$
$$= T\left(a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$
$$= aT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= a \cdot 1 + b \cdot x$$
$$= a + bx$$

In order for T to be linear we need

This is what the theorem said also.

Thus the only linear transformation $T: \mathbb{R}^2 \to P_1(\mathbb{R})$ where $T\begin{pmatrix}1\\0\end{pmatrix} = 1$ and $T\begin{pmatrix}0\\1\end{pmatrix} = x$ is given by the formula $T\begin{pmatrix}a\\b\end{pmatrix} = a + bx$



By Mondays theorem this is a linear transformation.

Furthermore, it is an isomorphism if and only if $\{1, x\}$ is a basis for $P_1(\mathbb{R})$ which it is!

Thus, T is an isomorphism and

$$\mathbb{R}^2 \cong P_1(\mathbb{R}).$$

pg 4

Ex: Let's consider the vector spaces $V = \mathbb{R}^4$ and $W = M_{2,2}(\mathbb{R})$.

Pick the standard basis for $V = \mathbb{R}^4$ which is $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

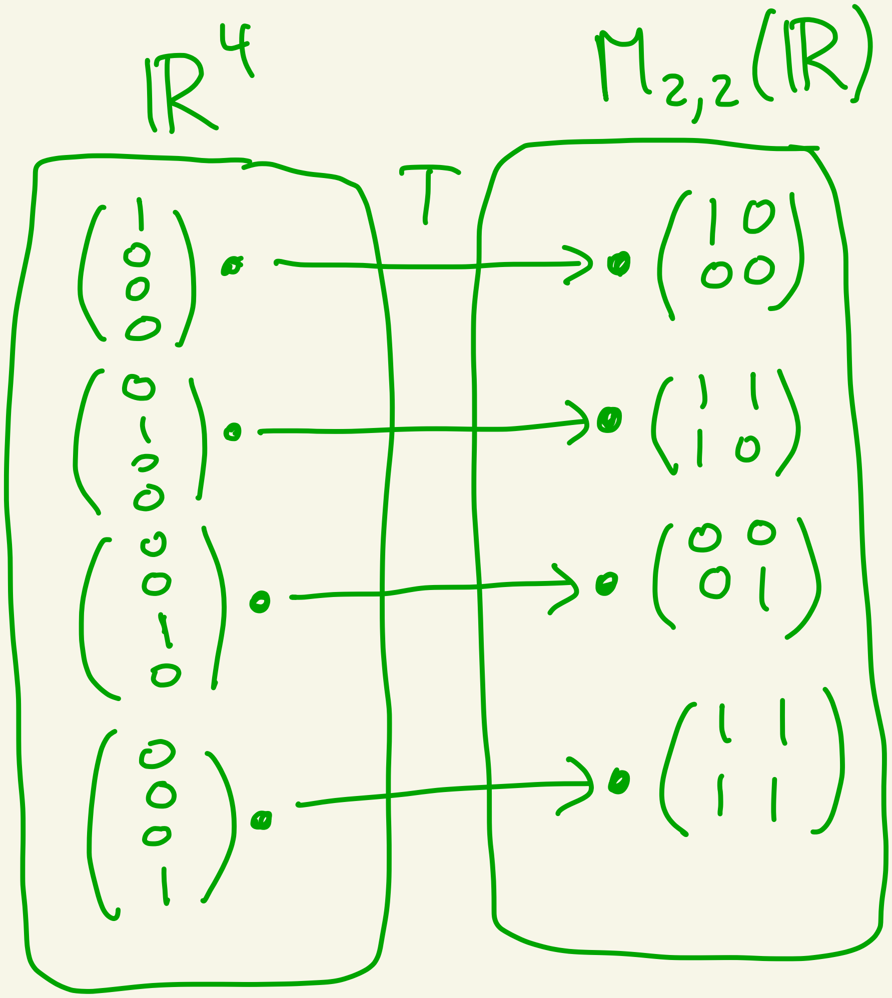
Let's create the linear transformation where

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



The formula for such a linear transformation is given by

pg 5

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = T\left(a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

$$= aT\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + bT\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + cT\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + dT\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

has to be true to make T linear as in theorem from Mon.

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + d\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

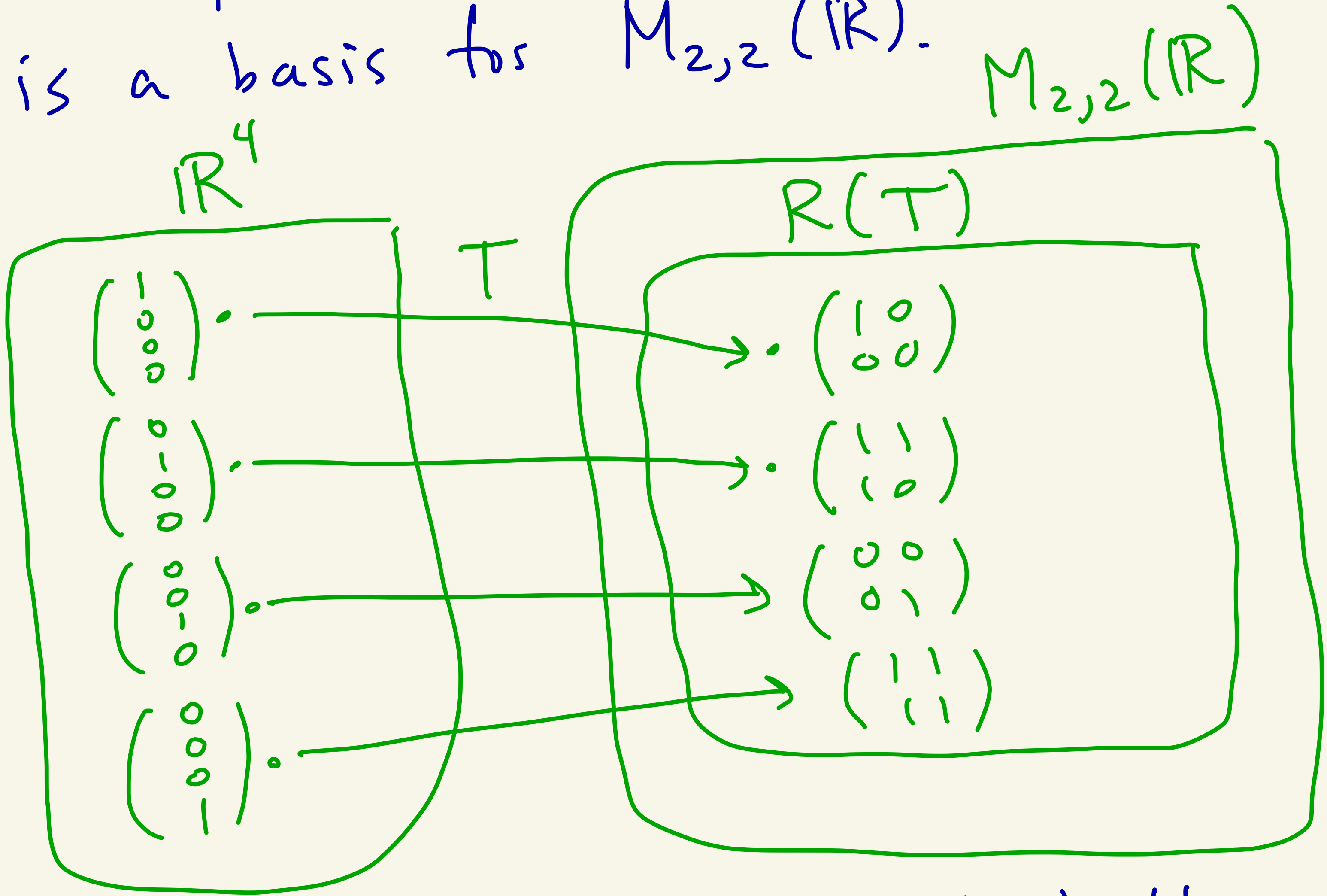
$$= \begin{pmatrix} a+b+d & b+d \\ b+d & c+d \end{pmatrix}$$

Thus from Mon theorem $T: \mathbb{R}^4 \to M_{2,2}(\mathbb{R})$ given by

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b+d & b+d \\ b+d & c+d \end{pmatrix}$$

is a linear transformation.

The theorem from Monday tells us that T is an isomorphism iff $\beta' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ is a basis for $M_{2,2}(\mathbb{R})$.



Idea is: If β' spans $M_{2,2}(\mathbb{R})$, then $R(T) = \text{span}(\beta') = M_{2,2}(\mathbb{R})$ and T will be onto. If in addition, β' is a lin. ind. set that will make T one-to-one.

β' is actually <u>not</u> lin. ind. because a solution to the equation

$$c_1\begin{pmatrix}1&0\\0&0\end{pmatrix}+c_2\begin{pmatrix}1&1\\1&0\end{pmatrix}+c_3\begin{pmatrix}0&0\\0&1\end{pmatrix}+c_4\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

is

$$0\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+1\cdot\begin{pmatrix}1&1\\1&0\end{pmatrix}+1\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}-1\cdot\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

which shows β' is a lin. dep. set.

So, T will not be an isomorphism.

If you wanted to, you could show that $\dim(N(T))=1$ by solving

$$T\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a+b+d & b+d\\ b+d & c+d\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

By rank-nullity,

$$\underbrace{\dim(\mathbb{R}^4)}_{4}=\underbrace{\dim(N(T))}_{1}+\dim(R(T))$$

So, $\dim(R(T))=3$ and so T is not onto.

<u>Theorem:</u> Let V and W be finite-dimensional vector spaces over a field F.
We have that $V \cong W$ if and only if $\dim(V) = \dim(W)$.

<u>proof:</u>

$(\Leftarrow)$ Suppose $\dim(V) = \dim(W)$.
Then there exist bases
$\beta = \{v_1, v_2, \ldots, v_n\}$ for V and
$\beta' = \{w_1, w_2, \ldots, w_n\}$ for W
where $n = \dim(V) = \dim(W)$.

Construct the linear transformation $T: V \to W$ given as follows:

Given $x \in V$, express x in terms of the basis β as follows:

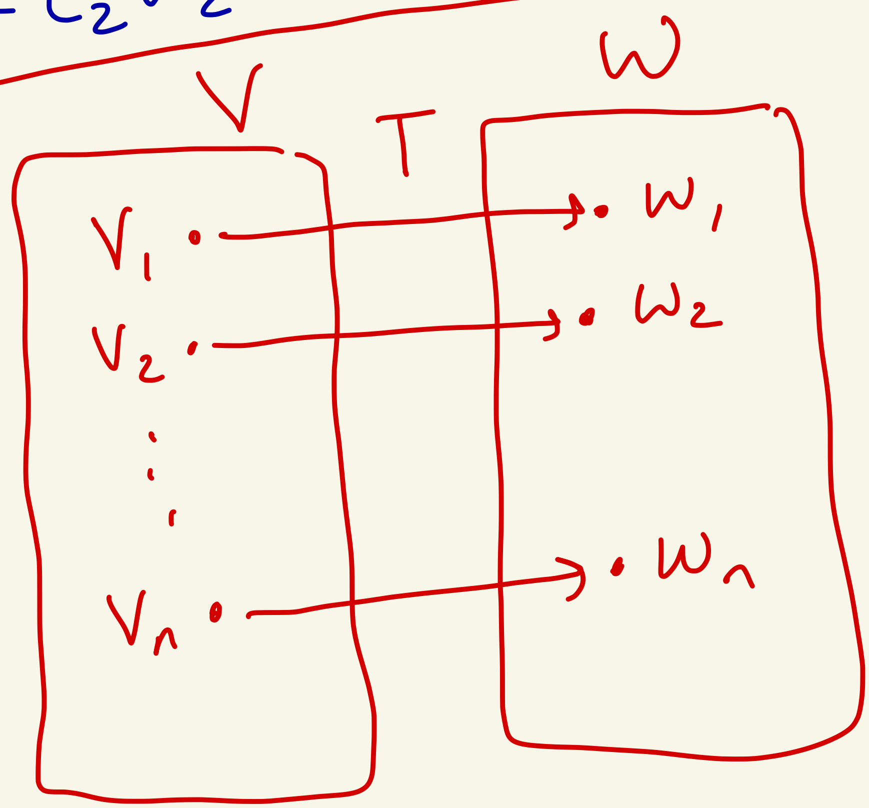
$$x = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Then, as in Mondays thm, define

$$T(x) = T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$$

$$= c_1 w_1 + c_2 w_2 + \dots + c_n w_n$$

So, β goes to β'. Since β' is a basis for W, by Mon. thm, T is an isomorphism.



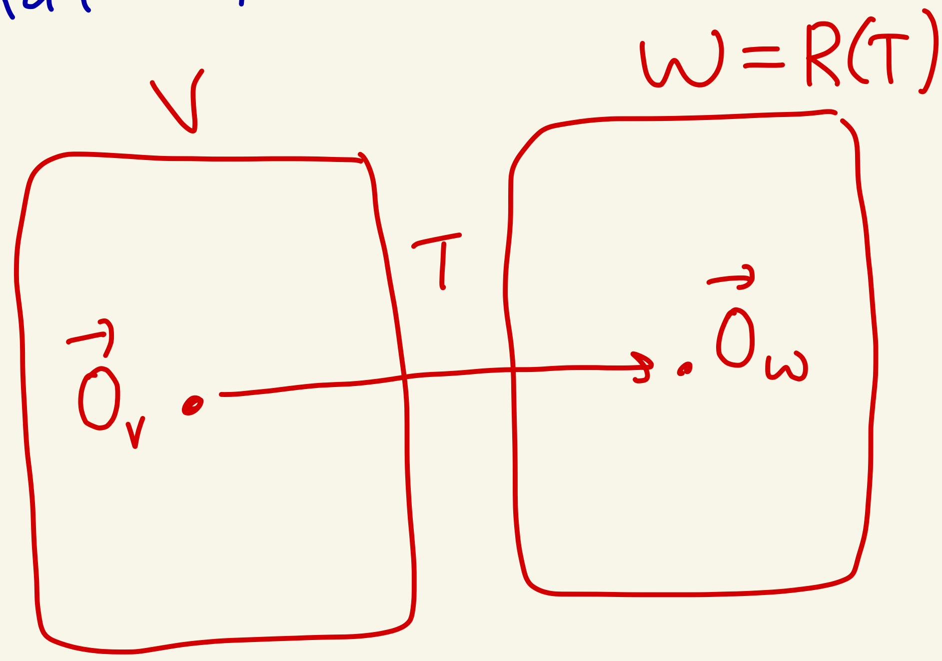
($\Rightarrow$) Suppose V and W are isomorphic.

This means there exists an isomorphism $T: V \longrightarrow W$.

So, T is a linear transformation that is 1-1 and onto.

By HW, because T is 1-1 we know that $N(T) = \{\vec{0}_V\}$.

Because T is onto we know $R(T) = W$.



By the rank-nullity theorem,

$$\begin{aligned}
\dim(V) &= \dim(N(T)) + \dim(R(T)) \\
&= \dim(\{\vec{0}_V\}) + \dim(W) \\
&= 0 + \dim(W) = \dim(W).
\end{aligned}$$

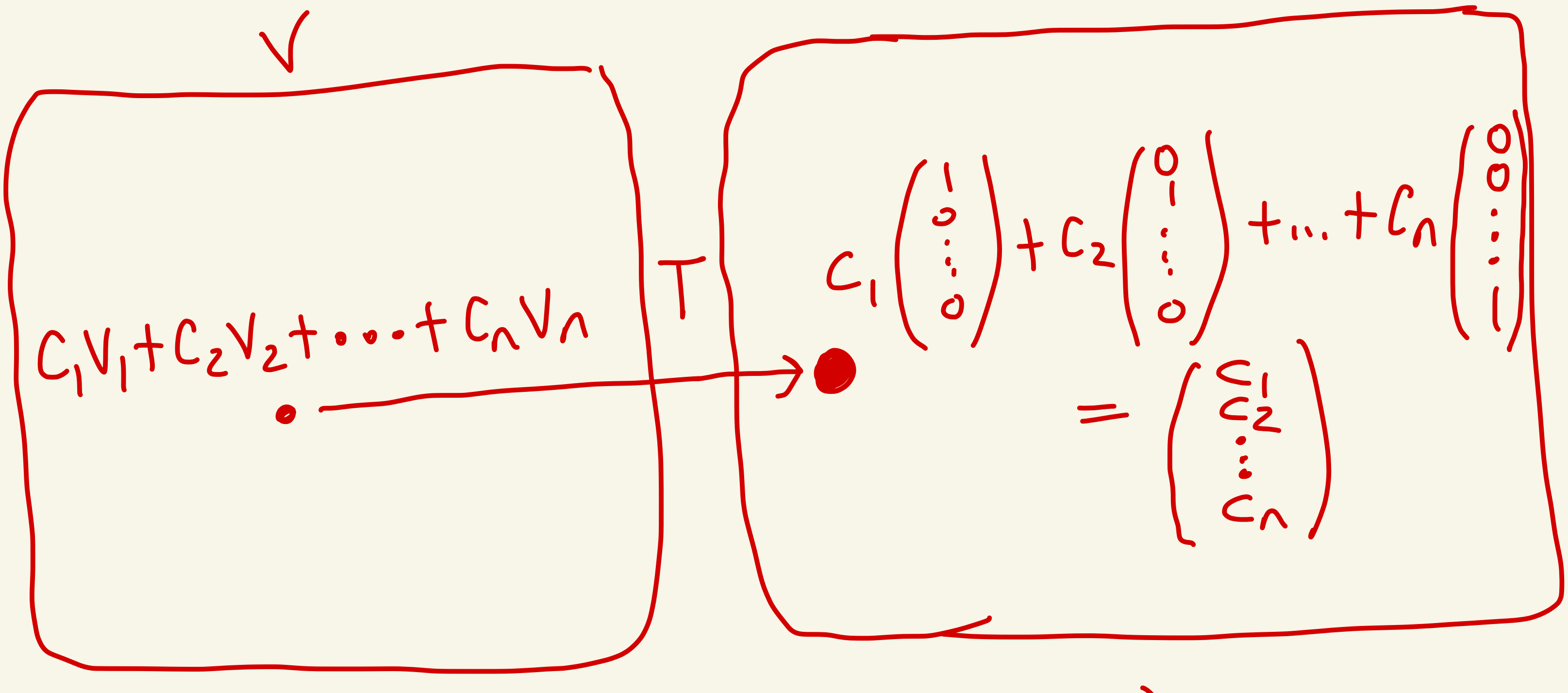
$\blacksquare$

<u>Corollary:</u> Let V be a finite-dimensional vector space over a field F. If $\dim(V) = n$, then $V \cong F^n$.

<u>proof:</u> Use the previous theorem and the fact that $\dim(F^n) = n = \dim(V)$. So, $V \cong F^n$. $\blacksquare$

Idea basis for V is $\{v_1, v_2, \ldots, v_n\}$

F^n



That is,

$$T(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = \begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}$$

is an isomorphism between V and F^n.

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Some people use the term "invertible" instead of "isomorphism"

Math 4570

10/11/21

Note: I fixed HW 1 # 1(c)

Recall: Test 1 is Monday 10/18.
See 10/4 notes pg 1 for some things to focus on.

HW 4 Topic
The Matrix of a linear Transformation

<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. We write $\beta = [v_1, v_2, \ldots, v_n]$ to mean that β is an <u>ordered basis</u> for V, that is, the order of the vectors in β is given and fixed.

<u>Def:</u> Let V be a vector space over a field F with an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$.

Let $x \in V$.

Then we can write x uniquely in the form

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

We write

$$[x]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

and call $[x]_\beta$ the <u>coordinates of x with respect to β</u>

(or the <u>coordinate vector of x with respect to β</u>)

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Consider $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

You can check that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are linearly independent.

Since are two linearly independent vectors and $\dim(V) = \dim(\mathbb{R}^2) = 2$ we know that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are a basis.

Thus, $\beta = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$ is an ordered basis.

Pick $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Let's find $[x]_\beta$

We need to solve

$$\underbrace{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}_{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

coordinates for x

This becomes

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix}$$

Which becomes

$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ 2c_1 + c_2 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} 5 &= c_1 - c_2 \\ 4 &= 2c_1 + c_2 \end{aligned}}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 1 & 4 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 3 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -2 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} c_1 - c_2 &= 5 \\ c_2 &= -2 \end{aligned}}$$

Thus, $c_2 = -2$.

And, $c_1 = 5 + c_2 = 5 - 2 = 3$.

So,

$$x = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus,

$$[x]_\beta = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

What if we kept $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ but changed to the standard basis

$$\beta' = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Then, $x = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So, $[x]_{\beta'} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Ex: Let

$V = P_2(\mathbb{R}) = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \}$

$F = \mathbb{R}$

Let

$\beta = [1, 1+x, 1+x+x^2]$

You can show that these 3 vectors are lin. ind. Since $\dim(P_2(\mathbb{R})) = 3$ they must be a basis

Consider

$v = 2 - x + 3x^2$

Let's find $[v]_\beta$.

We need to solve

$$\underbrace{2 - x + 3x^2}_{v} = c_1 \cdot 1 + c_2(1+x) + c_3(1+x+x^2)$$

v's coordinates with respect to β

This becomes

$$2 - x + 3x^2 = (c_1 + c_2 + c_3)\cdot 1 + (c_2 + c_3)\cdot x + c_3 x^2$$

So we get

$$\boxed{\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ c_2 + c_3 &= -1 \\ c_3 &= 3 \end{aligned}}$$

This is already a reduced system

We get

$$c_3 = 3$$

$$c_2 = -1 - c_3 = -1 - 3 = -4$$

$$c_1 = 2 - c_2 - c_3 = 2 - (-4) - 3 = 3$$

Thus,

$$2 - x + 3x^2 = 3\cdot 1 - 4\cdot(1+x) + 3\cdot(1+x+x^2)$$

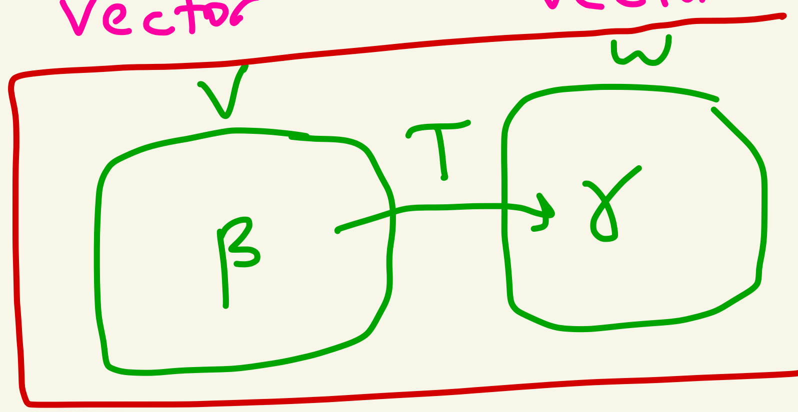
And $[2 - x + 3x^2]_\beta = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}$

Def: Let $T: V \to W$ be a linear transformation between two finite-dimensional vector spaces over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V and let γ be an ordered basis for W.

The matrix

$$[T]_\beta^\gamma = \left(\underbrace{[T(v_1)]_\gamma}_{\text{column vector}} \;\middle|\; \underbrace{[T(v_2)]_\gamma}_{\text{column vector}} \;\middle|\; \cdots \;\middle|\; \underbrace{[T(v_n)]_\gamma}_{\text{column vector}} \right)$$

is called the <u>matrix of T with respect to β and γ.</u>



If $V = W$ and $\beta = \gamma$, then we just write $[T]_\beta$ instead of $[T]_\beta^\beta$

Ex: Let $V = W = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

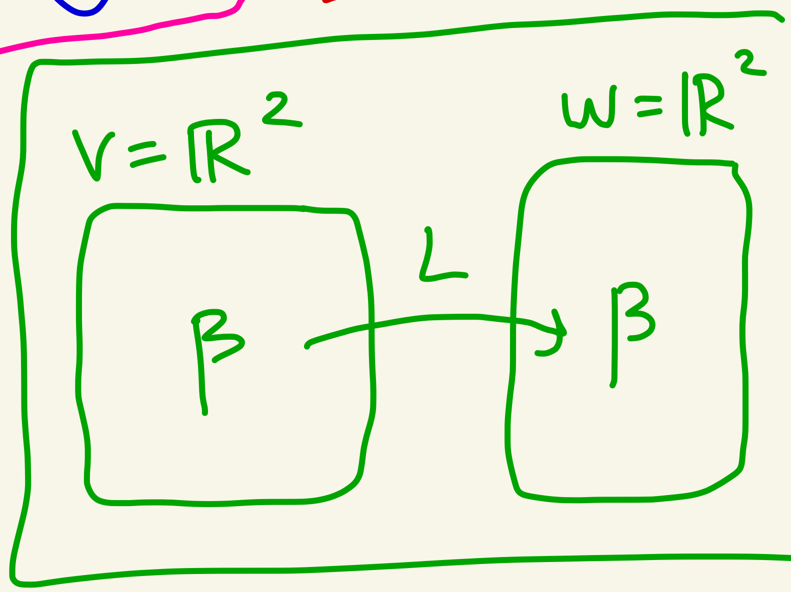
You can check that L is a linear transformation.

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

standard basis for $\mathbb{R}^2$

Let's compute

$$[L]_\beta = [L]_\beta^\beta$$



$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

plug in basis for $V = \mathbb{R}^2$

Find the coordinates of output in terms of basis for $W = \mathbb{R}^2$

Thus,

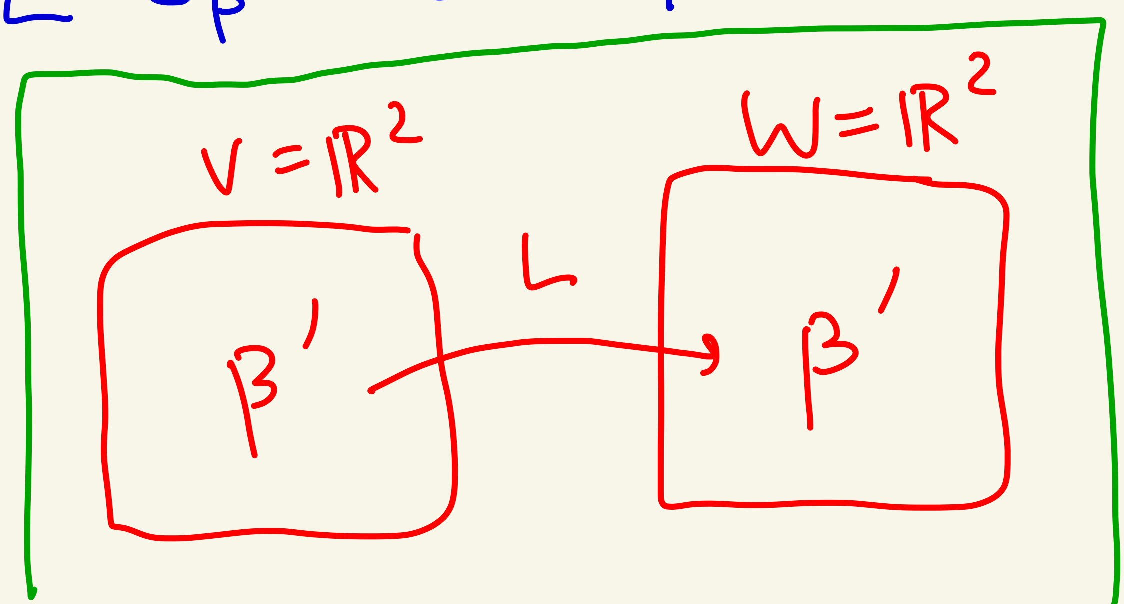
$$[L]_\beta = \left(\left[L\begin{pmatrix}1\\0\end{pmatrix}\right]_\beta \;\middle|\; \left[L\begin{pmatrix}0\\1\end{pmatrix}\right]_\beta \right)$$

$$= \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$$

Let $\beta' = \left[\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right]$ ← you can check that this is a basis for $\mathbb{R}^2$

Let's find

$$[L]_{\beta'} = [L]_{\beta'}^{\beta'}$$



Recall $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$

$$\underbrace{L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{\text{plug basis for } V=\mathbb{R}^2 \text{ into } L} = \underbrace{a\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{write output in terms of basis for } W=\mathbb{R}^2}$$

$$L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = b\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This becomes

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a-c \\ a+c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} b-d \\ b+d \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} 2 &= a-c \\ 1 &= a+c \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} 0 &= b-d \\ -3 &= b+d \end{aligned}}$$

If you solve these you will get

$$a = \frac{3}{2}, \ c = -\frac{1}{2}, \ b = -\frac{3}{2}, \ d = -\frac{3}{2}$$

So, $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

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$$L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underbrace{\frac{3}{2}}_{a} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underbrace{- \frac{1}{2}}_{c} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \underbrace{-\frac{3}{2}}_{b} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underbrace{- \frac{3}{2}}_{d} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus,

$$[L]_{\beta'} = [L]_{\beta'}^{\beta'}$$

$$= \left(\left[L\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\beta'} \middle| \left[L\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]_{\beta'} \right)$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

Math 4570

10/13/21

Math 4570 - Fall 2021 - Test 1

- You can only use your mind to take this exam. No help from any sources or people. No books, no notes, no online, etc.
- Use blank paper (like printer paper) if you have it please.
- On the first page of your exam, before any of your solutions, put these three things:
  (a) Your name.
  (b) The time period that you chose.
  (c) Copy this statement and then sign your signature after it:

  "Everything on this test is my own work. I did not use any sources or talk to anyone about this exam." **your signature**
- After your name and the above statement with signature, start putting your solutions to the problems. Please put them in order. That is, first problem 1, then problem 2, etc. You can put each one on its own page.
- Please scan your test using a scanner (such as a free one on your phone) and put it into one pdf document with your problems in order.
- To get a clean scan, make sure there is plenty of light, the phone is held flat directly above the paper, and the paper is placed on a flat object such as the floor or a table.
- Please upload your answer to canvas.

The problems are on the next page.

This is the cover page for Test 1 on Monday next week.

Test 1 - Monday

No class that day.

Do the test through canvas.

When you open the test, canvas will start timing you.

Format

- computations
- proofs

Ex: (Continued from Monday) [pg 2

Recap: $L: \mathbb{R}^2 \to \mathbb{R}^2$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-y \end{pmatrix}$$

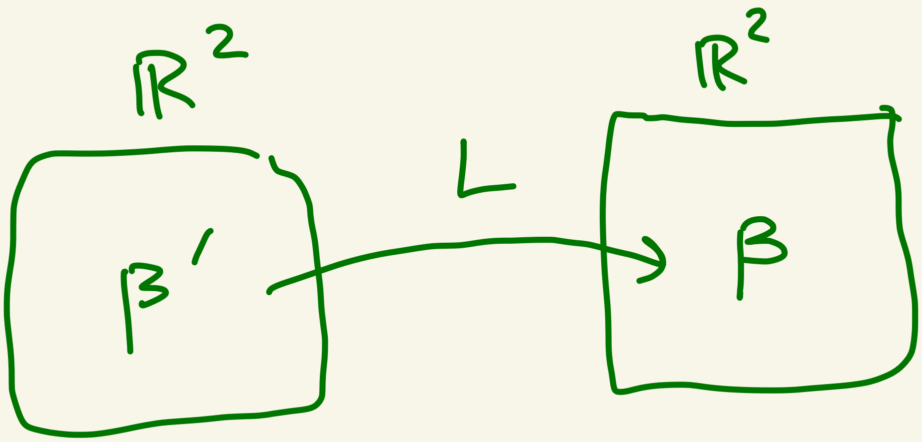
$$\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$$

$$[L]_\beta = [L]_\beta^\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$[L]_{\beta'} = [L]_{\beta'}^{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}$$

Let's calculate $[L]_{\beta'}^{\beta}$,



$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\2x-y\end{pmatrix}$$

$$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}$$

$$L\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\-3\end{pmatrix} = 0\begin{pmatrix}1\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\end{pmatrix}$$

plug β' into L

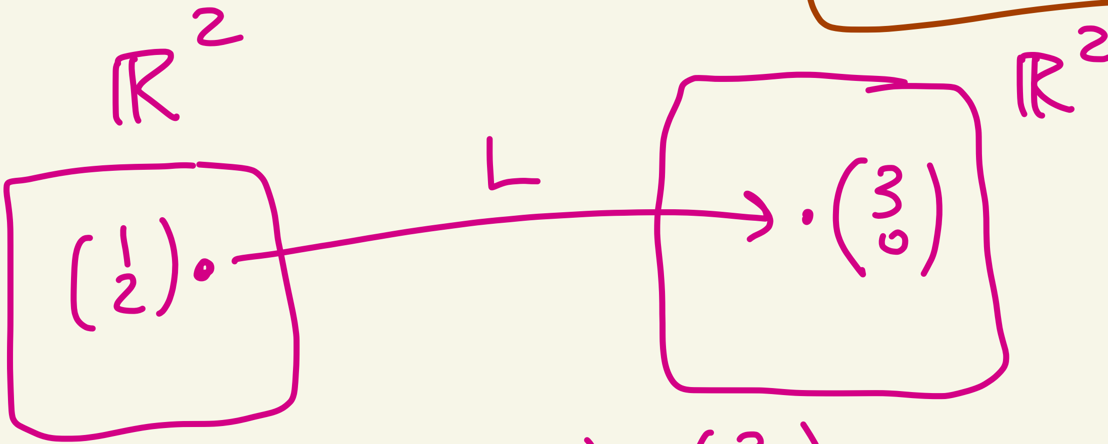
write the answers in terms of β

$$[L]_{\beta'}^{\beta} = \left(\left[L\begin{pmatrix}1\\1\end{pmatrix}\right]_{\beta} \;\middle|\; \left[L\begin{pmatrix}-1\\1\end{pmatrix}\right]_{\beta} \right)$$

$$= \begin{pmatrix}2 & 0\\1 & -3\end{pmatrix}$$

What do these matrices do?
Let's see with an example.

Pick $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ← these are the coordinates of v using the standard basis



$$L\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2\cdot 1 - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

The matrix that does the above is

$$[L]_\beta = [L]_\beta^\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Let's see:

$$[L]_\beta [v]_\beta = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [L(v)]_\beta$$

$\boxed{v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$\boxed{L(v) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Let's now see what $[L]_{\beta'} = [L]_{\beta'}^{\beta'}$ does to v.

We will show that

$$[L]_{\beta'} [v]_{\beta'} = [L(v)]_{\beta'}$$

So, $[L]_{\beta'} = [L]_{\beta'}^{\beta'}$ wants β' coordinates as its input and it computes L using the input and outputs the answer in β' coordinates.

What are v's β' coordinates?

Need to solve:

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underbrace{\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]}$$

This becomes

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} 1 &= \alpha_1 - \alpha_2 \\ 2 &= \alpha_1 + \alpha_2 \end{aligned}}$$

adding gives
$3 = 2\alpha_1$
$\alpha_1 = 3/2$
So, $\alpha_2 = 2 - \alpha_1$
$= 2 - 3/2 = 1/2$

The solution is

$\alpha_1 = 3/2$

$\alpha_2 = 1/2$

So, $v = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Thus, $[v]_{\beta'} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$

Then,

$$[L]_{\beta'}[v]_{\beta'} = \begin{pmatrix} 3/2 & -3/2 \\ -1/2 & -3/2 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (3/2)(3/2) - (3/2)(1/2) \\ (-1/2)(3/2) - (3/2)(\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

This should be $[L(v)]_{\beta'}$.

Whose β' coordinates are these?

$$\tfrac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} - \tfrac{3}{2}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3/2 + 3/2\\3/2 - 3/2\end{pmatrix}$$

$$= \begin{pmatrix}6/4\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$$

$$= L(v)$$

So, $[L(v)]_{\beta'} = \begin{pmatrix}3/2\\-3/2\end{pmatrix}$.

Thus, $[L]_{\beta'}[v]_{\beta'} = [L(v)]_{\beta'}$

Now let's see what $[L]^{\beta}_{\beta'}$ does.

I claim that

$$[L]^{\beta}_{\beta'}[v]_{\beta'} = [L(v)]_{\beta}$$

So, $[L]^{\beta}_{\beta'}$ wants β' coordinates as input, and computes L, but gives the answer in β coordinates

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We have that

$$[L]_{\beta}^{\beta'}[v]_{\beta'} = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(3/2) + (0)(1/2) \\ (1)(3/2) + (-3)(1/2) \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} = [L(v)]_{\beta}$$

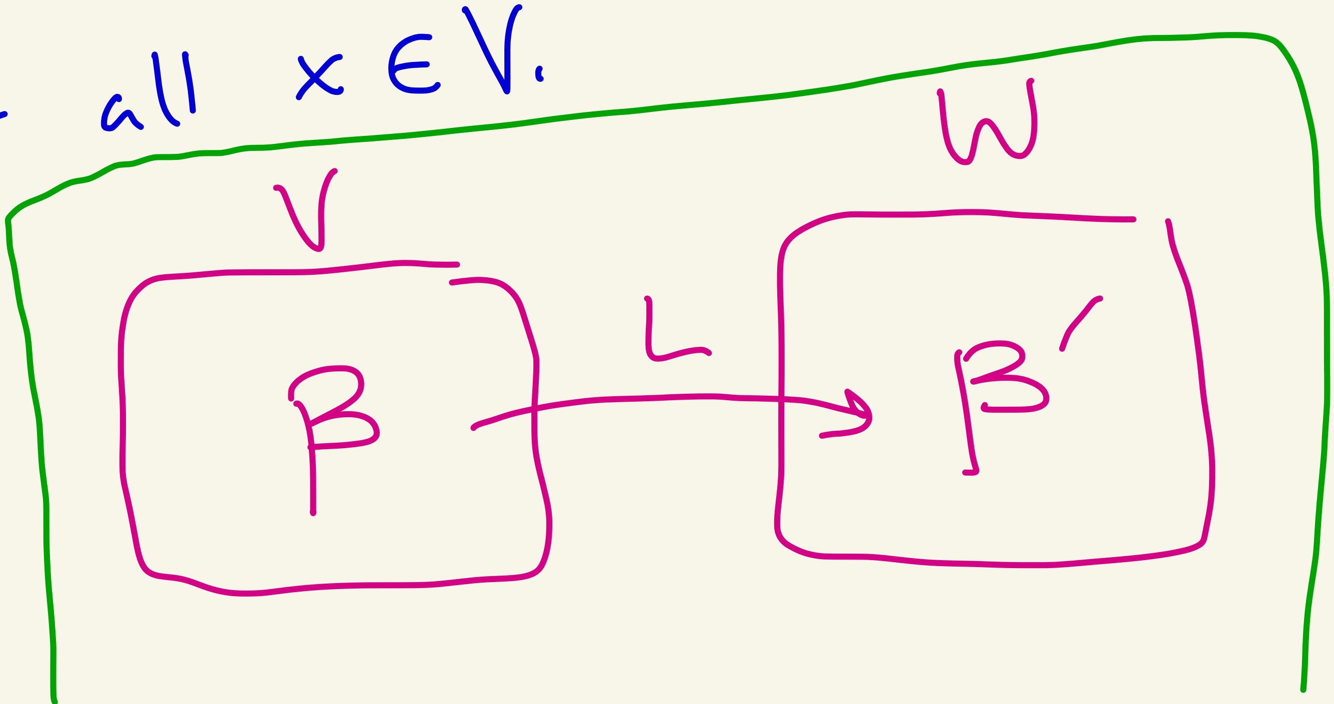
<u>Theorem:</u> Let V and W be finite-dimensional vector spaces over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V and $\beta' = [w_1, w_2, \ldots, w_m]$ be an ordered basis for W.

Let $L : V \to W$ be a linear transformation.

Then,

$$[L]_{\beta}^{\beta'} [x]_{\beta} = [L(x)]_{\beta'}$$

for all $x \in V$.



proof: Let $x \in V$.

pg 10

Since $\beta = [v_1, v_2, \ldots, v_n]$ is a basis for V, we may write

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Then, $[x]_\beta = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

Since $\beta' = [w_1, w_2, \ldots, w_m]$ is a basis for W we may write

$$L(v_1) = a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m$$
$$L(v_2) = a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_m$$
$$\vdots \qquad\qquad \vdots$$
$$L(v_n) = a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m$$

where $a_{ij} \in F$.

Thus,

$$[L]_{\beta}^{\beta'} = \left([L(v_1)]_{\beta'} \middle| [L(v_2)]_{\beta'} \middle| \cdots \middle| [L(v_n)]_{\beta'} \right)$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Now let's get $[L(x)]_{\beta'}$ and show that

$$[L]_{\beta}^{\beta'}[x]_{\beta} = [L(x)]_{\beta'}$$

To get $[L(x)]_{\beta'}$ we need to express $L(x)$ in terms of β'!

We have that

$$L(x) = L(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n)$$

$$\underset{\substack{L \text{ is} \\ \text{linear}}}{=} \alpha_1 L(v_1) + \alpha_2 L(v_2) + \dots + \alpha_n L(v_n)$$

$$= \alpha_1 \big(\overbrace{a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m}^{L(v_1)} \big)$$

$$+ \alpha_2 \big(\underbrace{a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m}_{L(v_2)} \big)$$

$$+ \dots +$$

$$+ \alpha_n \big(\underbrace{a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m}_{L(v_n)} \big)$$

$$=$$

$$= (\alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n}) w_1$$
$$+ (\alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n}) w_2$$
$$+ \dots +$$
$$+ (\alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots + \alpha_n a_{mn}) w_m$$

Thus,

$$[L(x)]_{\beta'} = \begin{pmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots \ \alpha_n a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$= [L]_{\beta}^{\beta'} [X]_{\beta} \quad \blacksquare$$

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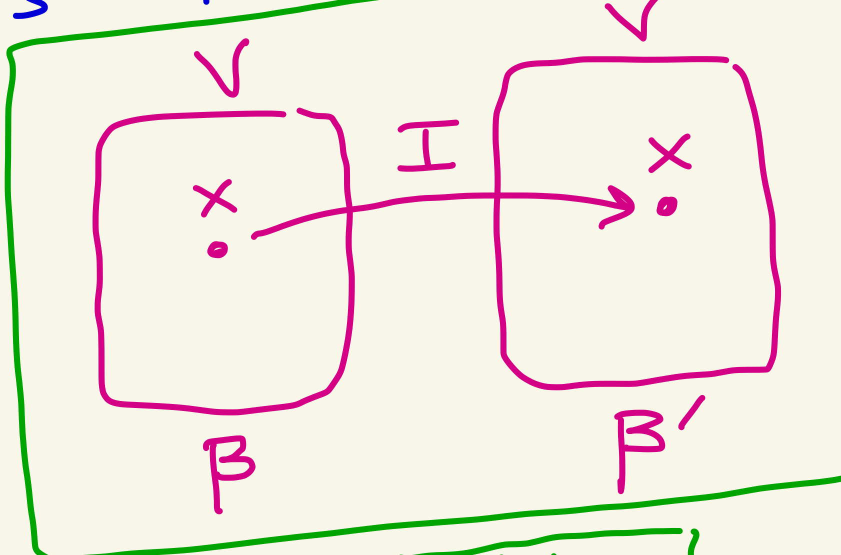
Last time we talked about $[L]_{\beta}^{\beta'}$ for a linear transformation L. We showed that

$$[L]_{\beta}^{\beta'}[x]_{\beta} = [L(x)]_{\beta'}$$

Today we will talk about a different matrix. It will be the matrix that converts one coordinate system to another.

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let β and β' be two ordered bases for V. Let $I: V \to V$ be the identity function, that is $I(x) = x$ for all $x \in V$.



Then,

$$[I]_{\beta}^{\beta'}[x]_{\beta} = [x]_{\beta'}$$

<u>proof:</u> We have that

$$[I]_{\beta}^{\beta'}[x]_{\beta} \underset{\text{thm from Weds or pg 1 today}}{=} [I(x)]_{\beta'} \overset{I(x)=x}{=} [x]_{\beta'}$$

$\blacksquare$

The matrix $[I]_{\beta}^{\beta'}$ is called the <u>change of basis matrix from β to β'.</u>

Ex: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$.

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ ← standard basis

and $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$ ← we used this basis last week

Lets calculate $[I]_{\beta}^{\beta'}$.

Recall
$I : \mathbb{R}^2 \to \mathbb{R}^2$
$I(v) = v$

We have that

$$\underbrace{I\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{plug } \beta \text{ into } I} = \underbrace{a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{express in terms of } \beta'}$$

$$I\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

This gives

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c-d \\ c+d \end{pmatrix}$$

This becomes

$$\boxed{\begin{aligned} 1 &= a - b \\ 0 &= a + b \end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} 0 &= c - d \\ 1 &= c + d \end{aligned}}$$

If you solve these you will get $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = \frac{1}{2}$, $d = \frac{1}{2}$.

Thus,

$$[I]_{\beta}^{\beta'} = \left(\left[I\begin{pmatrix}1\\0\end{pmatrix} \right]_{\beta'} \middle| \left[I\begin{pmatrix}0\\1\end{pmatrix} \right]_{\beta'} \right)$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Let's test this matrix.

Pick $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ ← random vector we picked

$$[v]_\beta = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\boxed{v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Then,

$$[v]_{\beta'} = \underbrace{[I]_\beta^{\beta'}}_{\substack{\text{this matrix turns } \beta\text{-coordinates} \\ \text{into } \beta'\text{-coordinates}}} [v]_\beta = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{1}{2}\right)(2) + \left(\frac{1}{2}\right)(5) \\ \left(-\frac{1}{2}\right)(2) + \left(\frac{1}{2}\right)(5) \end{pmatrix} = \begin{pmatrix} 7/2 \\ 3/2 \end{pmatrix}$$

Checking: $\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

$$\frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/2 - 3/2 \\ 7/2 + 3/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = v$$

What about $w = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

Then, $w = -3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So, $[w]_\beta = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

And,

$$[w]_{\beta'} = [I]_\beta^{\beta'} [w]_\beta$$

$$= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3/2 + 0 \\ 3/2 + 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} -3 \\ 0 \end{pmatrix} = w = -\frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Def: Let V be a finite-dimensional vector space over a field F. Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V.

So, $\dim(V) = n$. Define $\Phi : V \to F^n$ by $\Phi(x) = [x]_\beta$

[Note that Φ depends on the β that is chosen, so sometimes we will write Φ_β instead of just Φ]

We call Φ the <u>canonical isomorphism between V and F^n.</u>

Ex: $V = P_2(\mathbb{R})$, $F = \mathbb{R}$

Let $\beta = [1, x, x^2]$ standard basis

$\dim(P_2(\mathbb{R})) = 3$

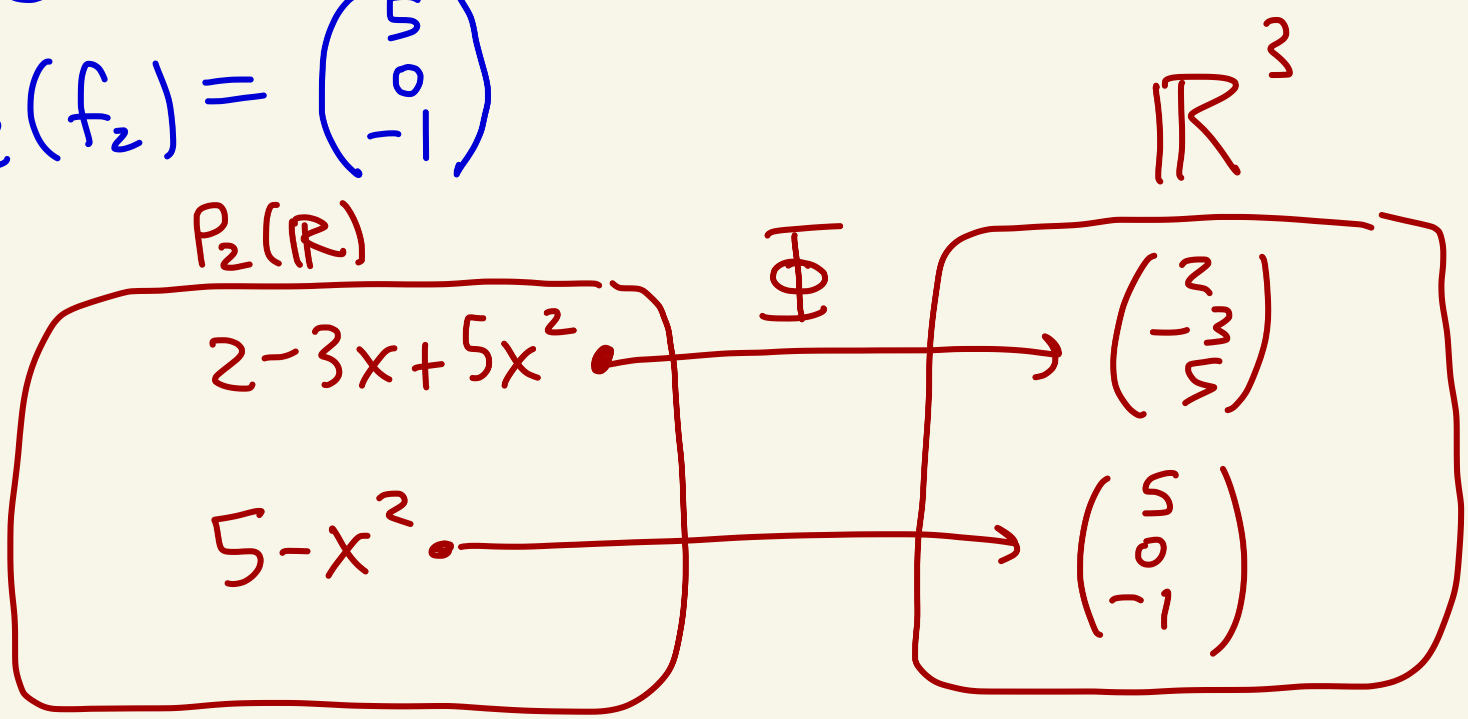
$\Phi : P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$

Let $f_1 = 2 - 3x + 5x^2$

$\Phi(f_1) = [f_1]_\beta = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$

Let $f_2 = 5 - x^2$

$\Phi(f_2) = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$



Let's show that Φ really is an isomorphism

Let V be a finite dimensional vector space over a field F.
Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V.
Pick the standard basis for F^n, ie

$$\beta' = \left[\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right]$$

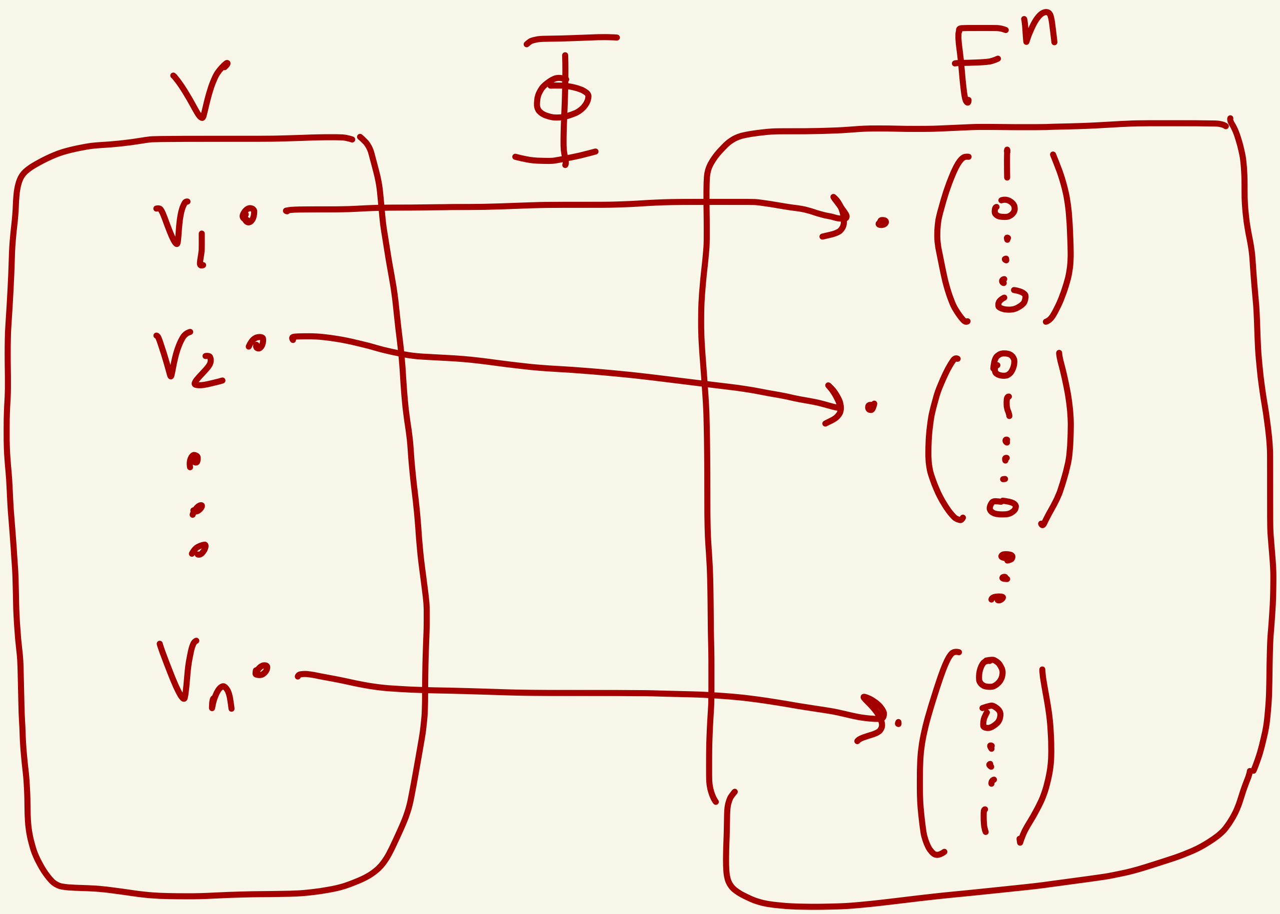
Then,

$$\Phi(v_1) = \Phi(1 \cdot v_1 + 0 v_2 + \cdots + 0 v_n) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Phi(v_2) = \Phi(0 \cdot v_1 + 1 \cdot v_2 + \cdots + 0 v_n) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\Phi(v_n) = \Phi(0 v_1 + 0 v_2 + \cdots + 1 v_n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$



Also, if $x \in V$ and $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

then

$$\Phi(x) = \Phi(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

This is the formula we had in a previous theorem about constructing linear transformations. It shows that Φ is a linear transformation.

The theorem also said that since

$$\{\Phi(v_1), \Phi(v_2), \dots, \Phi(v_n)\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

is a basis for F^n

We have that Φ is an isomorphism between V and F^n.

Commutative diagram

Let V and W be finite-dimensional vector spaces over a field F.
Let $L : V \to W$ be a linear transformation.
Let β be an ordered basis for V and γ be an ordered basis for W.
Let $n = \dim(V)$ and $m = \dim(W)$.

$$
\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & L(x) \arrow[d, mapsto] \\
{[x]_\beta} \arrow[r, mapsto] & {[L]_\beta^\gamma [x]_\beta = [L(x)]_\gamma}
\end{tikzcd}
$$

$$
\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "\Phi_\beta"', "\cong"] & W \arrow[d, "\Phi_\gamma", "\cong"'] \\
F^n \arrow[r, "{[L]_\beta^\gamma}"] & F^m
\end{tikzcd}
$$

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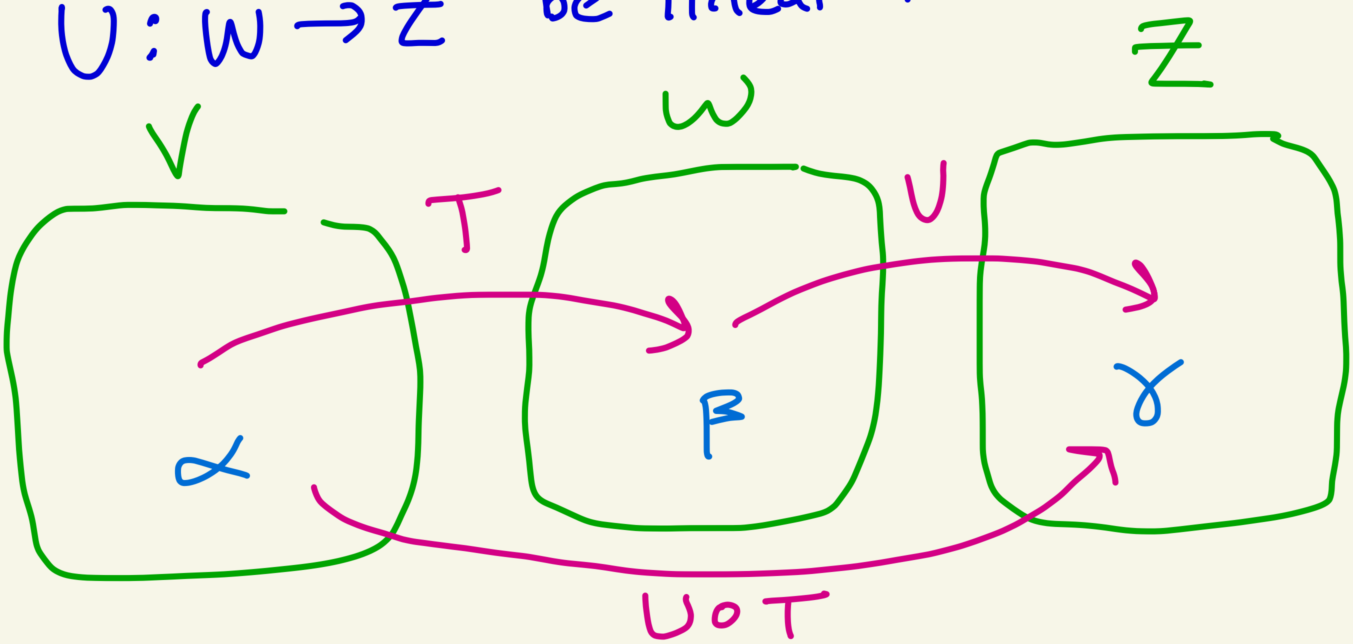
HW 4

③ Let V, W, Z be finite-dimensional vector spaces over a field F with ordered bases α, β, γ respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformations

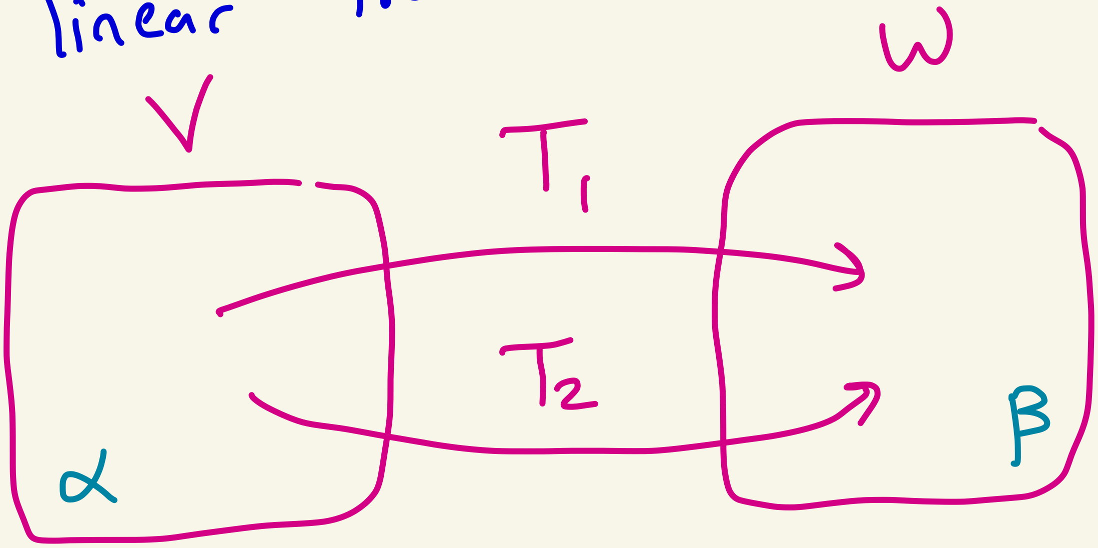


Then, $U \circ T : V \to Z$ is a linear transformation.

Also, $[U \circ T]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$

Proof: HW ▨

(4) Let V and W be finite-dimensional vector spaces over a field F. Let α and β be ordered bases for V and W. Let $T_1 : V \to W$ and $T_2 : V \to W$ be linear transformations.



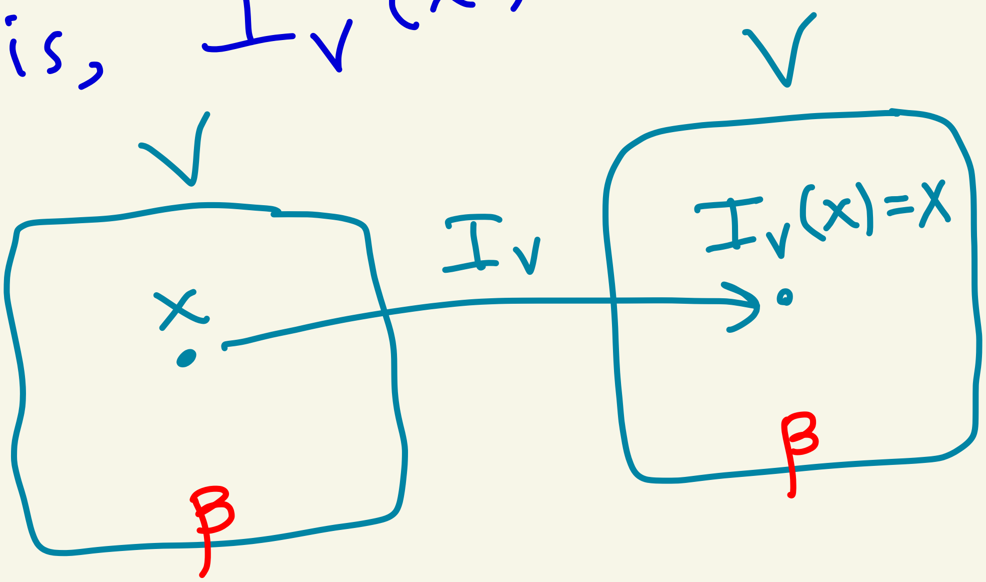
If $[T_1]_\alpha^\beta = [T_2]_\alpha^\beta$,

then $T_1 = T_2$.

HW 5

(2) Let V be a finite dimensional vector space over a field F. Let β be an ordered basis for V.

Let $I_V : V \to V$ be the identity linear transformation. That is, $I_V(x) = x$ for all $x \in V$.



Let $n = \dim(V)$.

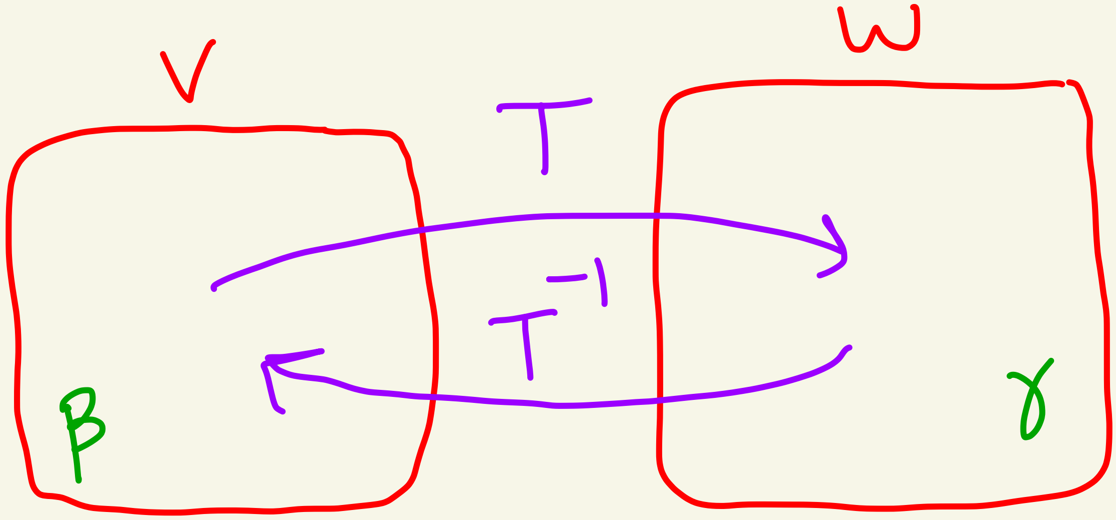
Then, $[I_V]_\beta = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

where I_n is the $n \times n$ identity matrix

Theorem: Let V and W be finite-dimensional vector spaces over a field F. Let $T: V \to W$ be a linear transformation. Let β and γ be ordered bases for V and W, respectively.

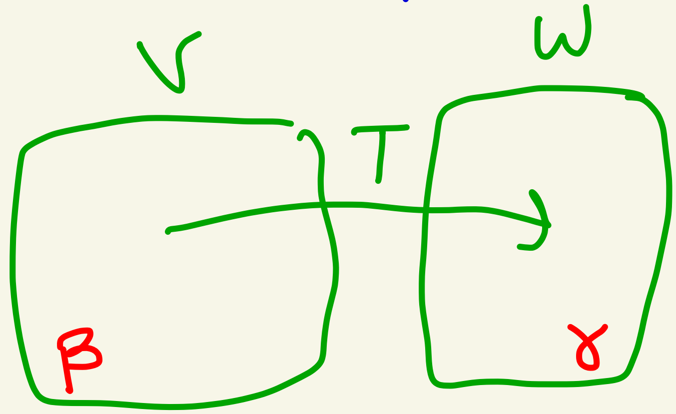
Then, T is an isomorphism (ie 1-1 and onto) iff $[T]_\beta^\gamma$ is invertible.

Furthermore, if this is the case then $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$



Proof:

($\Rightarrow$) Suppose T is an isomorphism.



Then T is one-to-one and onto, and from a theorem in class, $\dim(V) = \dim(W)$.

So, β and γ both have the same number of elements, lets say n elements each.

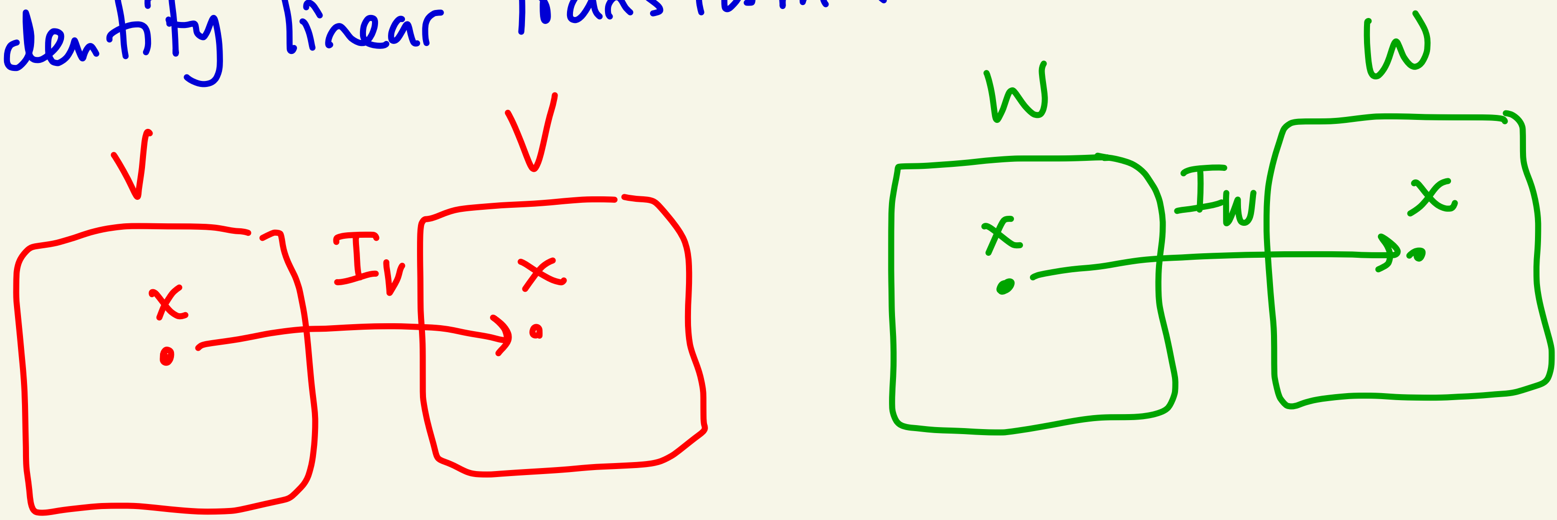
Then, $[T]_\beta^\gamma$ is $n \times n$.

# of elements in β is the # of columns, # of elements in γ is the # of rows

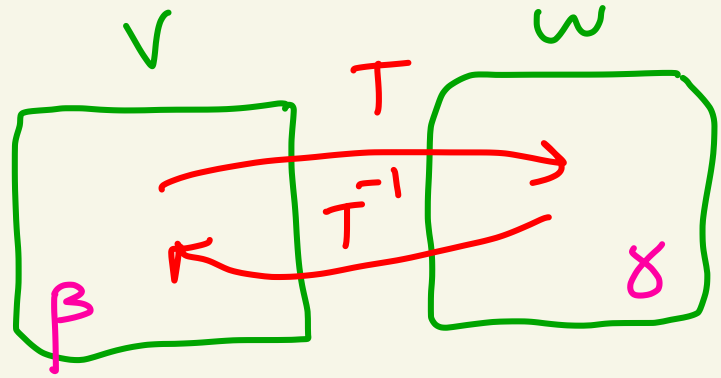
Let $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

be the $n \times n$ identity matrix.

Let $I_V : V \to V$ be the identity linear transformation and $I_W : W \to W$ be the identity linear transformation



Because T is an isomorphism, $T^{-1} : W \to V$ exists and is a linear transformation (we did this in class).



Then,

$$[T^{-1}]_{\gamma}^{\beta}\,[T]_{\beta}^{\gamma} \underset{\text{HW 4}}{=} [T^{-1}\circ T]_{\beta}^{\beta} \underset{T^{-1}\circ T = I_V}{=} [I_V]_{\beta} \underset{\text{HW 5}}{=} I_n$$

and

$$[T]_{\beta}^{\gamma}\,[T^{-1}]_{\gamma}^{\beta} \underset{\text{HW 4}}{=} [T\circ T^{-1}]_{\gamma}^{\gamma} \underset{T\circ T^{-1} = I_W}{=} [I_W]_{\gamma} \underset{\text{HW 5}}{=} I_n$$

Thus, $[T]_{\beta}^{\gamma}$ is invertible and

$$\left([T]_{\beta}^{\gamma}\right)^{-1} = [T^{-1}]_{\gamma}^{\beta}$$

($\Leftarrow$) Suppose that $[T]_\beta^\gamma$ is invertible.

We want to show that T is an isomorphism.

We will show that T^{-1} exists.

Since $[T]_\beta^\gamma$ is invertible it is a square matrix.

Let $A = [T]_\beta^\gamma$.

Suppose A is $n \times n$.

Then $\beta = [v_1, v_2, \dots, v_n]$ and $\gamma = [w_1, w_2, \dots, w_n]$

where $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$.

Let $B = A^{-1}$.

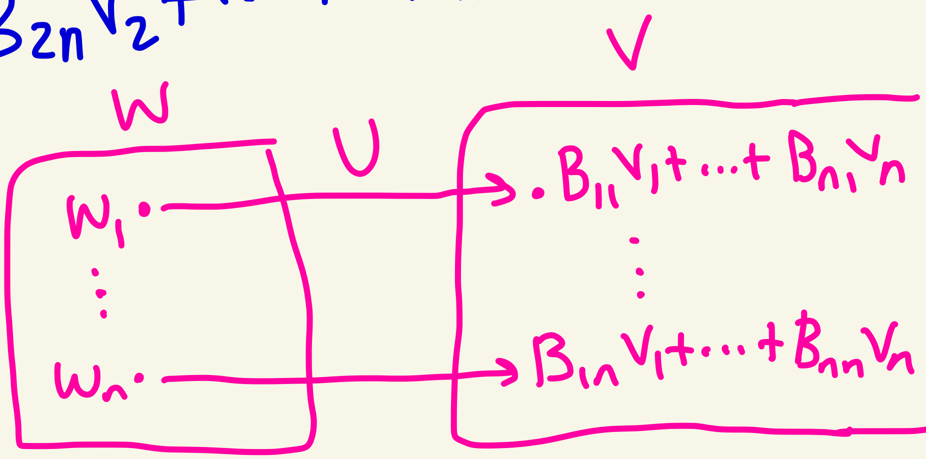
So, B is $n \times n$ also.

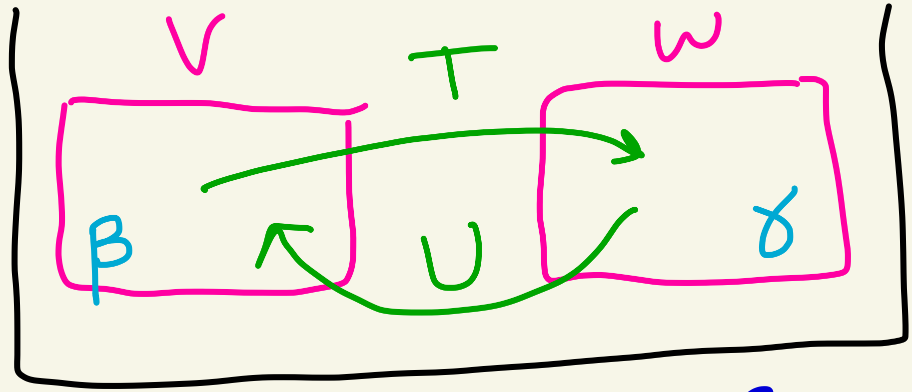
Let $B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}$

From a previous theorem in class we can construct a linear transformation $U : W \to V$ where

$$
\begin{aligned}
U(w_1) &= B_{11}v_1 + B_{21}v_2 + \cdots + B_{n1}v_n \\
U(w_2) &= B_{12}v_1 + B_{22}v_2 + \cdots + B_{n2}v_n \\
&\vdots \\
U(w_n) &= B_{1n}v_1 + B_{2n}v_2 + \cdots + B_{nn}v_n
\end{aligned}
$$

So, $[U]_\gamma^\beta = B$





Then,

$$[U \circ T]_\beta = [U \circ T]_\beta^\beta = [U]_\gamma^\beta [T]_\beta^\gamma$$

$\boxed{\text{HW } 4}$

$$= BA = A^{-1}A = I_n = [I_V]_\beta$$

Since $[U \circ T]_\beta = [I_V]_\beta$, by HW

$U \circ T = I_V$.

Similarly,

$$[T \circ U]_\gamma = [T]_\beta^\gamma [U]_\gamma^\beta = AB = AA^{-1} = I_n = [I_W]_\gamma$$

So, by HW $T \circ U = I_W$.

pg
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Since $U \circ T = I_V$

and $T \circ U = I_W$

we know that $U = T^{-1}$.

So, $T^{-1} : W \to V$ exists

and T is 1-1 and onto.

$\blacksquare$

Math 4570

11/1/21

Grading schemes

syllabus

test 1 - 1/3

test 2 - 1/3

final - 1/3

drop one

$\max\{\text{test } 1, \text{test } 2\}$ - 1/2

final - 1/2

I will pick the better of "syllabus" or "drop one" method for each student.

Test 2

Monday Nov 15

Covers HW 3 and HW 4

Corollary: Let V be a finite dimensional vector space over a field F. Let β and β' be ordered bases for V. Let $I: V \to V$ be the identity linear transformation $[I(x) = x$ for all $x \in V]$

Let $Q = [I]_{\beta}^{\beta'}$ be the change of basis matrix from β to β'.

Then:

1. Q is invertible and $Q^{-1} = [I]_{\beta'}^{\beta}$
2. If $T: V \to V$ is a linear transformation then

$$[T]_{\beta} = \underbrace{Q^{-1}[T]_{\beta'}Q}_{[I]_{\beta'}^{\beta}[T]_{\beta'}[I]_{\beta}^{\beta'}}$$

proof:

① I is invertible and $I^{-1} = I$.

$[I : V \to V \quad I(x) = x \text{ for all } x \in V]$

The theorem for Weds says that

$Q = [I]_{\beta}^{\beta'}$ is invertible and

$$Q^{-1} = \left([I]_{\beta}^{\beta'}\right)^{-1} = [I^{-1}]_{\beta'}^{\beta} = [I]_{\beta'}^{\beta}$$

② We have that

$$Q^{-1}[T]_{\beta'}Q = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[I]_{\beta}^{\beta'}$$

$$= [I]_{\beta'}^{\beta}[T \circ I]_{\beta}^{\beta'}$$

$$= [I]_{\beta'}^{\beta}[T]_{\beta}^{\beta'} = [I \circ T]_{\beta}^{\beta}$$

$$= [T]_{\beta}^{\beta} = [T]_{\beta} \quad \blacksquare$$

HW 4

$[U \circ T]_{\alpha}^{\gamma} = [U]_{\delta}^{\gamma}[T]_{\alpha}^{\delta}$

Def: Let A and B be $n \times n$ matrices with entries in a field F. We say that A and B are <u>similar</u> if there exists an $n \times n$ invertible matrix Q with entries from F where

$$B = Q^{-1} A Q$$

In the previous theorem we saw that $[T]_\beta$ and $[T]_{\beta'}$ are similar matrices.

Theorem: Let V be a finite-dimensional vector space over a field F. Let β be an ordered basis for V. Let $T: V \to V$ be a linear transformation.

Suppose $n = \dim(V)$.

If A is an $n \times n$ matrix with entries from F that is similar to $[T]_\beta$, then

$A = [T]_\gamma$ where γ is some ordered basis for V.

<u>Proof:</u> We have $n = \dim(V)$.

Then $\beta = [v_1, v_2, \ldots, v_n]$ where $v_1, v_2, \ldots, v_n \in V$.

Also, $[T]_\beta$ is $n \times n$.

Since A is similar to $[T]_\beta$ we know that there exists an invertible matrix Q that is $n \times n$ and has entries in F and

$$A = Q^{-1} [T]_\beta Q$$

Let Q_{ij} denote the entry in Q in row i and column j.

That is,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & & & \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix}$$

Define the vectors $w_1, w_2, \ldots, w_n$ as follows:

$$w_1 = Q_{11}v_1 + Q_{21}v_2 + \cdots + Q_{n1}v_n$$
$$w_2 = Q_{12}v_1 + Q_{22}v_2 + \cdots + Q_{n2}v_n$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$w_n = Q_{1n}v_1 + Q_{2n}v_2 + \cdots + Q_{nn}v_n$$

So, $$w_j = \sum_{i=1}^{n} Q_{ij} v_i \longleftarrow$$

this sum runs down the j-th column of Q

Let $\gamma = [w_1, w_2, \ldots, w_n]$

We will now show that γ is a basis for V.

Let's show γ is a linearly independent set

Suppose

$$c_1 w_1 + c_2 w_2 + \cdots + c_n w_n = \vec{0}$$

Where $c_1, c_2, \ldots, c_n \in F$.

Then,

$$c_1 \big(\overbrace{Q_{11} v_1 + Q_{21} v_2 + \cdots + Q_{n1} v_n}^{w_1} \big)$$
$$+ c_2 \big(\overbrace{Q_{12} v_1 + Q_{22} v_2 + \cdots + Q_{n2} v_n}^{w_2} \big)$$
$$+ \cdots +$$
$$+ c_n \big(\overbrace{Q_{1n} v_1 + Q_{2n} v_2 + \cdots + Q_{nn} v_n}^{w_n} \big)$$
$$= \vec{0}$$

Rearranging we get that

$$(c_1 Q_{11} + c_2 Q_{12} + \dots + c_n Q_{1n}) V_1$$
$$+ (c_1 Q_{21} + c_2 Q_{22} + \dots + c_n Q_{2n}) V_2$$
$$+ \dots +$$
$$+ (c_1 Q_{n1} + c_2 Q_{n2} + \dots + c_n Q_{nn}) V_n = \vec{0}$$

Since $\beta = [V_1, V_2, \dots, V_n]$ is a linearly independent set we have that

$$c_1 Q_{11} + c_2 Q_{12} + \dots + c_n Q_{1n} = 0$$
$$c_1 Q_{21} + c_2 Q_{22} + \dots + c_n Q_{2n} = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$c_1 Q_{n1} + c_2 Q_{n2} + \dots + c_n Q_{nn} = 0$$

Rewriting this as a matrix equation we get that

$$\begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus,

$$Q \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since Q is invertible, Q^{-1} exists and

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{Q^{-1} Q}_{I_n} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = Q^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, $c_1 = 0, c_2 = 0, \dots, c_n = 0.$

Thus $\gamma = [w_1, w_2, \ldots, w_n]$ is a linearly independent set.

Since γ contains n vectors and $\dim(V) = n$, we know γ is a basis for V.

By the definition of w_j, $Q = [I]_\gamma^\beta$

Why? $W_j = \sum_{i=1}^{n} Q_{ij} V_i$

$$I(w_j) = w_j = \sum_{i=1}^{n} Q_{ij} V_i$$

So the jth column of $[I]_\gamma^\beta$ is

$\begin{pmatrix} Q_{1j} \\ Q_{2j} \\ \vdots \\ Q_{nj} \end{pmatrix}$ which is the same as the jth column of Q.

Thus,

$$Q^{-1} = \left([I]_\gamma^\beta\right)^{-1} = [I^{-1}]_\beta^\gamma = [I]_\beta^\gamma$$

So,

$$A = Q^{-1}[T]_\beta Q$$
$$= [I]_\beta^\gamma [T]_\beta [I]_\gamma^\beta$$
$$= [T]_\gamma$$

$\blacksquare$

Math 4570

11/3/21

(In between HW 4/HW 5 Topic) Pg 1

$\boxed{\text{Review of determinants}}$

<u>Def:</u> Let A be an $n \times n$ matrix with coefficients from a field F. Let $1 \le i \le n$ and $1 \le j \le n$. The matrix A_{ij} is defined to be the $(n-1)\times(n-1)$ matrix obtained by removing the i-th row and j-th column of A.

<u>Ex:</u> $A = \begin{pmatrix} 1 & 5 & 7 \\ 0 & -1 & 2 \\ 3 & \pi & 10 \end{pmatrix}$

$$A_{32} = \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & -1 \\ 3 & \pi \end{pmatrix}$$

$$\begin{pmatrix} 1 & \cancel{5} & 7 \\ 0 & \cancel{-1} & 2 \\ \cancel{3} & \cancel{\pi} & \cancel{10} \end{pmatrix} \qquad \begin{pmatrix} \cancel{1} & \cancel{5} & \cancel{7} \\ 0 & -1 & \cancel{2} \\ 3 & \pi & \cancel{10} \end{pmatrix}$$

Def: Let A be an $n \times n$ matrix with coefficients from a field F. Let a_{ij} be the entry in the i-th row and j-th column of A.

① If $n=1$ and $A=(a_{11})$ then define $\det(A)=a_{11}$

② If $n=2$ and $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then define $\det(A)=a_{11}a_{22}-a_{12}a_{21}$

③ If $n \geq 3$, then define $\det(A)$ as follows. Pick a column j where $1 \leq j \leq n$. Define

$$\det(A) = \underbrace{\sum_{i=1}^{n}}_{\substack{\text{sum over rows } i \\ \text{column } j \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(\underbrace{A_{ij}}_{\substack{\text{this is} \\ \text{an} \\ (n-1)\times(n-1) \\ \text{matrix}}})$$

This is called <u>the expansion of the determinant along the j-th column</u>

Note: One can also expand along a row in part 3 of the previous definition. To do this, pick a row i with $1 \le i \le n$ and replace step 3 with

$$\det(A) = \underbrace{\sum_{j=1}^{n}}_{\substack{\text{sum over the columns } j \\ \text{row } i \text{ is fixed}}} (-1)^{i+j} a_{ij} \det(A_{ij})$$

This is called the expansion of the determinant along row i.

Fact: This def is well-defined. One can show that the final result is the same no matter what row or column you expand on in step 3.

Notation: One can also use bars instead of det. For example,

$$\det\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ \pi & 5 & 7 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ \pi & 5 & 7 \end{vmatrix}$$

Ex: $\det(10) = 10$

Ex: $\det\begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} = (-1)(7) - (0)(3) = -7$

Ex: Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

Expand on row $i = 1$

$$\begin{pmatrix} \boxed{3 \quad 1 \quad 0} \\ -2 \quad -4 \quad 3 \\ 5 \quad 4 \quad -2 \end{pmatrix}$$

$$\det(A) = \underbrace{(-1)^{1+1} a_{11} \det(A_{11})}_{i=1,\ j=1} + \underbrace{(-1)^{1+2} a_{12} \det(A_{12})}_{i=1,\ j=2}$$

$$+ \underbrace{(-1)^{1+3} a_{13} \det(A_{13})}_{i=1,\ j=3}$$

$$= \underbrace{(1)(3)\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix}} + \underbrace{(-1)(1)\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}} + \underbrace{(1)(0)\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}}$$

a_{11} $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$ a_{12} $\quad$ a_{13} $\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

$$= (3)\left[(-4)(-2) - (3)(4)\right] + (-1)\left[(-2)(-2) - (3)(5)\right] + 0$$

$$= (3)(-4) - [-11] = -1$$

pg 5

So,

$$\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} = -1$$

Useful tool

$$\begin{pmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \leftarrow \boxed{\text{put + in top left and alternate +/-}}$$

$$\left(\begin{array}{ccc} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{array}\right) \qquad \left(\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}\right)$$

<u>Ex:</u> Let $A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$

Lets expand on column 2.

$$\det \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$= \underbrace{\overset{-}{(-1)}(1) \begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix}}_{\begin{pmatrix} \cancel{3} & 1 & \cancel{0} \\ -2 & \cancel{-4} & 3 \\ 5 & \cancel{4} & -2 \end{pmatrix}\ \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}} + \underbrace{\overset{+}{(1)}(-4) \begin{vmatrix} 3 & 0 \\ 5 & -2 \end{vmatrix}}_{\begin{pmatrix} 3 & \cancel{1} & 0 \\ \cancel{-2} & -4 & \cancel{3} \\ 5 & \cancel{4} & -2 \end{pmatrix}\ \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}} + \underbrace{\overset{-}{(-1)}(4) \begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix}}_{\begin{pmatrix} 3 & \cancel{1} & 0 \\ -2 & \cancel{-4} & 3 \\ \cancel{5} & 4 & \cancel{-2} \end{pmatrix}\ \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$$

$$= (-1)\left[4 - 15\right] - 4\left[-6 - 0\right] - 4\left[9 - 0\right]$$

$$= (-1)(-11) + 24 - 36 = 35 - 36 = -1$$

Ex: Let $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}$

Let's expand on column $j=1$

$$\det(A) = \underbrace{\overset{+}{(1)}(1)\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}}$$

$$\underbrace{\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}}_{\text{gives the } (-1)^{i+j}}$$

$$\underbrace{\overset{-}{(-1)}(0)\begin{vmatrix} 2 & -1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}} + \underbrace{\overset{+}{(1)}(0)\begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \\ 5 & 4 & -2 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}} + \underbrace{\overset{-}{(-1)}(0)\begin{vmatrix} 2 & -1 & 0 \\ 3 & 1 & 0 \\ -2 & -4 & 3 \end{vmatrix}}_{\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & 5 & 4 & -2 \end{pmatrix}}$$

$$= \underbrace{\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}}_{\text{calculated previously}} + 0 + 0 + 0$$

$$= -1$$

Properties of the determinant

Let F be a field and A and B be $n \times n$ matrices with entries from F. Then:

① $\det(AB) = \det(A)\det(B)$

② A is invertible iff $\det(A) \neq 0$

If A is invertible then

$$\det(A^{-1}) = (\det(A))^{-1}$$

Math 4570

11/8/21

Test 2

Monday Nov 15

Same as before. No class that day. Test will appear on canvas at 5am on the 15th until 12 noon on Tuesday. Pick your 2.5 hour window in that time.

- I emailed the class a study guide on Friday for test 2.

I put it on the website for the class also.

HW 3 $\boxed{F=\mathbb{R}}$

② Let $T: M_{2,3}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ be given by

$$T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix}$$

<u>Show T is a linear transformation</u>

Let $\alpha, \beta \in \mathbb{R}$ and $x, y \in M_{2,3}(\mathbb{R})$.

Then, $x = \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{pmatrix}$

So,

$T(\alpha x + \beta y) =$

$$= T\begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 & \alpha c_1 + \beta c_2 \\ \alpha d_1 + \beta d_2 & \alpha e_1 + \beta e_2 & \alpha f_1 + \beta f_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 2(\alpha a_1 + \beta a_2) - (\alpha b_1 + \beta b_2) , & (\alpha c_1 + \beta c_2) + 2(\alpha d_1 + \beta d_2) \\ 0 , & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\alpha a_1 - \alpha b_1 & \alpha c_1 + 2\alpha d_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2\beta a_2 - \beta b_2 & \beta c_2 + 2\beta d_2 \\ 0 & 0 \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 2a_1 - b_1 & c_1 + 2d_1 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 2a_2 - b_2 & c_2 + 2d_2 \\ 0 & 0 \end{pmatrix}$$

$$= \alpha T \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \beta T \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$= \alpha T(x) + \beta T(y)$$

Thus, T is a linear transformation.

(i) Compute the nullspace of T / (iii) Compute the nullity of T:

pg 4

$$N(T) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; \begin{pmatrix} 2a-b & c+2d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; \begin{matrix} 2a-b=0 \\ c+2d=0 \end{matrix} \right\}$$

Note: e, f can be any real numbers

This is a system

$$\boxed{\begin{aligned} 2a-b &= 0 \\ c+2d &= 0 \end{aligned}}$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\boxed{\begin{aligned} a - \tfrac{1}{2}b &= 0 \\ c+2d &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} a - \tfrac{1}{2}b \quad &= 0 \\ c + 2d &= 0 \end{aligned}}$$

leading variables
a, c
free variables
b, d

Let
$b = t$
$d = s$ where $t, s \in \mathbb{R}$

Then

$$\boxed{\begin{aligned} a &= \tfrac{1}{2}b \\ c &= -2d \end{aligned}} \leftarrow \boxed{\begin{aligned} a &= \tfrac{1}{2}t \\ c &= -2s \end{aligned}}$$

Solutions:

$a = \frac{1}{2}t$
$b = t$
$c = -2s$
$d = s$

where $s, t \in \mathbb{R}$

Thus,

$$\boxed{\begin{aligned} N(T) &= \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \middle| \begin{aligned} 2a-b&=0 \\ c+2d&=0 \end{aligned} \right\} \\ &= \left\{ \begin{pmatrix} \tfrac{1}{2}t & t & -2s \\ s & e & f \end{pmatrix} \middle| s,t,e,f \in \mathbb{R} \right\} \end{aligned}}$$

Note that

$$\begin{pmatrix} \frac{1}{2}t & t & -2s \\ s & e & f \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}t & t & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -2s \\ s & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f \end{pmatrix}$$

$$= t \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + s \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$$

$$+ e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note:

$$T\begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2(\frac{1}{2})-1 & 0+2(0) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2(0)-0 & -2+2(1) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2(0)-0 & 0-2(0) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2(0)-0 & 0-2(0) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, $\begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix},$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

are in $N(T)$ and they span $N(T)$ by page 6.

pg 8

Let's show that

$\begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are linearly independent.

Suppose

$$\alpha_1 \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then,

$$\begin{pmatrix} \frac{1}{2}\alpha_1 & \boxed{\alpha_1} & -2\alpha_2 \\ \boxed{\alpha_2} & \boxed{\alpha_3} & \boxed{\alpha_4} \end{pmatrix} = \begin{pmatrix} 0 & \boxed{0} & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} \end{pmatrix}$$

So, $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$.

Thus, the above 4 matrices are lin. ind. and thus are a basis for $N(T)$. So, nullity$(T) = \dim(N(T)) = 4$.

(ii)/(iv)/(vi) Compute the range and rank of T; is T onto?

Since $T: \underbrace{M_{2,3}(\mathbb{R})}_{\text{dimension is } 6} \to M_{2,2}(\mathbb{R})$,

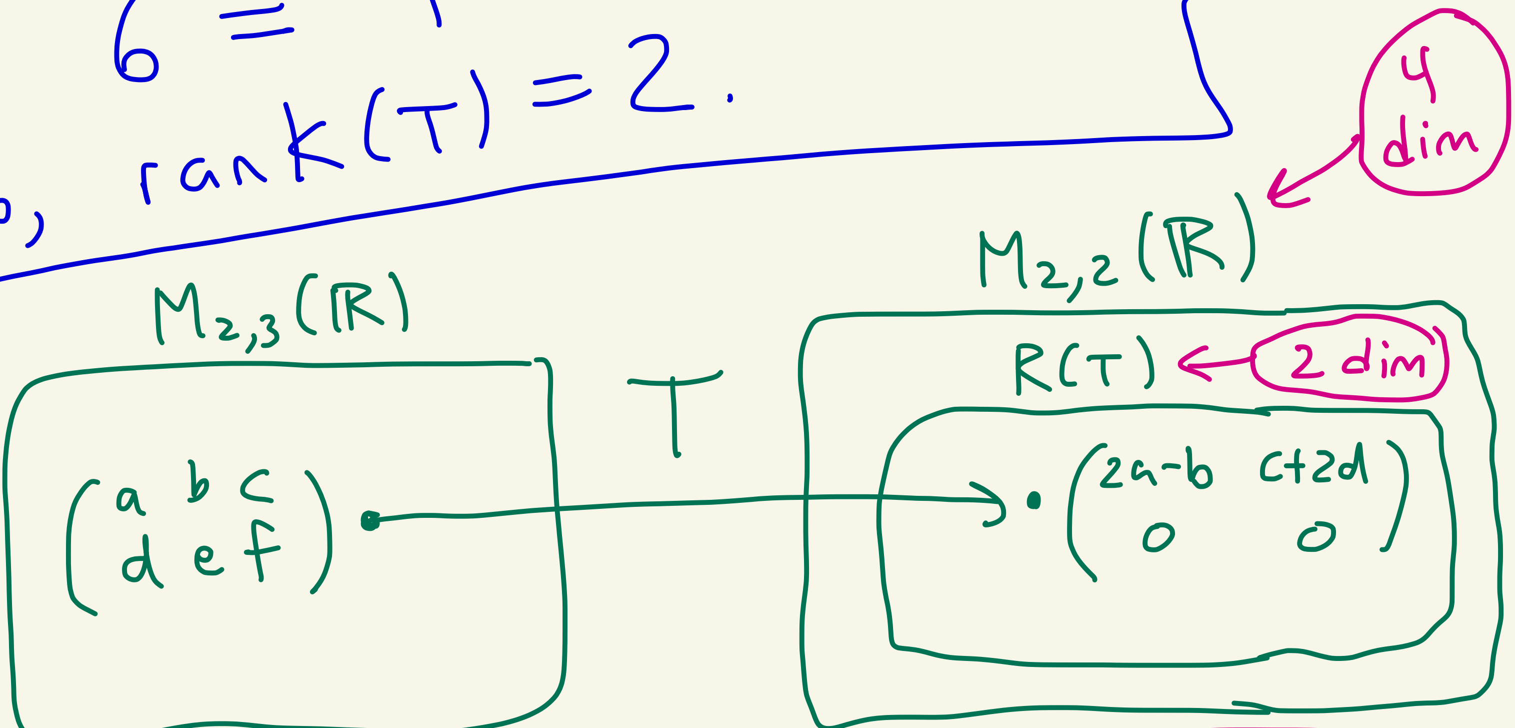
we have from rank/nullity theorem that

$$\dim(M_{2,3}(\mathbb{R})) = \underbrace{\dim(N(T))}_{\text{nullity}(T)} + \underbrace{\dim(R(T))}_{\text{rank}(T)}$$

Thus,

$$6 = 4 + \text{rank}(T)$$

So, $\text{rank}(T) = 2$.



R(T) = range of T

Since the range of T, ie $R(T)$, is 2-dimensional and $M_{2,2}(\mathbb{R})$ is 4-dimensional, T is not onto.

In the HW solutions we show

$$R(T) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

(v) Is T one-to-one?

By HW 3 #6, since $\text{nullity}(T) = 4 \neq 0$ we have that T is not one-to-one.

$\blacksquare$

HW 3 #6

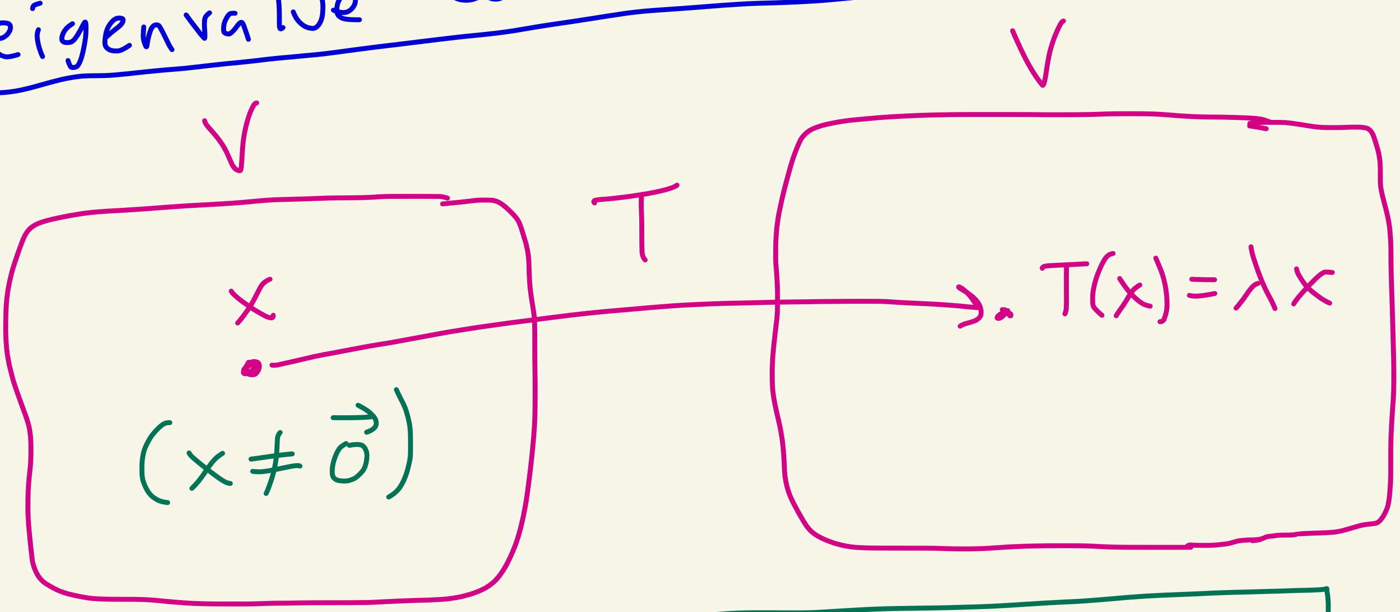
$T: V \to W$
T is linear trans.

T is 1-1 iff $N(T) = \{\vec{0}_V\}$ iff $\text{nullity}(T) = 0$

HW 5 Topic

Eigenvalues, Eigenvectors, and Diagonalization

Def: Let V be a vector space over a field F. Let $T: V \to V$ be a linear transformation. If $x \in V$ with $x \neq \vec{0}$ and $T(x) = \lambda x$ where $\lambda \in F$, then we call x an <u>eigenvector</u> of T and λ the <u>eigenvalue corresponding to x</u>.



$\boxed{\lambda = 0 \text{ is okay}}$

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$$

You can check that T is a lin. trans.

We have that

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+3(-1) \\ 4(1)+2(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So, $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = -2$ [because $T(x) = -2x$]

Also,

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3+3(4) \\ 4(3)+2(4) \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So, $y = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 5$ [because $T(y) = 5y$]

Math 4570

11/10/21

$\boxed{\text{Test 2 on Monday}}$

- See notes from previous class and study guide in email (also on website)

Ex: Let

$V = P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

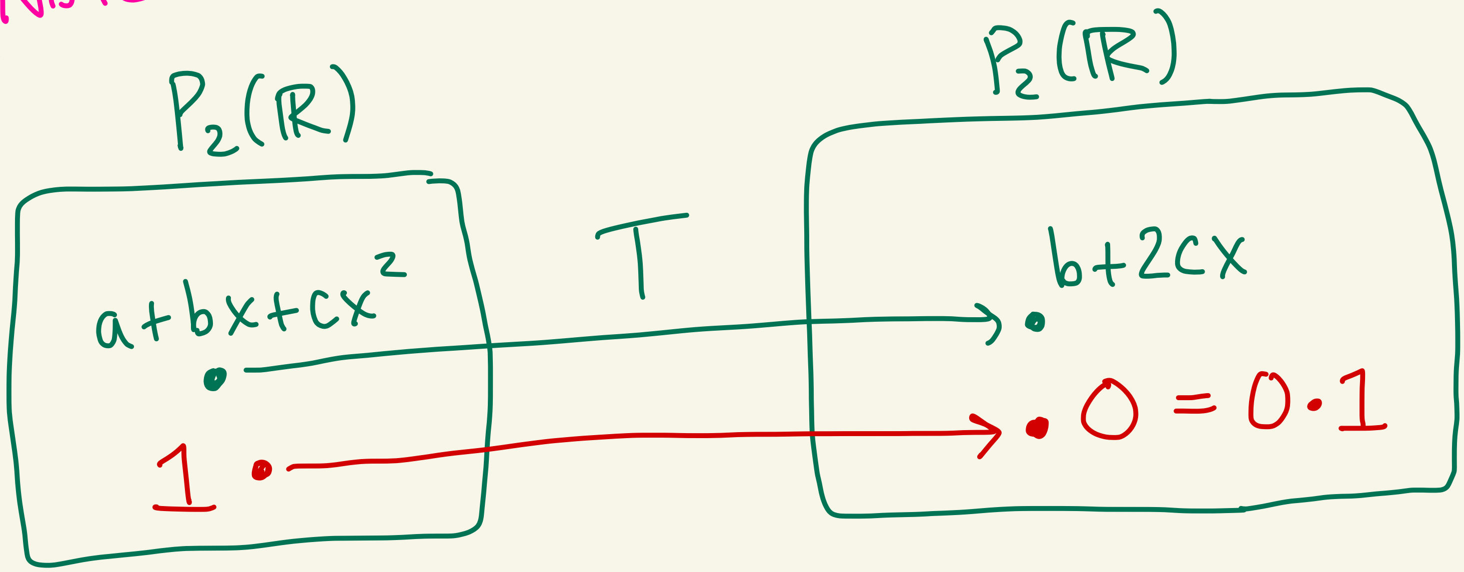
$F = \mathbb{R}$

$T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$T(a + bx + cx^2) = b + 2cx$

You can check this is a linear transformation

[Note that $T(f) = f'$]



Note that

$$T(1) = 0 = 0 \cdot 1$$

So, 1 is an eigenvector with eigenvalue $\lambda = 0$.

__Def:__ Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.
We say that T is __diagonalizable__ if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

__Recall:__ A diagonal matrix has the form
$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$

We saw on Monday that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are eigenvectors for T.

You can check that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are linearly independent and thus since there are two of them and $\dim(\mathbb{R}^2) = 2$ they form a basis for $\mathbb{R}^2$.

Let $\beta = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]$.

Let's compute $[T]_\beta$. $\boxed{[T]_\beta^\beta}$

$$\underbrace{T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}}_{} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

plug β into T write answer in terms of β

Thus, $[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

So, T is diagonalizable.

Why is this useful?

Let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. We know $\beta = [v_1, v_2]$ is a basis for $\mathbb{R}^2$.

Given any $x \in \mathbb{R}^2$ we can write $x = c_1 v_1 + c_2 v_2$. Then,

$$
\begin{aligned}
T(x) &= T(c_1 v_1 + c_2 v_2) \\
&\stackrel{\downarrow}{=} c_1 T(v_1) + c_2 T(v_2) \\
&\stackrel{\downarrow}{=} c_1(-2v_1) + c_2(5v_2) \\
&= -2c_1 v_1 + 5c_2 v_2
\end{aligned}
$$

T is linear

$T(v_1) = -2v_1$
$T(v_2) = 5v_2$

In matrix notation we have

$$[T(x)]_\beta = [T]_\beta [x]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2c_1 \\ 5c_2 \end{pmatrix}$$

Theorem: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

T is diagonalizable iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V consisting of eigenvectors of T.

Moreover, if this is the case then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where λ_i is the eigenvalue corresponding to v_i.

proof: T is diagonalizable

iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

ie $[T]_\beta$ is diagonal

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$

iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V such that

$$T(v_1) = \lambda_1 v_1 + 0v_2 + 0v_3 + \cdots + 0v_n$$
$$T(v_2) = 0v_1 + \lambda_2 v_2 + 0v_3 + \cdots + 0v_n$$
$$T(v_3) = 0v_1 + 0v_2 + \lambda_3 v_3 + \cdots + 0v_n$$
$$\vdots$$
$$T(v_n) = 0v_1 + 0v_2 + 0v_3 + \cdots + \lambda_n v_n$$

iff

iff there exists an ordered basis
$\beta = [v_1, v_2, \ldots, v_n]$ of V
consisting of eigenvectors of
T where $T(v_i) = \lambda_i v_i$
[So each λ_i is an eigenvalue
for v_i]. $\blacksquare$

Why is this useful?
Let $T: V \to V$ be a linear transformation
and $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered
basis of eigenvectors with eigenvalues λ_i.
Let $x \in V$.
Express $x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.
So, $T(x) = T(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)$
$= c_1 T(v_1) + c_2 T(v_2) + \ldots + c_n T(v_n)$
$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n$

T linear

$T(v_i) = \lambda_i v_i$

Let's learn how to find the eigenvalues and eigenvectors

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Let β and γ be ordered bases for V. Then,

$$\det([T]_\beta) = \det([T]_\gamma)$$

$I: V \to V$
$I(x) = x$
identity transformation

<u>proof:</u> [HW 5 #4] We have that

$$\det([T]_\beta) = \det\left([I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma\right)$$

$$= \det([I]_\gamma^\beta)\det([T]_\gamma)\det([I]_\beta^\gamma)$$

$$= \det([T]_\gamma)\det([I]_\gamma^\beta)\det([I]_\beta^\gamma)$$

$$= \det([T]_\gamma)\det\left([I]_\gamma^\beta [I]_\beta^\gamma\right)$$

$\det(AB) = \det(A)\det(B)$

$=$

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$$= \det([T]_\gamma)\det\Big(\underbrace{[I]_\gamma^\beta [I]_\beta^\gamma}\Big)$$

$$[I]_\gamma^\beta = \left([I]_\beta^\gamma\right)^{-1}$$

these are matrices $n \times n$

$$= \det([T]_\gamma)\det(I_n)$$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$n = \#$ of elements in β and γ

$$= \det([T]_\gamma) \cdot 1$$

$$= \det([T]_\gamma).$$

$\blacksquare$

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The theorem from last week showing $\det([T]_\beta) = \det([T]_\gamma)$ makes the next definition well-defined.

<u>Def:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. The <u>determinant</u> of T is defined to be

$$\det(T) = \det([T]_\beta)$$

where β is any ordered basis for V.

<u>Ex:</u> Recall

$P_2(\mathbb{R}) = \{a+bx+cx^2 \mid a,b,c \in \mathbb{R}\}$

Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$

be given by $T(a+bx+cx^2) = b+2cx$

T is a linear transformation.

Let's calculate $\det(T)$.

Let's pick $\beta = [1, x, x^2]$

(ie the standard basis)

$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

Thus, $[T]_\beta = [T]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Then,

$$\det(T) = \det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

expand on column 1

If a matrix has a row or column of zeros, then its determinant is zero

We will need the following:

Let V be a finite-dimensional vector space over a field F and $T: V \to V$ be a linear transformation. T is 1-1 iff $\det(T) \neq 0$.

<u>proof:</u> By HW 3 #6(b), since $T: V \to V$ we know T is 1-1 iff T is onto. By HW 5 #5(a), $\det(T) \neq 0$ iff T is 1-1 and onto. ∎

Theorem: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

Then, $\underbrace{\text{the following are equivalent}}_{\text{TFAE}}$:

① There exists an eigenvector $x \in V$, $x \neq \vec{0}$, of T with eigenvalue λ.

② $\det(T - \lambda I) = 0$

③ $N(T - \lambda I) \neq \{\vec{0}\}$

$T - \lambda I : V \to V$

$(T - \lambda I)(x)$

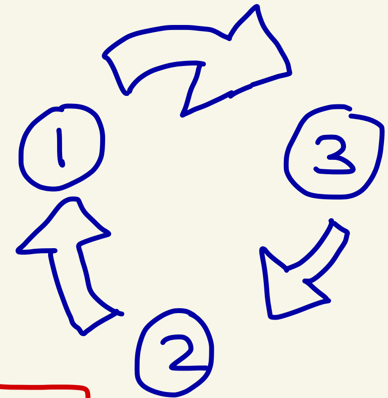
$= T(x) - \lambda I(x) = T(x) - \lambda x$

$I: V \to V$ is the identity transformation

TFAE means if one of ①, ②, or ③ is true then they are all true

Proof:

We will prove this like this



Proof that 1 $\Rightarrow$ 3 :

Suppose 1 is true. That is, there exists $x \in V$, $x \neq \vec{0}$, where $T(x) = \lambda x$ and $\lambda \in F$.

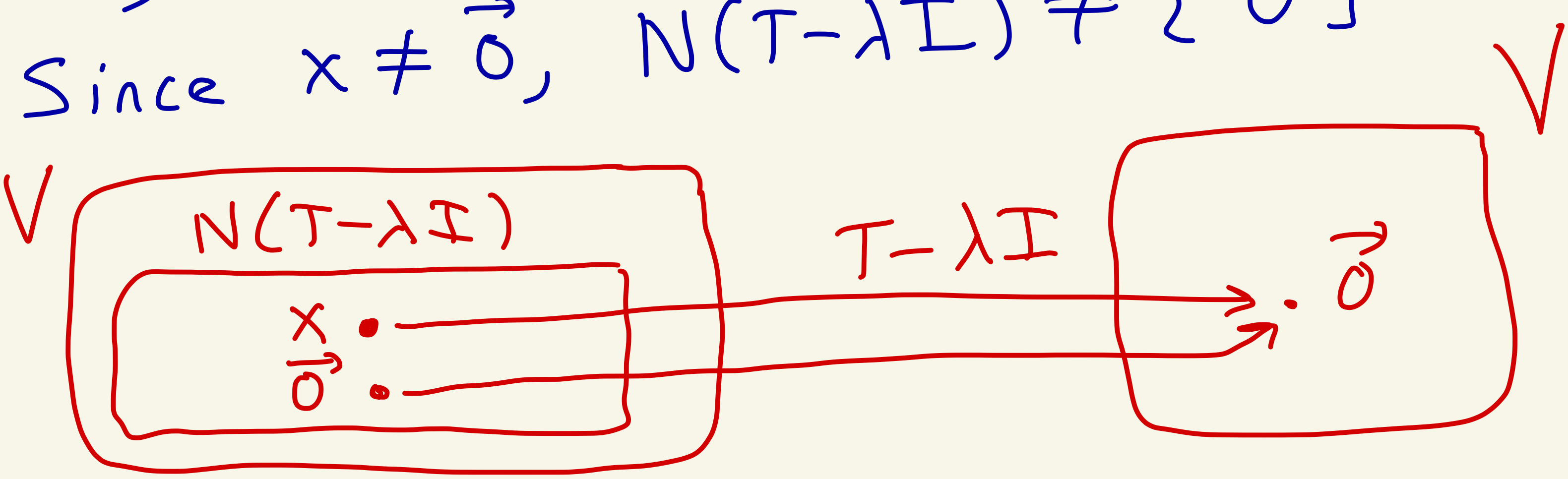
Then, $T(x) = \lambda I(x)$ $\quad \boxed{I(x) = x}$

So, $T(x) - \lambda I(x) = \vec{0}$.

Thus, $(T - \lambda I)(x) = \vec{0}$.

So, $x \in N(T - \lambda I)$.

Since $x \neq \vec{0}$, $N(T - \lambda I) \neq \{\vec{0}\}$



proof that ③ $\Rightarrow$ ②:

Suppose ③ is true, that is $N(T-\lambda I) \neq \{\vec{0}\}$ for some $\lambda \in F$.

Recall that $\vec{0} \in N(T-\lambda I)$ because $T-\lambda I$ is a linear transformation and so by HW 3 #1(a), $(T-\lambda I)(\vec{0}) = \vec{0}$.

Since $N(T-\lambda I) \neq \{\vec{0}\}$ there exists $x \in V$ with $x \neq \vec{0}$ and $x \in N(T-\lambda I)$.

Then, $(T-\lambda I)(x) = \vec{0}$.

Thus, $(T-\lambda I)(x) = \vec{0} = (T-\lambda I)(\vec{0})$.

Since $x \neq \vec{0}$ this shows that $T-\lambda I$ is not one-to-one.

By our earlier discussion, $\det(T-\lambda I) = 0$.

proof that ② $\Rightarrow$ ① :

Suppose ② is true, that is
$\det(T-\lambda I) = 0$ for some $\lambda \in F$.

By our previous discussion $T-\lambda I$ is not one-to-one.

This will lead to $N(T-\lambda I) \neq \{\vec{0}\}$.

Why?

Since $T-\lambda I$ is not one-to-one there exists x_1, x_2 with $x_1 \neq x_2$ and $(T-\lambda I)(x_1) = (T-\lambda I)(x_2)$.

Then, $(T-\lambda I)(x_1) - (T-\lambda I)(x_2) = \vec{0}$

Since $T-\lambda I$ is a linear transformation,
$(T-\lambda I)(x_1 - x_2) = \vec{0}$

Thus, $x_1 - x_2 \in N(T-\lambda I)$ and since $x_1 \neq x_2$ we have $x_1 - x_2 \neq \vec{0}$.

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Let $x = x_1 - x_2$.

Then, $x \neq \vec{0}$ and $(T - \lambda I)(x) = \vec{0}$.

So, $T(x) - \lambda I(x) = \vec{0}$.

Thus, $T(x) = \lambda I(x)$ $\quad I(x) = x$

Hence, $T(x) = \lambda x$

So, $x \neq \vec{0}$ is an eigenvector of T with eigenvalue λ.

$\blacksquare$

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Theorem: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Let β be an ordered basis for V.

Then,

$$\det(T - \lambda I) = \det\left([T]_\beta - \lambda I_n\right)$$

where I_n is the identity matrix with $n = \dim(V)$.

Recall $I: V \to V$ where $I(x) = x$ for all $x \in V$.

Proof: We have that

def of det

$$\det(T - \lambda I) = \det([T - \lambda I]_\beta)$$

$$= \det([T]_\beta + [-\lambda I]_\beta)$$

HW 4 #2

$[T+S]_\beta = [T]_\beta + [S]_\beta$

$[cT]_\beta = c[T]_\beta$

$$= \det([T]_\beta - \lambda [I]_\beta)$$

$$= \det([T]_\beta - \lambda I_n)$$

HW 5 #2

$[I]_\beta = I_n$

$\blacksquare$

<u>Def:</u> Let V be a finite-dimensional vector space over a field F and let $T: V \to V$ be a linear transformation. Let λ be an eigenvalue of T.

Define

$$E_\lambda(T) = \{ x \in V \mid T(x) = \lambda x \}$$

$$= N(T - \lambda I)$$

$$T(x) = \lambda x$$
$$T(x) - \lambda x = \vec{0}$$
$$T(x) - \lambda I(x) = \vec{0}$$
$$(T - \lambda I)(x) = \vec{0}$$

$E_\lambda(T)$ is called the <u>eigenspace of T corresponding to λ</u>. The dimension of $E_\lambda(T)$ is called the <u>geometric multiplicity of λ</u>.

- $E_\lambda(T)$ is a subspace of V [HW 5]
- $E_\lambda(T)$ consists of $\vec{0}$ and all the eigenvectors corresponding to λ.

Def: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Let β be an ordered basis for V. Let $n = \dim(V)$. Then the function

$$f_T(\lambda) = \det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

is called the characteristic polynomial of T. The roots of $f_T(\lambda)$ are the eigenvalues of T. If λ_0 is a root of $f_T(\lambda)$ then it's multiplicity as a root is called the algebraic multiplicity of λ_0.

That is, the alg. mult. of λ_0 is the largest positive integer k such that $(\lambda - \lambda_0)^k$ is a factor of $f_T(\lambda)$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}$

You can that T is a linear transformation.

Let's find the eigenvalues, eigenvectors, etc for T.

Let's find the eigenvalues first, ie the roots of $f_T(\lambda)$.

We need to pick a basis for $V = \mathbb{R}^3$.

Let $\beta = [v_1, v_2, v_3]$ where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

β is the standard basis for $\mathbb{R}^3$.

Let's calculate $[T]_\beta$

We have

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}=0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}=0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+0\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\3\end{pmatrix}=\underbrace{-2\begin{pmatrix}1\\0\\0\end{pmatrix}+1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+3\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{put in terms of }\beta}$$

$$\boxed{T\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}-2c\\a+2b+c\\a+3c\end{pmatrix}}$$

Thus, $[T]_\beta=\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}$

So,

$$f_T(\lambda)=\det\left([T]_\beta-\lambda I_3\right)$$

I_3 since $\dim(\mathbb{R}^3)=3$

$$=\det\left(\begin{pmatrix}0&0&-2\\1&2&1\\1&0&3\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & \boxed{0} & -2 \\ 1 & \boxed{2-\lambda} & 1 \\ 1 & \boxed{0} & 3-\lambda \end{pmatrix}$$

expand on column 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix} + (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} - 0 \cdot \begin{vmatrix} -\lambda & -2 \\ 1 & 1 \end{vmatrix}$$

$$\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} \quad \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} \quad \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= 0 + (2-\lambda)\left[\underbrace{(-\lambda)(3-\lambda) - (-2)(1)}_{-3\lambda+\lambda^2+2}\right] + 0$$

$$= -6\lambda + 2\lambda^2 + 4 + 3\lambda^2 - \lambda^3 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Recall the rational roots theorem

Let
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
where $a_n, a_{n-1}, \ldots, a_1, a_0$ are integers, $a_n \neq 0$, $a_0 \neq 0$. If a rational number $\frac{p}{q}$ is a root of $f(x)$, then p divides a_0 and q divides a_n

This theorem gives you a list of the possible rational roots

The possible rational roots of

$$f_T(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

are $\frac{p}{q}$ where p divides 4

and q divides -1.

So, $p = \pm 1, \pm 2, \pm 4$ and $q = \pm 1$.

This gives that possible rational roots are

$$\frac{p}{q} = \pm 1, \pm 2, \pm 4.$$

check:

$f_T(1) = -(1)^3 + 5(1)^2 - 8(1) + 4 = 0$

$f_T(-1) = -(-1)^3 + 5(-1)^2 - 8(-1) + 4 = 16 \neq 0$

$f_T(2) = 0$

$f_T(-2) \neq 0$

$f_T(\pm 4) \neq 0$

So the only rational roots of $f_T(\lambda)$ are $\lambda = 1$ and $\lambda = 2$.

Since $\lambda = 1$ is a root of $f_T(\lambda)$ we know $(\lambda - 1)$ is a factor of $f_T(\lambda)$. Let's divide!

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$$
\begin{array}{r}
-\lambda^2 + 4\lambda - 4 \\
\lambda - 1 \overline{\left) -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \right.} \\
-(-\lambda^3 + \lambda^2) \\
\hline
4\lambda^2 - 8\lambda + 4 \\
-(4\lambda^2 - 4\lambda) \\
\hline
-4\lambda + 4 \\
-(-4\lambda + 4) \\
\hline
0
\end{array}
$$

no remainder

Thus,

$$\underbrace{-\lambda^3 + 5\lambda^2 - 8\lambda + 4}_{f_T(\lambda)} = (\lambda - 1)(-\lambda^2 + 4\lambda - 4)$$

Recall: If r_1, r_2 are roots of $ax^2+bx+c=0$ then
$$ax^2+bx+c = a(x-r_1)(x-r_2)$$

The roots of $-\lambda^2+4\lambda-4$ are
$$\lambda = \frac{-4 \pm \sqrt{4^2-4(-1)(-4)}}{2(-1)} = \frac{-4 \pm \sqrt{0}}{-2} = 2$$

Thus, 2 is a root twice!

So, $-\lambda^2+4\lambda-4 = \boxed{-(\lambda-2)(\lambda-2)}$.

Thus,
$$f_T(\lambda) = (\lambda-1)(-\lambda^2+4\lambda-4)$$
$$= -(\lambda-1)(\lambda-2)^2$$

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Ex:

From last time:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}$

$$f_T(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$
$$= -(\lambda - 1)(\lambda - 2)^2$$

eigenvalue of T	$\lambda = 1$	$\lambda = 2$
algebraic multiplicity	1	2

multiplicity as a root of $f_T(\lambda)$

pg 2

Let's calculate $E_1(T)$

$$E_1(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{T(x) = 1 \cdot x} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

add $\begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}$ to both sides

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -a-2c \\ a+b+c \\ a+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} -a-2c = 0 \\ a+b+c = 0 \\ a+2c = 0 \end{matrix} \right\}$$

Let's solve the following system:

$$\boxed{\begin{array}{rrrl} -a & & -2c & = 0 \\ a & +b & +c & = 0 \\ a & & +2c & = 0 \end{array}}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\text{reduced}}$$

This gives

$$\boxed{\begin{array}{l} a \quad +2c = 0 \\ \quad b - c = 0 \\ \qquad\quad 0 = 0 \end{array}}$$

leading variables
a, b

free variable
c

Give free variable new name.

Let $c = t$.

Solve eqns for leading variables.

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$$\boxed{\begin{aligned} a &= -2c \quad (1) \\ b &= c \quad (2) \end{aligned}}$$

Back substitute:

$$c = t$$

(2) $b = c = t$

(1) $a = -2c = -2t$

Thus,

$$E_1(T) = \left\{ \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

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Let $\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$.

Then β_1 spans $E_1(T)$ and since β_1 consists of one non-zero vector, β_1 is a lin. ind. set. (HW)

So, β_1 is a basis for $E_1(T)$

The geometric multiplicity of $\lambda = 1$ is $\dim(E_1(T)) = 1$

size of β_1

Let's calculate $E_2(T)$.

$$E_2(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{T(x) = 2x} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2a & -2c \\ a & +c \\ a & +c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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add $\begin{pmatrix} -2a \\ -2b \\ -2c \end{pmatrix}$ to both sides

Let's solve

$$\boxed{\begin{array}{lll} -2a & -2c & = 0 \\ a & +c & = 0 \\ a & +c & = 0 \end{array}}$$

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{array}{lr} a & +c = 0 \\ & 0 = 0 \\ & 0 = 0 \end{array}}$$

leading variables
a

free variable
b, c

Set $b = t$

$c = s$

Then,

$$\boxed{\begin{aligned} a &= -c = -s \\ b &= t \\ c &= s \end{aligned} \quad \text{where } s, t \in \mathbb{R}}$$

So,

$$E_2(T) = \left\{ \begin{pmatrix} -s \\ t \\ s \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$$

Let $\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$.

Since $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are not multiples of each other, by HW they form a linearly independent set.

So, β_2 is a basis for $E_2(T)$.

Thus, the geometric multiplicity of $\lambda = 2$ is $\dim(E_2(T)) = 2$

Eigenvalues	$\lambda = 1$	$\lambda = 2$
algebraic multiplicity	1	2
geometric multiplicity	1	2
basis for $E_\lambda(T)$	$\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$	$\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

Let $\beta = \beta_1 \cup \beta_2 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$ pg 9

One can show β is a basis for $\mathbb{R}^3$.

What is $[T]_\beta$?

$$T\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So,

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus, T is diagonalizable

Ex: Let

$T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$T(f) = f'$

$T(a+bx+cx^2) = b+2cx$

Let's find the eigenvalues of T.

Let $\gamma = [1, x, x^2]$

Then,

$$T(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$

Thus,

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$f_T(\lambda) = \det\left([T]_\gamma - \lambda I_3\right)$$

$$= \det\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}$$

expand on this column

$$\boxed{\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$$

$$= -\lambda \cdot \underbrace{\begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix}} + 0 + 0$$

$$\begin{pmatrix} \cancel{-\lambda} & \cancel{1} & \cancel{0} \\ \cancel{0} & -\lambda & 2 \\ \cancel{0} & 0 & -\lambda \end{pmatrix}$$

$$= -\lambda\left[\lambda^2 - 0\right]$$

$$= -\lambda^3$$

$$= -(\lambda - 0)^3$$

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Since $f_T(\lambda) = -(\lambda - 0)^3$,

$\lambda = 0$ is the only eigenvalue of T and it has algebraic multiplicity 3.

Let's calculate $E_0(T)$.

$$E_0(T) = \left\{ a+bx+cx^2 \in P_2(\mathbb{R}) \;\middle|\; T(a+bx+cx^2) = 0(a+bx+cx^2) \right\}$$

$$= \left\{ a+bx+cx^2 \in P_2(\mathbb{R}) \;\middle|\; \underline{b+2cx = 0} \right\}$$

$$\boxed{\begin{array}{c} b=0 \\ 2c=0 \end{array}} \quad \text{or} \quad \boxed{\begin{array}{c} b=0 \\ c=0 \end{array}}$$

$$= \left\{ a \;\middle|\; a \in \mathbb{R} \right\}$$

$$= \left\{ a \cdot 1 \;\middle|\; a \in \mathbb{R} \right\} = \operatorname{span}(\{1\})$$

Thus, $\beta = [1]$ is a basis for $E_0(T)$. pg 13

So, $\lambda = 0$ has geometric multiplicity $\dim(E_0(T)) = 1$

\# elements in β

Eigenvalue	$\lambda = 0$
algebraic multiplicity	3
geometric multiplicity	1
basis for $E_\lambda(T)$	$[1]$

note: geo. mult. $\leq$ alg. mult.

In this example there aren't enough eigenvectors to diagonalize T. It turns out that T is <u>not</u> diagonalizable. We need 3 lin. ind. eigenvectors and we only have 1.

M	W
12/6 Finish Topic 5	12/8 Review
	12/15 Final

HW 5 will be on Final.
HW 6 not on final.

Start doing problem 1 of HW 5 and any others in HW 5 you can do

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Final exam

- Weds Dec 15
- opens at 5am on Weds 12/15 and closes at 12pm noon on Thursday 12/16.
- You will get a 3 hr window to take the exam

- covers:
  - Test 1 material
  - Test 2 material
  - Hw 5 - Eigenvalue/Eigenvectors

<u>Lemma:</u> Let $T: V \to V$ be a linear transformation where V is a vector space over a field F.
Let $v_1, v_2, \ldots, v_r$ be eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_i \neq \lambda_j$ when $i \neq j$.
Then, $v_1, v_2, \ldots, v_r$ are linearly independent.

[So, eigenvectors from different/distinct eigenspaces are linearly independent]

<u>proof:</u> We prove by induction on r.

<u>Base case:</u> Suppose $r = 1$.
Suppose v_1 is an eigenvector of T.
By def of eigenvector $v_1 \neq \vec{0}$
By Hw 2 # 6, $\{v_1\}$ is a linearly independent set.

Induction hypothesis: Suppose any k eigenvectors of T with distinct eigenvalues are linearly independent.

Now we prove for $k+1$:

Suppose $v_1, v_2, \dots, v_k, v_{k+1}$ are eigenvectors of T with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_{k+1}$ where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Consider the equation

$$\boxed{c_1 v_1 + c_2 v_2 + \dots + c_k v_k + c_{k+1} v_{k+1} = \vec{0}} \quad (*)$$

where $c_1, c_2, \dots, c_{k+1}$ can be in F.

Apply T to $(*)$ and use the formulas $T(v_i) = \lambda_i v_i$ and $T(\vec{0}) = \vec{0}$.

This gives

We get

$$T(c_1 v_1 + \dots + c_{k+1} v_{k+1}) = T(\vec{0})$$

which becomes

$$c_1 T(v_1) + \dots + c_{k+1} T(v_{k+1}) = \vec{0}$$

which becomes

$$\boxed{c_1 \lambda_1 v_1 + \dots + c_k \lambda_k v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0}} \quad (**)$$

Multiply $(*)$ by λ_{k+1} to get:

$$\boxed{c_1 \lambda_{k+1} v_1 + \dots + c_k \lambda_{k+1} v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0}} \quad (***)$$

Computing $(**) - (***)$ we get

$$\boxed{c_1(\lambda_1 - \lambda_{k+1}) v_1 + c_2(\lambda_2 - \lambda_{k+1}) v_2 + \dots + c_k(\lambda_k - \lambda_{k+1}) v_k = \vec{0}} \quad (****)$$

Since we have k eigenvectors $v_1, \dots, v_k$ with distinct eigenvalues we can apply the induction hypothesis and get that $v_1, v_2, \dots, v_k$ are lin. ind.

Thus (****) gives

$$c_1(\lambda_1 - \lambda_{k+1}) = 0$$
$$c_2(\lambda_2 - \lambda_{k+1}) = 0$$
$$\vdots$$
$$c_k(\lambda_k - \lambda_{k+1}) = 0$$

Since

$$\lambda_1 - \lambda_{k+1} \neq 0,\ \lambda_2 - \lambda_{k+1} \neq 0, \dots, \lambda_k - \lambda_{k+1} \neq 0$$

we must have

$$c_1 = c_2 = \dots = c_k = 0.$$

Plug this back into (*) and get

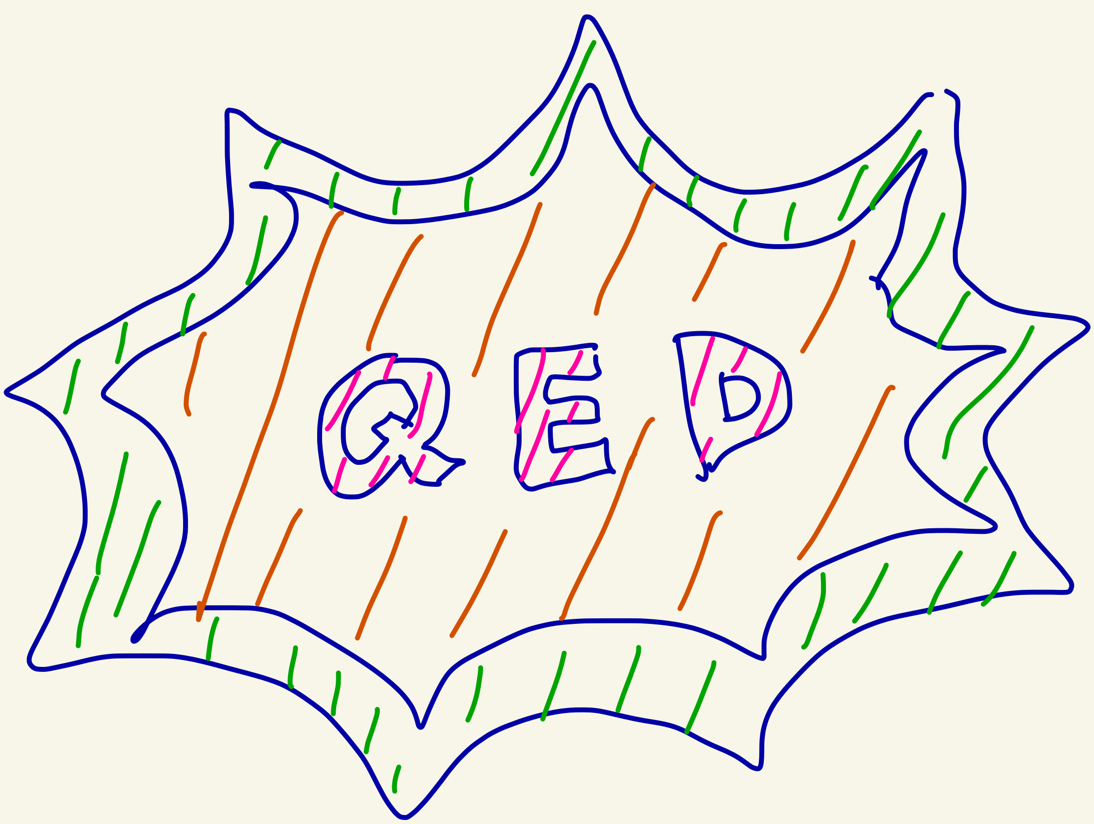
$$c_{k+1} v_{k+1} = \vec{0}$$

Since $v_{k+1} \neq \vec{0}$ the above equation gives $c_{k+1} = 0$.

Thus, $c_1 = c_2 = \cdots = c_k = c_{k+1} = 0$ are the only solutions to

$$c_1 v_1 + \cdots + c_k v_k + c_{k+1} v_{k+1} = \vec{0}.$$

So, $v_1, v_2, \ldots, v_k$ are linearly independent.



<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let $n = \dim(V)$.

Let $T: V \to V$ be a linear transformation.

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the distinct eigenvalues of T.

Let $n_1, \dots, n_r$ be their geometric multiplicities, ie $n_i = \dim(E_{\lambda_i}(T))$

For each i, let

$$\beta_i = [v_{i,1}, v_{i,2}, \dots, v_{i,n_i}]$$

be an ordered basis for $E_{\lambda_i}(T)$

Let

$\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_r$

$= [v_{1,1}, v_{1,2}, \ldots, v_{1,n_1},$ ← basis for $E_{\lambda_1}(T)$

$v_{2,1}, v_{2,2}, \ldots, v_{2,n_2},$ ← basis for $E_{\lambda_2}(T)$

$\vdots$

$v_{r,1}, v_{r,2}, \ldots, v_{r,n_r}]$ ← basis for $E_{\lambda_r}(T)$

Then, β is a linearly independent set.

However, β might not be a basis for V.

Moreover,

β is a basis for V

iff $n_1 + \cdots + n_r = |\beta| = n$

iff T is diagonalizable.

<u>proof:</u>

We first show β is a lin. ind. set.

Suppose

$$\boxed{\sum_{i=1}^{r} \sum_{k=1}^{n_i} c_{i,k} v_{i,k} = \vec{0} \qquad (*)}$$

Where $c_{i,k} \in F$.

<u>Goal</u>: Show $c_{i,k} = 0$ for all i,k.

By HW 5 #6, $E_{\lambda_i}(T)$ is a subspace of V.

Thus, since $v_{i,1}, \ldots, v_{i,n_i} \in E_{\lambda_i}(T)$ we know

$$w_i = \sum_{k=1}^{n_i} c_{i,k} v_{i,k}$$

is in $E_{\lambda_i}(T)$.

So, (*) becomes

$$\boxed{\underbrace{w_1}_{\text{in } E_{\lambda_1}(T)} + \underbrace{w_2}_{\text{in } E_{\lambda_2}(T)} + \cdots + \underbrace{w_r}_{\text{in } E_{\lambda_r}(T)} = \vec{0} \qquad (**)}$$

We will now show that

$$w_1 = w_2 = \cdots = w_r = \vec{0}.$$

Suppose this isn't the case. By reordering/renumbering if necessary, there must then exist m with $1 \leq m \leq r$ and $w_i \neq \vec{0}$ if $1 \leq i \leq m$ and $w_i = \vec{0}$ if $m < i \leq r$

$$\boxed{\underbrace{w_1, w_2, \ldots, w_m}_{\text{all } \neq \vec{0}}, \underbrace{w_{m+1}, \ldots, w_r}_{\text{all } = \vec{0}}}$$

Thus (**) becomes

$$\boxed{w_1 + w_2 + \dots + w_m = \vec{0} \qquad (***)}$$

But then since each w_i is in $E_{\lambda_i}(T)$ and non-zero, we have m eigenvectors $w_1, \dots, w_m$ with distinct eigenvalues $\lambda_1, \dots, \lambda_m$ satisfying the dependency relation (***)

ie $1 \cdot w_1 + 1 \cdot w_2 + \dots + 1 \cdot w_m = \vec{0}$.

This would contradict the previous lemma.

Thus, $w_1 = w_2 = \dots = w_r = \vec{0}$

So,

$$\boxed{w_i = \sum_{k=1}^{n_i} c_{i,k} v_{i,k} = \vec{0} \qquad (****)}$$

for each i

But by assumption,

$\beta_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}]$

is a basis for $E_{\lambda_i}(T)$ and

hence β is a lin. ind. set.

Thus from $(\ast\ast\ast\ast)$,

$c_{i,k} = 0$ for all i, k.

Thus, we've done it!

$\beta = \beta_1 \cup \cdots \cup \beta_r$ is a lin. ind. set.

<u>Moreover part</u>:

Since β is a lin. ind. set and $n = \dim(V)$, β will be a basis for V iff $\underbrace{|\beta|}_{n_1+n_2+\cdots+n_r} = n = \dim(V)$

Now we will show $n = n_1 + \dots + n_r$ iff T is diagonalizable.

[Recall: $n_i = \dim(E_{\lambda_i}(T))$, $n = \dim(V)$]

($\Leftarrow$) Suppose T is diagonalizable.

This means there exists an ordered basis γ of V of eigenvectors of T.

Let $\gamma_i = \gamma \cap E_{\lambda_i}(T)$ for $i = 1, \dots, r$.

Then, $\gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_r$.

Then,

$$\underset{\substack{\uparrow \\ \dim(V)}}{n} = \dim(\underbrace{\text{span}(\gamma)}_{V}) = \sum_{i=1}^{r} \dim(\text{span}(\gamma_i))$$

And $\dim(\underbrace{\text{span}(\gamma_i)}_{\substack{\text{subspace} \\ \text{of } E_{\lambda_i}(T)}}) \le \dim(E_{\lambda_i}(T)) = n_i$

Thus,

$$\boxed{n = \sum_{i=1}^{r} \dim(\operatorname{span}(\gamma_i)) \leq \sum_{i=1}^{r} n_i = n_1 + \dots + n_r}$$

But since β is a lin. ind. set with $n_1 + n_2 + \dots + n_r$ elements and they sit inside V with $\dim(V) = n$ we must have

$$\boxed{n_1 + n_2 + \dots + n_r \leq n.}$$

By the above two equations

$$n = n_1 + n_2 + \dots + n_r.$$

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($\Longrightarrow$) Suppose that

$$\underbrace{n}_{\dim(V)} = \underbrace{n_1 + \dots + n_r}_{\#\text{ elements in } \beta}.$$

Then, β is a basis for V consisting of eigenvectors of T.

[Because we know β is a lin. ind. set. and if $|\beta| = \dim(V)$ it must span V also.]

Thus T is diagonalizable.

$\blacksquare$

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One more thing about eigenvalues

Let V be a finite-dimensional vector space over a field F.
Let $T : V \to V$ be a linear transformation.
Then:

① Let λ be an eigenvalue of T.
Then,

$$1 \leq \underbrace{\begin{pmatrix}\text{geometric} \\ \text{multiplicity} \\ \text{of } \lambda\end{pmatrix}}_{\dim(E_\lambda(T))} \leq \underbrace{\begin{pmatrix}\text{algebraic} \\ \text{multiplicity} \\ \text{of } \lambda\end{pmatrix}}_{\substack{\text{multiplicity of } \lambda \\ \text{as a root} \\ \text{of characteristic} \\ \text{polynomial of } T}}$$

② T is diagonalizable iff

$$\begin{pmatrix}\text{geometric mult.} \\ \text{of } \lambda\end{pmatrix} = \begin{pmatrix}\text{algebraic} \\ \text{mult. of } \lambda\end{pmatrix}$$

for all eigenvalues λ.

HW 5 1(e)

$T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$

$T(f) = f' + f''$

You can check this is a linear transformation

Find eigenvalues

Pick a basis for $P_3(\mathbb{R})$

$\beta = [1, x, x^2, x^3]$

standard basis

Make $[T]_\beta$

$T(1) = 0 + 0 = 0 \cdot 1 + 0x + 0x^2 + 0x^3$

$T(x) = 1 + 0 = 1 \cdot 1 + 0x + 0x^2 + 0x^3$

$T(x^2) = 2x + 2 = 2 \cdot 1 + 2x + 0x^2 + 0x^3$

$T(x^3) = 3x^2 + 6x = 0 \cdot 1 + 6x + 3x^2 + 0x^3$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$f_T(\lambda) = \det\left([T]_\beta - \lambda I_4\right)$$

$$= \det\left(\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$$

expand on column 1

$$= -\lambda \underbrace{\begin{vmatrix} -\lambda & 2 & 6 \\ 0 & -\lambda & 3 \\ 0 & 0 & -\lambda \end{vmatrix}}_{\begin{pmatrix} \cancel{-\lambda} & \cancel{1} & \cancel{2} & \cancel{0} \\ \cancel{0} & -\lambda & 2 & 6 \\ \cancel{0} & 0 & -\lambda & 3 \\ \cancel{0} & 0 & 0 & -\lambda \end{pmatrix}} + 0 + 0 + 0$$

$$= (-\lambda)(-\lambda)\underbrace{\begin{vmatrix} -\lambda & 3 \\ 0 & -\lambda \end{vmatrix}}_{\begin{pmatrix} \cancel{-\lambda} & \cancel{2} & \cancel{6} \\ \cancel{0} & -\lambda & 3 \\ \cancel{0} & 0 & -\lambda \end{pmatrix}} + 0 + 0$$

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$$= (-\lambda)(-\lambda)\left[(-\lambda)(-\lambda) - (3)(0)\right]$$

$$= \lambda^4 = (\lambda - 0)^4$$

So, $\lambda = 0$ is the only eigenvalue with algebraic multiplicity of 4.

$\boxed{\text{Eigenspace time!}}$

$$E_0(T) = \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} T(a+bx+cx^2+dx^3) \\ = 0\cdot(a+bx+cx^2+dx^3) \end{array} \right\}$$

$$= \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} (b+2cx+3dx^2)+(2c+6dx) \\ = 0+0x+0x^2+0x^3 \end{array} \right\}$$

$$= \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} (b+2c)+(2c+6d)x+3dx^2 \\ = 0+0x+0x^2+0x^3 \end{array} \right\}$$

We need to solve

$$\boxed{\begin{array}{r} b+2c \qquad = 0 \\ 2c+6d = 0 \\ 3d = 0 \end{array}}$$

$$\boxed{\begin{aligned} b+2c \quad\;\; &= 0 \\ 2c+6d &= 0 \\ 3d &= 0 \end{aligned}}$$

divide
R_2 by 2
R_3 by 3

$$\Rightarrow \boxed{\begin{aligned} b+2c \quad\; &= 0 \quad (1) \\ c+3d &= 0 \quad (2) \\ d &= 0 \quad (3) \end{aligned}}$$

leading variables
b, c, d
free variable
a

$a = t$

(3) $d = 0$

(2) $c = -3d = -3(0) = 0$

(1) $b = -2c = -2(0) = 0$

Solutions: $a = t$, $b = 0$, $c = 0$, $d = 0$

$$E_0(T) = \{t \mid t \in \mathbb{R}\} = \{t \cdot 1 \mid t \in \mathbb{R}\}$$
$$= \operatorname{span}(\{1\})$$

So, $\beta = [1]$ is a basis for $E_0(\lambda)$

Thus, geometric mult. of λ is 1.

eigenvalues	$\lambda = 0$
alg. mult.	4
basis for $E_0(\lambda)$	$[1]$
geometric mult.	1

not equal

Is T diagonalizable?
Not enough eigenvectors.
We only have 1 lin. ind. eigenvector. We need 4 to diagonalize T because $\dim(P_3(\mathbb{R})) = 4$.

$\boxed{\text{HW 5}}$

(2) Let V be a finite-dimensional vector space over a field F.

Let $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis for V.

Let $I_V : V \to V$ be the identity transformation. $[I_V(x) = x \quad \forall x \in V]$

Show $[I_V]_\beta = I_n$ where $n = \dim(V)$.

proof: We have

$$
\begin{aligned}
I_V(v_1) &= v_1 = 1 \cdot v_1 + 0v_2 + 0v_3 + \ldots + 0v_n \\
I_V(v_2) &= v_2 = 0v_1 + 1v_2 + 0v_3 + \ldots + 0v_n \\
I_V(v_3) &= v_3 = 0v_1 + 0v_2 + 1v_3 + \ldots + 0v_n \\
&\vdots \qquad\qquad \vdots \\
I_V(v_n) &= v_n = 0v_1 + 0v_2 + 0v_3 + \ldots + 1v_n
\end{aligned}
$$

Thus,

$$[I_V]_\beta = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

$\blacksquare$

---

⑥ V is. f.d.v.s. over F

$T : V \to V$ lin. trans.

(a) Show $E_\lambda(T)$ is a subspace of V

Shorter way: $E_\lambda(T) = N(T - \lambda I)$

showed in class

and nullspace is always a subspace.