Multi-colouring the Mycielskian of Graphs

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Abstract

A k-fold colouring of a graph is a function that assigns to each vertex a set of k colours, so that the colour sets assigned to adjacent vertices are disjoint. The k-th chromatic number of a graph G, denoted by $\chi_k(G)$, is the minimum total number of colours used in a k-fold colouring of G. Let $\mu(G)$ denote the Mycielskian of G. For any positive integer k, it holds that $\chi_k(G)+1 \leq \chi_k(\mu(G)) \leq \chi_k(G)+k$ [5]. Although both bounds are attainable, it was proved in [7] that if $k \geq 2$ and $\chi_k(G) \leq 3k-2$, then the upper bound can be reduced by 1, i.e., $\chi_k(\mu(G)) \leq \chi_k(G)+k-1$. We conjecture that for any $n \geq 3k-1$, there is a graph G with $\chi_k(G)=n$ and $\chi_k(\mu(G))=n+k$. This is equivalent to conjecturing that the equality $\chi_k(\mu(K(n,k)))=n+k$ holds for Kneser graphs $K(n,k)$ with $n \geq 3k-1$. We confirm this conjecture for $k=2,3$, or when n is a multiple of k or $n \geq 3k^2/\ln k$. Moreover, we determine the values of $\chi_k(\mu(C_{2q+1}))$ for $1 \leq k \leq q$.

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1 Introduction

In search of graphs with large chromatic number but small clique size, Mycielski [6]introduced the following construction: Let G be a graph with vertex set V and edge set E. Let $\overline{V}$ be a copy of V, $\overline{V} = \{\overline{x} : x \in V\}$, and let u be a new vertex. The *Mycielskian* of G, denoted by $\mu(G)$, is the graph with vertex set $V \cup \overline{V} \cup \{u\}$ and edge set $E' = E \cup \{x\overline{y} : xy \in E\} \cup \{u\overline{x} : \overline{x} \in \overline{V}\}$. The vertex u is called the *root* of $\mu(G)$; and for any $x \in V$, $\overline{x}$ is called the *twin* of x. For a graph G, denote $\chi(G)$ and $\omega(G)$, respectively, the *chromatic number* and the *clique size* of G. It is straightforward to verify that for any graph G with $\omega(G) \geq 2$, we have $\omega(\mu(G)) = \omega(G)$ and $\chi(\mu(G)) = \chi(G) + 1$. Hence, one can obtain triangle free graphs with arbitrarily large chromatic number, by repeatedly applying the Mycielski construction to K_2.

Multiple-colouring of graphs was introduced by Stahl [10], and has been studied extensively in the literature. For any positive integers n and k, we denote by $[n]$ the set $\{0, 1, \ldots, n-1\}$ and $\binom{[n]}{k}$ the set of all k-subsets of $[n]$. A *k-fold n-colouring* of a graph G is a mapping, $f : V \to \binom{[n]}{k}$, such that for any edge xy of G, $f(x) \cap f(y) = \emptyset$. In other words, a k-fold colouring assigns to each vertex a set of k colours, where no colour is assigned to any adjacent vertices. Moreover, if all the colours assigned are from a set of n colours, then it is a k-fold n-colouring. The *k-th chromatic number* of G is defined as

$$\chi_k(G) = \min\{n : G \text{ admits a } k\text{-fold } n\text{-colouring}\}.$$

The k-fold colouring is an extension of conventional vertex colouring. A 1-fold n-colouring of G is simply a proper n-colouring of G, so $\chi_1(G) = \chi(G)$.

It is known [8] and easy to see that for any $k, k' \geq 1$, $\chi_{k+k'}(G) \leq \chi_k(G) + \chi_{k'}(G)$. This implies $\frac{\chi_k(G)}{k} \leq \chi(G)$. The *fractional chromatic number* of G is defined by

$$\chi_f(G) = \inf\{\frac{\chi_k(G)}{k} : k = 1, 2, \ldots\}.$$

Thus $\chi_f(G) \leq \chi(G)$ (cf. [8]).

For a graph G, it is natural to ask the following two questions:

1. What is the relation between the fractional chromatic number of G and the fractional chromatic number of the Mycielskian of G?

2. What is the relation between the k-th chromatic number of G and the k-th chromatic number of the Mycielskian of G?

The first question was answered by Larsen, Propp and Ullman [4]. It turned out that the fractional chromatic number of $\mu(G)$ is determined by the fractional chromatic number of G: For any graph G,

$$\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}.$$

The second question is largely open. Contrary to the answer of the first question in the above equality, the k-th chromatic number of $\mu(G)$ is not determined by $\chi_k(G)$. There are graphs G and G' with $\chi_k(G) = \chi_k(G')$ but $\chi_k(\mu(G)) \neq \chi_k(\mu(G'))$. So it is impossible to express $\chi_k(\mu(G))$ in terms of $\chi_k(G)$. Hence, we aim at establishing sharp bounds for $\chi_k(\mu(G))$ in terms of $\chi_k(G)$. Obviously, for any graph G and any positive integer k, $\chi_k(\mu(G)) \leq \chi_k(G) + k$. Combining this with a lower bound established in [5] we have:

$$\chi_k(G) + 1 \leq \chi_k(\mu(G)) \leq \chi_k(G) + k. \tag{1}$$

Moreover, it is proved in [5] that for any k both the upper and the lower bounds in (1) can be attained. On the other hand, it is proved in [7] that if $\chi_k(G)$ is relatively small compared to k, then the upper bound can be reduced.

Theorem 1 [7] *If $k \geq 2$ and $\chi_k(G) = n \leq 3k-2$, then $\chi_k(\mu(G)) \leq n+k-1$.*

In this article, we prove that for graphs G with $\chi_k(G)$ relatively large compared to k, then the upper bound in (1) cannot be improved. We conjecture that the condition $n \leq 3k-2$ in Theorem 1 is sharp.

Conjecture 1 *If $n \geq 3k-1$, then there is a graph G with $\chi_k(G) = n$ and $\chi_k(\mu(G)) = n+k$.*

A *homomorphism* from a graph G to a graph G' is a mapping $f : V(G) \to V(G')$ such that $f(x)f(y) \in E(G')$ whenever $xy \in E(G)$. If f is a homomorphism from G to G' and c' is a k-fold n-colouring for G', then the mapping defined as $c(x) = c'(f(x))$ is a k-fold n-colouring of G. Thus $\chi_k(G) \leq \chi_k(G')$.

For positive integers $n \geq 2k$, the *Kneser graph* $K(n,k)$ has vertex set $\binom{[n]}{k}$ in which $x \sim y$ if $x \cap y = \emptyset$. It follows from the definition that a graph G has a k-fold n-colouring if and only if there is a homomorphism from G to $K(n,k)$. In particular, if $k' = qk$ for some integer q, then it is easy to show that $\chi_{k'}(K(n,k)) = qn$. If k' is not a multiple of k, then determining $\chi_{k'}(K(n,k))$ is usually a difficult problem. The well-known Kneser-Lovász Theorem [3] gives the answer to the case for $k' = 1$: $\chi(K(n,k)) = n-2k+2$. For $k' \geq 2$, the values of $\chi_{k'}(K(n,k))$ are still widely open.

Notice that, a homomorphism from G to G' induces a homomorphism from $\mu(G)$ to $\mu(G')$. Hence, we have

$$\max\{\chi_k(\mu(G)) : \chi_k(G) = n\} = \chi_k(\mu(K(n,k))).$$

Therefore, Conjecture 1 is equivalent to

Conjecture 2 *If $n \geq 3k-1$, then $\chi_k(\mu(K(n,k))) = n+k$.*

In this paper, we confirm Conjecture 2 for the following cases:

- n is a multiple of k (Section 2),
- $n \geq 3k^2/\ln k$ (Section 2),
- $k \leq 3$ (Section 3).

It was proved in [5] that the lower bound in (1) is sharp for complete graphs K_n with $k \leq n$. That is, if $k \leq n$, then $\chi_k(\mu(K_n)) = \chi_k(K_n) + 1 = kn + 1$. In Section 4, we generalize this result to circular complete graphs $K_{p/q}$ (Corollary 10). Also included in Section 4 are complete solutions of the k-th chromatic number for the Mycielskian of odd cycles C_{2q+1} with $k \leq q$.

2 Kneser graphs with large order

In this section, we prove for any k, if $n = qk$ for some integer $q \geq 3$ or $n \geq 3k^2/\ln k$, then $\chi_k(\mu(K(n,k))) = n + k$.

In the following, the vertex set of $K(n,k)$ is denoted by V. The Mycielskian $\mu(K(n,k))$ has the vertex set $V \cup \overline{V} \cup \{u\}$. For two integers $a \leq b$, let $[a,b]$ denote the set of integers i with $a \leq i \leq b$.

Lemma 2 *For any positive integer k, $\chi_k(\mu(K(3k,k))) = 4k$.*

Proof. Suppose to the contrary, $\chi_k(\mu(K(3k,k))) \leq 4k-1$. Let c be a k-fold colouring of $\mu(K(3k,k))$ using colours from the set $[0, 4k-2]$. Without loss of generality, assume $c(u) = [0, k-1]$. Let $X = \{x \in V : c(x) \cap c(u) = \emptyset\}$. Then X is an independent set in $K(3k,k)$; for if $v, w \in X$ and $v \sim w$, then v, w have a common neighbor, say $\overline{x}$, in $\overline{V}$, implying that $c(v), c(w), c(\overline{x})$ and $c(u)$ are pairwise disjoint. So $|c(u)| + |c(\overline{x})| + |c(v)| + |c(w)| = 4k$, a contradiction. Hence, the vertices of V can be partitioned into $k+1$ independent sets: X and $A_i = \{v \in V : i = \min\ c(v)\}$, $i = 0, 1, \ldots, k-1$, contradicting the fact that $\chi(K(3k,k)) = k+2$. ∎

Lemma 3 *For any $n \geq 3k-1$,*

$$\chi_k(\mu(K(n,k))) \geq \chi_k(\mu(K(n-k,k))) + k.$$

Proof. Suppose $\chi_k(\mu(K(n,k))) = m$. Let c be a k-fold colouring for $\mu(K(n,k))$ using colours from $[0, m-1]$. Assume $c(u) = [0, k-1]$. Since $\chi(K(n,k)) = n - 2k + 2 > k$, there exists some vertex v in V with $c(v) \cap [0, k-1] = \emptyset$. Without loss of generality, assume $c(v) = [k, 2k-1]$. Let N be the set of neighbors of v in V, and let $\overline{N} = \{\overline{w} \in \overline{V} : w \in N\}$. Then the subgraph of $\mu(K(n,k))$ induced by $N \cup \overline{N} \cup \{u\}$ is isomorphic to $\mu(K(n-k,k))$. Denote this subgraph by G'. The colouring c restricted to G' is a k-fold colouring using colours from $[0, m-1] \setminus [k, 2k-1]$, which implies $\chi_k(G') = \chi_k(\mu(K(n-k,k))) \leq m - k$. ∎

Corollary 4 *For any integers $q \geq 3$ and $k \geq 1$, $\chi_k(\mu(K(qk,k))) = (q+1)k$.*

Next we prove that $\chi_k(\mu(K(n,k))) = n+k$ holds for $n \geq 3k^2/\ln k$. It was proved by Hilton and Milner [2] that if X is an independent set of $K(n,k)$ and $\cap_{x \in X} x = \emptyset$, then $|X| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$.

For any positive integer k, let $\phi(k)$ be the minimum n such that

$$\frac{n\left((n-k-1)(n-k-2)\ldots(n-2k+1) - (k-1)!\right)}{k(n-1)(n-2)\ldots(n-k+1)} > 1. \tag{2}$$

Theorem 5 *Let n and k be integers with $n \geq \phi(k)$. Then*

$$\chi_k(\mu(K(n-1,k))) \leq \chi_k(\mu(K(n,k))) - 1.$$

Proof. Let $t = \chi_k(\mu(K(n,k)))$ and let c be a k-fold t-colouring of $\mu(K(n,k))$ using colours from $[0, t-1]$. Assume $c(u) = [0, k-1]$. For $i \in [0, t-1]$, let $S_i = \{x \in V : i \in c(x)\}$. Then $\sum_{i=0}^{t-1} |S_i| = k\binom{n}{k}$, since each vertex appears in exactly k of the S_i's.

Since $t \leq n+k$, by a straightforward calculation, inequality (2) implies that

$$k\binom{n}{k} > (t-k)\left(1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1}\right)+k\binom{n-1}{k-1}.$$

Therefore, at least $k+1$ of the S_i's satisfy the following:

$$|S_i| > 1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1}.$$

Hence there exists $i^* \notin [0, k-1]$ with $|S_{i^*}| > 1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1}$. This implies $\cap_{x\in S_{i^*}} x \neq \emptyset$. Note that the intersection $\cap_{x\in S_{i^*}} x$ contains only one integer. For otherwise, assume $a \in W = \cap_{x\in S_{i^*}} x$ and $W\setminus\{a\} \neq \emptyset$. Let x' be a vertex containing a, and y' be a vertex such that $y' \cap W = W \setminus \{a\}$ and $y' \cap x' \neq \emptyset$. Then $S' = S_{i^*} \cup \{x', y'\}$ is an independent set with $|S'| > 1+\binom{n-1}{k-1}-\binom{n-k-1}{k-1}$ and $\cap_{x\in S'} x = \emptyset$, a contradiction.

Assume $\cap_{x\in S_{i^*}} x = \{a\}$. If $y \in K(n,k)$ and y intersects every $x \in S_{i^*}$, then $a \in y$. For otherwise, $S' = S_{i^*} \cup \{y\}$ is an independent set with $S_{i^*} \subset S'$ and $\cap_{x\in S'} x = \emptyset$, a contradiction. We conclude that for any $y \in K(n,k)$, if $a \notin y$, then none of $S_{i^*} \cup \{y\}$ and $S_{i^*} \cup \{\overline{y}\}$ is an independent set in $\mu(K(n,k))$, which implies that $i^* \notin c(y)$ and $i^* \notin c(\overline{y})$.

By letting $a = n$, the restriction of c to the subgraph $\mu(K(n-1,k))$ gives a k-fold $(t-1)$-colouring of $\mu(K(n-1,k))$. ∎

Corollary 6 *For any $n \geq \max\{2k+1, N\}$, $\chi_k(\mu(K(n,k))) = n+k$, where N is defined as follows. If $\phi(k) = qk+1$, then $N = qk$; otherwise, N is the smallest integer such that N is a multiple of k and $N \geq \phi(k)$.*

Proof. By Corollary 4, $\chi_k(\mu(K(N,k))) = N+k$. By Theorem 5,

$$\chi_k(\mu(K(n,k))) \geq (n-N) + \chi_k(\mu(K(N,k))) = n+k.$$

∎

Although it might be hard to find a simple formula for the function $\phi(k)$ defined in the above, one can easily learn that $\phi(k)$ has order $k^2/\ln k$.

Corollary 7 *If $k \geq 4$ and $n \geq 3k^2/\ln k$, then $\chi_k(\mu(K(n,k))) = n + k$.*

Proof. Assume $n \geq 3k^2/\ln k$. Then

$$\begin{aligned}
& \frac{n[(n-k-1)(n-k-2)\dots(n-2k+1)-(k-1)!]}{k(n-1)(n-2)\dots(n-k+1)} \\
> & \frac{(n-1)(n-k-1)(n-k-2)\dots(n-2k+1)}{k(n-1)(n-2)\dots(n-k+1)} \\
> & \frac{n-1}{k}\left(\frac{n-2k}{n-k}\right)^{k-1} \\
> & \frac{n-1}{k}e^{-k(k-1)/(n-2k)} \\
> & \frac{2k}{\ln k}e^{-k(k-1)\ln k/2k^2} \\
> & \frac{2k}{\sqrt{k}\ln k} > 1.
\end{aligned}$$

Therefore, $n \geq N$ for the N defined in Corollary 6, so the result follows. ∎

In Corollary 7, $3k^2/\ln k$ can be replaced by $(1+\epsilon)k^2/\ln k$ for any $\epsilon > 0$, provided that k is large enough.

3 $K(n,2)$ and $K(n,3)$

In this section, we confirm Conjecture 2 for $k \leq 3$. The case $k = 1$ was proved by Mycielski. For $k = 2, 3$, the value of $\phi(k)$ defined in (2) in Section 2 can be easily determined: $\phi(2) = 6$ and $\phi(3) = 10$. Thus to prove Conjecture 2 for $k = 2, 3$, by Corollary 6 it suffices to show that $\chi_2(\mu(K(5,2))) = 7$ and $\chi_3(K(8,3)) = 11$. As it was proved in [5] that $\chi_2(\mu(K(5,2))) = 7$, the case $k = 2$ is settled.

In the following, we confirm the case $k = 3$.

Theorem 8 $\chi_3(\mu(K(8,3))) = 11$.

Proof. As $\chi_k(K(8,3)) \leq 11$, it suffices to show $\chi_k(K(8,3)) > 10$. Assume to the contrary, there exists a 3-fold 10-colouring c of $\mu(K(8,3))$, using colours from the set $\{a_0, a_1, \ldots, a_9\}$. For simplicity, we denote each vertex in V by (ijk), where $i, j, k \in \{0, 1, 2, \ldots, 7\}$, and its twin by $(\overline{ijk})$; and for $s \leq t$, we denote the set of colours $\{a_s, a_{s+1}, \ldots, a_t\}$ by $a[s,t]$.

Assume $c(u) = a[0,2]$. Let $X = \{x \in V : c(x) \cap c(u) = \emptyset\}$. For $x \in X$ and $i \notin x$, let $M_i(x) = \{v \in V : v \setminus x = \{i\}\}$. For a set A of vertices, let $c\langle A\rangle = \cup_{x \in A} c(x)$.

Claim 1 *For any $x \in X$, there is at most one integer $i \notin x$ for which $c\langle M_i(x)\rangle \not\subseteq c(x) \cup c(u)$.*

Proof. Assume the claim is not true. Without loss of generality, assume that $x = (012)$, $c(x) = a[3,5]$ and $c\langle M_3(x)\rangle, c\langle M_7(x)\rangle \not\subseteq c(x) \cup c(u) = a[0,5]$. We may assume $a_6 \in c\langle M_3(x)\rangle$ and $a_t \in c\langle M_7(x)\rangle$ for some $t \in [6,9]$. For any $i, j, k \in [4,7]$, $(\overline{ijk}) \sim x, u, M_3(x)$. Hence $c(\overline{ijk}) = a[7,9]$. Similarly, for any $i, j, k \in [3,6]$, $c(\overline{ijk}) = a[6,9] - \{a_t\}$. As $c(\overline{456}) = a[7,9] = a[6,9] - \{a_t\}$, we conclude that $t = 6$.

Let $W := \{(034), (157), (026), (134), (257)\}$. Every vertex in W is adjacent to some $(\overline{ijk})$, with $i, j, k \in [4,7]$ or $i, j, k \in [3,6]$. Hence, $c\langle W\rangle \subseteq a[0,6]$. This is impossible, as W induces a C_5 while it is known [10] that $\chi_3(C_5) = 8$. ∎

Claim 2 *Let $x, y \in X$. If $x \neq y$, then $c(x) \neq c(y)$. Moreover, if $x \cap y \neq \emptyset$, then $|c(x) \cap c(y)| = 2$.*

Proof. Let $x, y \in X$, $x \neq y$. Assume to the contrary, $c(x) = c(y)$. Then $x \cap y \neq \emptyset$. Assume $|x \cap y| = 2$, say $x = (012), y = (013) \in X$ and $c(y) = c(x) = a[3,5]$. Then $c(\overline{245}), c(\overline{367}) \subseteq a[6,9]$, implying $|c(245) \cap a[0,2]| \geq 2$ and $|c(367) \cap a[0,2]| \geq 2$. This is impossible as $(367) \sim (245)$.

Next, assume $|x \cap y| = 1$, say $x = (012), y = (234)$ and $c(x) = c(y) = a[3,5]$. By Claim 1, there exists $i \in \{5,6,7\}$, say $i = 5$, $c\langle M_i(x)\rangle \subseteq c(x) \cup c(u) = a[0,5]$. Hence $c(015) = a[0,2]$ (as $(015) \sim (234)$). Then $c(346), c(\overline{015}) \subseteq a[6,9]$, a contradiction, as $345 \sim \overline{015}$. Hence, $c(x) \neq c(y)$.

To prove the moreover part, assume $x \cap y \neq \emptyset$. Then there is some $z \in V$ with $z \sim x, y$. Thus $c(x) \cup c(y) \cup c(u)$ is disjoint from $c(\overline{z})$. This implies $|c(x) \cap c(y)| = 2$. ∎

In the remainder of the proof, we use Schrijver graphs. For $n \geq k$, the *Schrijver graph*, denoted by $S(n,k)$, is a subgraph of $K(n,k)$ induced by the vertices that do not contain any pair of consecutive integers in the cyclic order of $[n]$. Schrijver [9] proved that $\chi(K(n,k)) = \chi(S(n,k))$ and $S(n,k)$ is vertex critical.

Denote the subgraph of $K(8,3)$ induced by $V - X$ by $K(8,3) \setminus X$. Then $K(8,3) \setminus X$ has a 3-vertex-colouring f, defined by $f(v) = \min\{c(v)\}$. Hence, $S(8,3)$ can not be a subgraph of $K(8,3) \setminus X$. In what follows, we frequently use the fact that if, for some ordering of $\{0,1,\ldots,7\}$, each vertex $x \in X$ contains a pair of cyclically consecutive integers in $\{0,1,\ldots,7\}$, then $K(8,3) \setminus X$ contains $S(8,3)$ as a subgraph, which is a contradiction.

Claim 3 *For any $x, y \in X$, $x \cap y \neq \emptyset$.*

Proof. Assume to the contrary, $x = (012), y = (567) \in X$. Suppose there is a vertex $z \in X \setminus \{x, y\}$ which intersects both x, y. By Claim 2, $|c(z) \cap c(x)| = 2$ and $|c(z) \cap c(y)| = 2$, which is a contradiction, as $c(x) \cap c(y) = \emptyset$. Therefore, any $z \in X \setminus \{x, y\}$ is either disjoint from x or disjoint from y. We partition X into two sets, A_x and A_y, that include vertices disjoint from x or from y, respectively.

Next we claim $A_x = \{y\}$ or $A_y = \{x\}$. For each $z \in A_y$, applying the above discussion on x and y to z and y, one can show that for any $z' \in A_x$, $z \cap z' = \emptyset$. Hence, if $A_y - \{x\} \neq \emptyset$ and $A_x - \{y\} \neq \emptyset$, then we may assume $z \subseteq [0, 3]$ for all $z \in A_y$, and $z' \subseteq [4, 7]$ for all $z' \in A_x$. This implies that every vertex of X contains two consecutive integers. Thus, $A_x = \{y\}$ or $A_y = \{x\}$.

Assume $A_x = \{y\}$. If $(024) \notin X$, then clearly every vertex of X contains two consecutive integers. Suppose $z_1 = (024) \in X$. If $(023) \notin X$, then by exchanging 3 and 4 in the cyclic ordering, every vertex in X contains two consecutive integers. Assume $z_2 = (023) \in X$. By Claim 1, for some $i \in \{1, 2\}$, $c(z_i) \subseteq c(x) \cup c(u)$, and hence $c(x) = c(z_i)$ (since $z_i \in X$), contradicting Claim 2. ∎

It follows from Claims 2 and 3 that for any distinct $x, y \in X$, $|c(x) \cap c(y)| = 2$. There are at most five 3-subsets of $a[3, 9]$ that pairwisely have two elements in common. Thus $|X| \leq 5$. By Claim 3, it is straightforward to verify that there exists an ordering of $\{0, 1, 2, \ldots, 7\}$ such that each $x \in X$ contains a pair of cyclic consecutive integers. The details are omitted, as they are a bit tedious yet apparent. ∎

4 Circular cliques and odd cycles

For any positive integers $p \geq 2q$, the *circular complete graph* (or *circular clique*) $K_{p/q}$ has vertex set $[p]$ in which ij is an edge if and only if $q \leq |i - j| \leq p - q$. Circular cliques play an essential role in the study of circular chromatic number of graphs (cf. [12, 13]). A homomorphism from G to $K_{p/q}$ is also called a (p, q)-*colouring* of G. The *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf\{p/q : G \text{ has a } (p,q)\text{-colouring}\}.$$

It is known [12] that for any graph G, $\chi_f(G) \le \chi_c(G)$. Moreover, a result in [1] implies that if $\chi_f(G) = \chi_c(G)$ then for any positive integer k,

$$\chi_k(G) = \lceil k\chi_f(G) \rceil.$$

As $\chi_c(K_{p/q}) = \chi_f(K_{p/q}) = p/q$, we have

$$\chi_k(K_{p/q}) = \lceil kp/q \rceil.$$

Let $m = \lceil kp/q \rceil$. Indeed, a k-fold m-colouring c of $K_{p/q}$, using colours $a_0, a_1, \ldots, a_{m-1}$, can be easily constructed as follows. For $j = 0, 1, \ldots, m-1$, assign colour a_j to vertices $jq, jq+1, \ldots, (j+1)q-1$. Here the calculations are modulo p. Observe that c is a k-fold colouring for $K_{p/q}$, because each colour a_j is assigned to an independent set of $K_{p/q}$, and the union $\cup_{j=0}^{m-1}\{jq, jq+1, \ldots, (j+1)q-1\} = [0, mq-1]$ is an interval of mq consecutive integers. As $mq \ge kp$, for each integer i, there are at least k integers $t \in [0, mq-1]$ that are congruent to i modulo p, i.e., there are at least k colours assigned to each vertex i of $K_{p/q}$. (Here, for convenience, we modify the definition of a k-fold colouring to be a colouring which assigns to each vertex a set of *at least* k colours.)

Now we extend the above k-fold colouring c of $K_{p/q}$ to a k-fold colouring for $\mu(K_{p/q})$ by assigning at least k colours to each vertex in $\overline{V} \cup \{u\}$. Let $S = a[m-k, m-1]$ and let $c(u) = S$. For $i \in V(K_{p/q})$, let $g(\bar{i}) = c(i) \setminus S$. Then $|g(\bar{i})|$ is equal to the number of integers in the interval $[0, (m-k)q-1]$ that are congruent to i modulo p. Hence $|g(\bar{i})| \ge \lfloor (m-k)q/p \rfloor$. Let $b = k - \lfloor (m-k)q/p \rfloor$, and let $c(\bar{i}) = g(\bar{i}) \cup \{a_m, a_{m+1}, \ldots, a_{m+b-1}\}$. Then c is a k-fold $(m+b)$-colouring of $\mu(K_{p/q})$, implying $\chi_k(\mu(K_{p/q})) \le m+b$.

Theorem 9 *Suppose p, q, k are positive integers with $p \geq 2q$. Then*

$$\lceil kp/q + kq/p \rceil \leq \chi_k(\mu(K_{p/q})) \leq \lceil kp/q \rceil + \lceil kq/p \rceil.$$

Proof. The lower bound follows from the result that $\chi_f(\mu(K_{p/q})) = \chi_f(K_{p/q}) + \frac{1}{\chi_f(K_{p/q})} = \frac{p}{q} + \frac{q}{p}$. For the upper bound, we have shown in the previous paragraph that $\chi_k(\mu(K_{p/q})) \leq m+b$, where $m = \lceil kq/p \rceil$ and $b = k - \lfloor (m-k)q/p \rfloor$. By letting $m = (kp+s)/q$, easy calculation shows that $b = \lceil (kq-s)/p \rceil \leq \lceil kq/p \rceil$. ∎

It was proved in [5] that $\chi_k(\mu(K_n)) = \chi_k(K_n) + 1 = kn + 1$ holds for $k \leq n$. By Theorem 9, this result can be generalized to circular cliques.

Corollary 10 *If $k \leq p/q$, then $\chi_k(\mu(K_{p/q})) = \chi_k(K_{p/q}) + 1$.*

Proof. As $\chi_k(\mu(G)) \geq \chi_k(G) + 1$ holds for any graph G, it suffices to note that when $k \leq p/q$, Theorem 9 implies that $\chi_k(\mu(K_{p/q})) \leq \chi_k(K_{p/q}) + 1$. ∎

Corollary 11 *If $k = tq$ is a multiple of q, then $\chi_k(\mu(K_{p/q})) = tp + \lceil kq/p \rceil$; if $k = sp$ is a multiple of p, then $\chi_k(\mu(K_{p/q})) = sq + \lceil kp/q \rceil$.*

Corollary 11 implies that for any integer s with $1 \leq s \leq \lceil k/2 \rceil$, there is a graph G with $\chi_k(\mu(G)) = \chi_k(G) + s$.

If $p = 2q+1$, then $K_{p/q}$ is the odd cycle C_{2q+1}. Assume $k \leq q$. By Theorem 9,

$$2k + \lceil (k+1)/2 \rceil \leq \chi_k(\mu(C_{2q+1})) \leq 2k + \lceil (k+2)/2 \rceil.$$

In particular, if k is even, then $\chi_k(\mu(C_{2q+1})) = 5k/2 + 1$; if k is odd, then $\chi_k(\mu(C_{2q+1})) \in \{2k + \frac{k+1}{2}, 2k + \frac{k+3}{2}\}$. It was proved in [5] that $\chi_k(\mu(C_{2q+1})) = 2k + \frac{k+3}{2}$ if k is odd and $k \leq q \leq \frac{3k-1}{2}$. In the next theorem, we completely determine the value of $\chi_k(\mu(C_{2q+1}))$ for $3 \leq k \leq q$.

Theorem 12 *Let k be an odd integer, $k \geq 3$. Then*

$$\chi_k(\mu(C_{2q+1})) = \begin{cases} 2k + \frac{k+3}{2}, & \textit{if } k \leq q \leq \frac{3k+3}{2}; \\ 2k + \frac{k+1}{2}, & \textit{if } q \geq \frac{3k+5}{2}. \end{cases}$$

Proof. Denote $V(C_{2q+1}) = \{v_0, v_1, \ldots, v_{2q}\}$, where $v_i \sim v_{i+1}$. Throughout the proof, all the subindices are taken modulo $2q+1$.

We first consider the case $k \leq q \leq \frac{3k+3}{2}$. Assume to the contrary, $\chi_k(\mu(C_{2q+1})) = 2k + \frac{k+1}{2}$. Let c be a k-fold colouring of $\mu(C_{2q+1})$ using colours from the set $a[0, 2k + \frac{k-1}{2}]$. Without loss of generality, assume $c(u) = a[0, k-1]$.

Denote by X the colour set $a[k, 2k + \frac{k-1}{2}]$. For $i = 0, 1, \ldots, 2q$, let $W_i = c(v_i)$, $X_i = W_i \cap X$, and $Y_i = W_i \cap a[0, k-1]$. Then $W_i = Y_i \cup X_i$ and $|X_i| + |Y_i| = k$. For each i, since $c(\overline{v_i}) \subseteq X$ and $(c(v_{i-1}) \cup c(v_{i+1})) \cap c(\overline{v_i}) = \emptyset$, we have $|X_{i-1} \cup X_{i+1}| \leq |X| - k = (k+1)/2$. As $|W_i \cup W_{i+1}| = 2k$, we have $|X_i \cup X_{i+1}| \geq k$. Hence, for each i, $\frac{k-1}{2} \leq |X_i| \leq \frac{k+1}{2}$.

Partition $V = \{v_0, v_1, \ldots, v_{2q}\}$ into the following two sets:

$$\begin{aligned} A_1 &= \{v_i \in V : \ |X_i| = \tfrac{k-1}{2}\}, \\ A_2 &= \{v_i \in V : \ |X_i| = \tfrac{k+1}{2}\}. \end{aligned}$$

Observation A. All the following hold for every $i \in [0, 2q]$:

1. If $v_i \in A_1$, then $v_{i-1}, v_{i+1} \in A_2$.
2. If $v_i, v_{i+2} \in A_2$, then $X_i = X_{i+2}$; if $v_i, v_{i+2} \in A_1$, then $|X_i \setminus X_{i+2}| \leq 1$ and $|X_{i+2} \setminus X_i| \leq 1$.
3. Assume $v_i \in A_1$ for some i. If $v_{i+2} \in A_2$ (or $v_{i-2} \in A_2$, respectively), then $X_i \subseteq X_{i+2}$ (or $X_i \subseteq X_{i-2}$, respectively).

For each i, as $|X_i| + |Y_i| = k$, one has $\frac{k-1}{2} \leq |Y_i| \leq \frac{k+1}{2}$. Similar to the above discussion on X_i's, we have:

Observation B. The following hold for all $i \in [0, 2q]$:

1. If $v_i, v_{i+2} \in A_1$, then $Y_i = Y_{i+2}$.

2. Assume $v_i \in A_1$ for some i. If $v_{i+2} \in A_2$ (or $v_{i-2} \in A_2$, respectively), then $Y_{i+2} \subseteq Y_i$ (or $Y_{i-2} \subseteq Y_i$, respectively).

3. Assume $v_i, v_{i+2} \in A_2$ for some i. If $v_{i+1} \in A_1$, then $Y_i = Y_{i+2}$; if $v_{i+1} \in A_2$, then $|Y_{i+2} \setminus Y_i| \leq 1$ and $|Y_i \setminus Y_{i+2}| \leq 1$.

By Observation A (1), there exists some i such that $v_i, v_{i+1} \in A_2$. Without loss of generality, assume $v_0, v_1 \in A_2$.

Claim 1. $|A_1| = k + 2$. Moreover, all the following hold:

1. $\bigcup_{i=0}^{2q} X_i = X_0 \cup X_1 \cup \{w^*\}$ for some $w^* \notin X_0 \cup X_1$.

2. For each $v_i \in A_1$, $i \in [0, 2q]$, there exists some $x \in X_{i-2} \setminus X_i$. In addition, if $x \neq w^*$, then $x \in X_0$ if i is even; and $x \in X_1$ if i is odd.

3. For each $x \in X_0 \cup X_1 \cup \{w^*\}$ there exists a unique $i \in [0, 2q]$ such that $x \in X_i \setminus X_{i+2}$. In addition,

   – if $x = w^*$, then $x \notin X_{i+2} \cup X_{i+3} \cup \ldots \cup X_{2q}$;

   – if $x \in X_0$, then i is even and $x \notin X_{i+2} \cup X_{i+4} \cup \ldots \cup X_{2q}$; and

   – if $x \in X_1$, then i is odd and $x \notin X_{i+2} \cup X_{i+4} \cup \ldots \cup X_{2q-1}$.

Proof. Consider the sequence $(X_0, X_2, \ldots, X_{2q}, X_1)$. Because $X_0 \cap X_1 = \emptyset$, for each $x \in X_0$, there exists some even number $i \in [0, 2q]$ such that $x \in X_i \setminus X_{i+2}$. By Observation A, $X_i \setminus X_{i+2} = \{x\}$ and $v_{i+2} \in A_1$. Since $|X_0| = \frac{k+1}{2}$, we conclude that there exist $\frac{k+1}{2}$ even integers $i \in [0, 2q]$ with $|X_i \setminus X_{i+2}| = 1$ and $v_{i+2} \in A_1$. Similarly, by considering the sequence $(X_1, X_3, \ldots, X_{2q-1}, X_0)$, there exist $\frac{k+1}{2}$ odd integers $i \in [0, 2q]$ with $|X_i \setminus X_{i+2}| = 1$ and $v_{i+2} \in A_1$. Hence, $|A_1| \geq k + 1$.

Let i^* be the smallest nonnegative integer such that $|X_{i^*+2} \setminus X_{i^*}| = 1$. Note, by the above discussion, i^* exists. Let $X_{i^*+2} \setminus X_{i^*} = \{w^*\}$. It can be seen that $w^* \notin X_0 \cup X_1$. By the same argument as in the previous paragraph (using either the even or the odd sequence depending on the parity of i^*), there exists some $i \geq i^*$ such that $w^* \in X_i \setminus X_{i+2}$ and $v_{i+2} \in A_1$. Moreover, this i is different from the i's observed in the previous paragraph. So, $|A_1| \geq k + 2$.

By a similar discussion applied to Y_0 and Y_1 one can show that there are at least k integers i such that $|Y_i \setminus Y_{i+2}| = 1$.

Combining all the above discussion, to complete the proof (including the moreover part) it is enough to show $|A_1| \leq k+2$. Consider a sequence $v_i, v_{i+1}, \ldots, v_{i+s}, v_{i+s+1}$ with $v_i, v_{i+s+1} \in A_1$ and $v_{i+1}, \ldots, v_{i+s} \in A_2$. Then $s > 0$ holds, and by Observation B, there are at most $s-1$ integers j in $[i, i+s]$ such that $|Y_j \setminus Y_{j+2}| = 1$. Hence, there are at most $|A_2| - |A_1|$ integers i in $[0, 2q]$ with $|Y_i \setminus Y_{i+2}| = 1$. This implies, by the previous paragraph, $|A_2| - |A_1| \geq k$. Recall, $|A_2| + |A_1| = 2q + 1 \leq 3k + 4$. Therefore, $|A_1| \leq k + 2$. ∎

Claim 2. For any $v_i, v_j \in A_1$ with $i \neq j$, we have $|i - j| \geq 3$.

Proof. Suppose the claim fails. Without loss of generality, by Observation A (1), we may assume there exists some $i \in [0, 2q]$ such that $v_{i-1}, v_{i+1} \in A_1$ and $v_{i-3}, v_{i-2}, v_i, v_{i+2} \in A_2$. By Observation A (2), $X_{i-2} = X_i = X_{i+2}$. Assume i is odd. (The proof for i even is symmetric.) By Claim 1 (2), there exist $w_1 \in X_{i-3} \setminus X_{i-1}$ and $w_2 \in X_{i-1} \setminus X_{i+1}$, where $\{w_1, w_2\} \subseteq X_0 \cup \{w^*\}$. From $w_1 \in X_{i-3}$ and $w_2 \in X_{i-1}$, it follows $w_1, w_2 \notin X_{i-2}$. By Claim 1 (3), $w_1, w_2 \notin X_{i+1} \cup X_{i+3}$. Hence,

$$X_{i+2} \cup X_{i+1} = X_i \cup X_{i+1} = (X_0 \cup X_1 \cup \{w^*\}) \setminus \{w_1, w_2\}.$$

If $v_{i+3} \in A_2$, by Observation A (3), we have $X_{i+1} \subseteq X_{i+3}$, implying w_1 or w_2 is in $X_{i+3} \setminus X_{i+1}$, a contradiction. Hence, $v_{i+3} \in A_1$. Again by Claim 1 (2),

w_1 or w_2 must be in $X_{i+3} \setminus X_{i+1}$, a contradiction. $\blacksquare$

By Claims 1 and 2, we have $2q + 1 = |A_1| + |A_2| \geq 3(k + 2) = 3k + 6$, contradicting $q \leq \frac{3k+3}{2}$. This completes the proof for $q \leq \frac{3k+3}{2}$.

Now consider $q \geq \frac{3k+5}{2}$. Observe that if $q' \leq q$, then $\mu(C_{2q+1})$ admits a homomorphism to $\mu(C_{2q'+1})$, which implies that $\chi_k(\mu(C_{2q+1})) \leq \chi_k(\mu(C_{2q'+1}))$. Thus to prove the case $q \geq \frac{3k+5}{2}$, it suffices to give a k-fold colouring f for $\mu(C_{3k+6})$ using colours from the set $[0, 2k + \frac{k-1}{2}]$. We give such a colouring f below by using the above proof. For instance, combining Claims 1 and 2, there are exactly $k+2$ vertices in A_1; and these vertices are evenly distributed on C_{3k+6}.

Let $f(u) = [0, k-1]$, where u is the root of $\mu(C_{3k+6})$. Next, we extend f to a k-fold colouring for C_{3k+6} using colours from $[0, 2k+1]$. For $a, b \in [0, 3k+5]$ with appropriate parities, denote $< a, b >$ as the set of integers $\{a, a+2, a+4, \ldots, b-2, b\}$ (mod $3k+6$). For $0 \leq j \leq 2k+1$, define:

$$V[j] = \begin{cases} < 5+6j, 2+6j >, & j = 0, 1, \ldots, \frac{k-3}{2}; \\ < 8+6(j-\frac{k-1}{2}), 5+6(j-\frac{k-1}{2}) >, & j = \frac{k-1}{2}, \ldots, k-2; \\ < 2, 3k-1 > \cup \{3k+2, 3k+5\}, & j = k-1; \\ < 7+6(j-k), 6(j-k) >, & j = k, k+1, \ldots, k+\frac{k-1}{2}; \\ < 10+6(j-k-\frac{k+1}{2}), 3+6(j-k-\frac{k+1}{2}) >, & j = k+\frac{k+1}{2}, \ldots, 2k; \\ < 4, 3k+3 >, & j = 2k+1. \end{cases}$$

Define f on C_{3k+6} by $j \in f(v_i)$ whenever $i \in V[j]$. Observe, for each i, $|(f(v_{i-1}) \cup f(v_{i+1})) \cap [k, 2k+1]| \leq \frac{k+1}{2}$.

Finally, let $f(\overline{v_i})$ be any k colours from $[k, 2k + \frac{k-1}{2}] \setminus (f(v_{i-1}) \cup f(v_{i+1}))$. It is straightforward to verify that f is a k-fold $(2k + \frac{k+1}{2})$-colouring for $\mu(C_{3k+6})$. We shall leave the details to the reader. This completes the proof of Theorem 12. $\blacksquare$

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