California State University – Los Angeles

Department of Mathematics

Master's Degree Comprehensive Examination

Analysis   Sample Test

Do at least five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only.
(3) Begin each problem on a new page.
(4) Assemble the problems you hand in in numerical order.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) problems from this section. If you attempt all three, then the best two will be used for your grade.

Sample Test #1.

(a) Define what it means for a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers to be (i) Cauchy; (ii) convergent; (iii) bounded, using ε-definition.

Prove or give a counterexample for the statements (b) - (e) below:

(b) Every convergent sequence is Cauchy.

(c) Every Cauchy sequence is convergent.

(d) Every Cauchy sequence is bounded.

(e) Every bounded sequence is Cauchy.

Sample Test #2.

Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and let $y_n = (-1)^n x_n$.

(a) Show that $\limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} |x_n|$.

(b) Does the equality have to hold? If true, prove it; or, if false, provide a counterexample.

Sample Test #3.

For the following subsets of $\mathbb{R}$, state whether or not they are (i) open; (ii) closed; (iii) compact, and justify

(a) $[0, 1)$

(b) $A = \{\frac{1}{n} \mid n \in Z^+\} \cup \{0\}$

(c) $\mathbb{Q}$, which is the set of rational numbers

(d) $\mathbb{Z}$, which is the set of integers.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Sample Test #4. Let (x, y) and (a, b) represent points in $\mathbb{R}^2$.

(a) For each of the following decide if the formula given for $\|(x, y)\|$ defines a norm on $\mathbb{R}^2$. If it does, prove it. If it does not, explain your reason.

(i) $\|(x, y)\|_{(i)} = \sqrt{x^4 + y^4}$
(ii) $\|(x, y)\|_{(ii)} = |x| + 5|y|$

(b) For each of the following decide if the formula given for $\langle (a,b),(x,y)\rangle$ defines an inner product on $\mathbb{R}^2$. If it does, prove it. If it does not, explain your reason.

(i) $\langle (a,b),(x,y)\rangle_{(i)} = 2ax + 3by$
(ii) $\langle (a,b),(x,y)\rangle_{(ii)} = ax^2 - by^2$

Sample Test #5. For each continuous function f on the interval $[0,2]$ define a function Tf by

$$(Tf)(x) = x + \lambda \int_0^x xtf(t)\,dt.$$

(a) Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0,2])$ with respect to the supremum norm, $\|f\|_\infty = \sup\{|f(t)| : t \in [0,2]\}$. Justify your answer.

(b) Describe the iterative process for solving the integral equation

$$f(x) = x + \lambda \int_0^x xtf(t)\,dt$$

by specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first two iterates, $f_1(x)$ and $f_2(x)$.

(c) Show that if f is a solution to the integral equation of Part (b), then it is also a solution to the differential equation

$$f''(x) - \lambda x^2 f'(x) - 3\lambda x f(x) = 0 \qquad \text{with } f(0) = 0, f'(0) = 1.$$

Sample Test #6. Let $\mathcal{P}^1$ be the space of all polynomials of degree no more than 1 with the inner product

$$\langle f, g\rangle = \int_0^2 f(t)\overline{g(t)}\,dt.$$

(a) Find a basis for $\mathcal{P}^1$ which is orthonormal with respect the inner product above.

(b) Find constants a and b which make the quantity

$$J = \int_0^2 |t^3 - a - bt|^2 \, dt$$

as small as possible.

Sample Test #7. Let $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} 1, & \text{for} 0 < x < \pi, \\ -1, & \text{for} - \pi < x < 0, \\ 0, & \text{for} x = -\pi, 0, \pi, \end{cases}$$

and extending 2π-periodically.

(a) Find the Fourier series for $f(x)$. (Exponential form or trigonometric form, your choice).

(b) Use the result of Part (a) to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$