Compositions of *n* with no occurrence of *k*

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Abstract

A *composition of n* is an ordered collection of one or more positive integers whose sum is *n*. The number of summands is called the number of *parts* of the composition. A *palindromic composition* or *palindrome* is a composition in which the summands are the same in the given or in reverse order. Compositions may be viewed as tilings of 1-by-*n* rectangles with 1-by-*i* rectangles, $1 \le i \le n$. We count the number of compositions and the number of palindromes of *n* that do not contain any occurrence of a particular positive integer *k*. We also count the total number of occurrences of each positive integer among all the compositions of *n* without occurrences of *k*. This counting problem corresponds to the number of rectangles of each allowable size among the tilings of length *n* without 1-by-*k* tiles. Finally we count the number of compositions without *k* having a fixed number of parts, and explore some patterns involving the number of parts in compositions without *k*.

Key words: Compositions, tilings, palindromes.

1. Introduction

A *composition of n* is an ordered collection of one or more positive integers whose sum is *n*. The number of summands is called the number of *parts* of the composition. A *palindromic composition* or *palindrome* is a composition in which the summands are the same in the given or in reverse order. Compositions may be viewed as tilings of 1-by-*n* rectangles with 1-by-*i* rectangles, $1 \le i \le n$. In this view, a palindromic composition is one corresponding to a symmetric tiling. Because of the relation of compositions to tilings, we sometimes refer to a composition of *n* as a *composition of length n.*

Grimaldi [5] explores the question of how many compositions of *n* exist when no 1's are allowed in the composition. In [4], the authors explore the question of how many compositions of *n* exist when no 2's are allowed in the composition. In this paper we explore the general question of how many compositions of *n* exist when no *k*'s are allowed in the composition. Related to this question we will also explore how many of these compositions are palindromes.

We count the number of compositions and the number of palindromes without *k*, as well as the total number of occurrences of each positive integer among all the compositions of *n* with no *k*'s. The preceding two counting problems correspond respectively to the number of 1-by-*n* tilings and the total number of tiles of a specific size used among all the tilings of length *n* without 1-by-*k* tiles. Finally, we explore particular patterns involving the number of parts among all the compositions of n without occurrences of k. We will use the following notation.

$C(n)$	is the number of compositions of n
$C(n,\hat{k})$	is the number of compositions of n with no k's
$P(n,\hat{k})$	is the number of palindromes of n with no k's
$x(n,i)$	is the number of occurrences of i among all compositions of n
$x(n,i,\hat{k})$	is the number of occurrences of i among all compositions of n with no k's
$C_j(n)$	is the number of compositions of n with j parts
$C_j(n,\hat{k})$	is the number of compositions of n with j parts and no k's.

2. The number of compositions without *k*'s.

In the following theorem we will present three different ways to generate the compositions without k's, each of which gives rise to a different formula.

Theorem 1. The number of compositions of n without k's is given by

$$C(n,\hat{k}) = 2\cdot C(n-1,\hat{k}) + C(n-(k+1),\hat{k}) - C(n-k,\hat{k}) \quad \text{for } n \ge k+1 \tag{1}$$

or

$$C(n,\hat{k}) = \left(\sum_{i=0}^{n-1} C(i,\hat{k})\right) - C(n-k,\hat{k}) \text{ for } n \ge 1 \tag{2}$$

or

$$C(n,\hat{k}) = \left(\sum_{i=1}^{k} C(n-i,\hat{k})\right) + C(n-2k,\hat{k}) \text{ for } n \ge k+1, \tag{3}$$

with initial conditions $C(i,\hat{k}) = 0$ for $i < 0$, $C(i,\hat{k}) = 2^{i-1}$ for $0 < i < k$, $C(k,\hat{k})$ $= 2^{k-1} - 1$, and we define C(0, $\hat{k}$) = 1. The generating function for $C(n,\hat{k})$ is given by $\sum_{n=0}^{\infty} C(n,\hat{k})\cdot t^n = \dfrac{1-t}{1-2t+t^k-t^{k+1}}$.

Proof: The initial conditions follow from the fact that $C(n,\hat{k}) = C(n)$ for $n < k$, as no forbidden k's can occur in the compositions of n, and $C(n) = 2^{n-1}$ (see for example [6], p. 33). If $n = k$, then there is one composition of n consisting of just k, which has to be eliminated, hence $C(k,\hat{k}) =$ $C(k) - 1 = 2^{k-1} - 1$. We now derive the individual formulas.

To show Eq. (1), we generate the compositions of n without k's recursively by the following process: to any such composition of $n-1$, one can add a 1 or increase the last summand by 1. However, this process needs two corrections. First, we must separately generate those compositions that end in $k+1$, since they will not be generated by this recursive method. They come from adding a $k+1$ to compositions of $n-(k+1)$. Secondly, we must subtract the compositions of $n-1$ that initially ended in k-1, since they would now end in k under this process. The number of such compositions corresponds to compositions of length $n-k$.

Eq. (2) follows readily from the following alternate creation method: append a 1 to all allowable compositions of $n-1$, append a 2 to those of $n-2$, and in general, appending j to all allowable compositions of $n-j$ except when $j=k$. This method gives rise to the second formula.

Note that in the summation of Eq. (2), the number of terms being added increases as n increases. A third way of generating the compositions of n without k's requires only a fixed number of summands. It is useful to express this method in terms of tilings. One can either add a tile of length i to any composition of length $n-i$ for $i=1,\ldots,k-1$ or extend the last tile in any composition of length $n-k$ by k units. Because none of the tilings used in this process end in k, we need to add in those that will not be created by the extension methods, namely the $C(n-2k,\hat{k})$ tilings that now end with a tile of length 2k, which leads to Eq. (3).

Finally, to derive the generating function $G_C(t)=\sum_{n=0}^{\infty} C(n,\hat{k})\cdot t^n$, we multiply the Eq. (3) by t^n, then sum over $n \geq k+1$. Thus,

$$\sum_{n=k+1}^{\infty} C(n,\hat{k})\cdot t^n = \sum_{n=k+1}^{\infty}\left(\sum_{i=0}^{k} C(n-i,\hat{k})\right)\cdot t^n + \sum_{n=k+1}^{\infty} C(n-2k,\hat{k})\cdot t^n. \tag{4}$$

Factoring out appropriate powers of t, then re-indexing the infinite series and expressing the resulting series in terms of $G_C(t)$ reduces Eq. (4) to

$$G_C(t)-\sum_{n=0}^{k} C(n,\hat{k})\cdot t^n = \sum_{i=1}^{k} t^i\left(G_C(t)-\sum_{n=0}^{k-i} C(n,\hat{k})\cdot t^n\right)+t^{2k}G_C(t).$$

Collecting the terms containing $G_C(t)$ and then combining terms according to powers of t yields

$$\begin{aligned} G_C(t)\left(1-\sum_{i=1}^{k} t^i - t^{2k}\right) &= \sum_{n=0}^{k} C(n,\hat{k})\cdot t^n - \sum_{i=1}^{k}\sum_{n=0}^{k-i} C(n,\hat{k})\cdot t^{n+i} \\ &= \sum_{j=0}^{k} t^j\left(C(j,\hat{k})-\sum_{i=1}^{j-1} C(i,\hat{k})\right). \end{aligned}$$

We now look at the summands on the right hand side. For $j = 0$, we get $t^0 C(0,\hat{k}) = 1$. For $j = k$, we get

$$t^k (C(k,\hat{k}) - \sum_{i=1}^{k-1} C(i,\hat{k})) = t^k(-C(0,\hat{k})) = -t^k$$

using Eq. (2). If $0 < j < k$ then $C(j,\hat{k}) = \sum_{i=1}^{j-1} C(i,\hat{k})$, thus all these powers of t have zero factors. Thus, $G_C(t)\left(1 - \sum_{i=1}^{k} t^i - t^{2k}\right) = 1 - t^k$, and

$$G_C(t) = \frac{1-t^k}{\left(1 - \sum_{i=1}^{k} t^i - t^{2k}\right)} = \frac{(1-t)(1+t+t^2+\cdots+t^{k-1})}{(1-2t+t^k-t^{k+1})(1+t+t^2+\cdots+t^{k-1})}$$

which gives the desired result. ■

Table 1 gives values for the number of compositions with no k's for $k \le 6$ and $n \le 17$, as well as the number of compositions without restrictions.

	No restrictions	No 1's	No 2's	No 3's	No 4's	No 5's	No 6's
$n = 0$	1	1	1	1	1	1	1
1	1	0	1	1	1	1	1
2	2	1	1	2	2	2	2
3	4	1	2	3	4	4	4
4	8	2	4	6	7	8	8
5	16	3	7	11	14	15	16
6	32	5	12	21	27	30	31
7	64	8	21	39	52	59	62
8	128	13	37	73	101	116	123
9	256	21	65	136	195	228	244
10	512	34	114	254	377	449	484
11	1024	55	200	474	729	883	960
12	2048	89	351	885	1409	1737	1905
13	4096	144	616	1652	2724	3417	3779
14	8192	233	1081	3084	5266	6722	7497
15	16384	377	1897	5757	10180	13223	14873
16	32768	610	3329	10747	19680	26012	29506
17	65536	987	5842	20062	38045	51170	58536

Table 1: Values of $C(n,\hat{k})$ for $k \le 6$

Some patterns that occur in this table are a result of the initial conditions and were already mentioned in the proof of Theorem 1, for example that the column representing compositions without restrictions contains powers of 2, that each column agrees with the number of compositions with no restrictions through the entry $C(k-1,\hat{k})$ and that $C(k,\hat{k}) = C(k)-1 = 2^{k-1}-1$. Furthermore, it is easy to see that $C(k+1,\hat{k}) = C(k)-2 = 2^{k}-2$, since the only missing compositions of $k+1$ are the two involving an occurrence of k, namely $k+1$ and $1+k$.

The second column in Table 1 contains the Fibonacci numbers and was thoroughly investigated in the context of compositions with no 1's in [5]. Likewise, the column with no 2's appeared in [4]. The column representing the number of compositions with no 3's occurs in [7] as A049856 where it is given by the same recurrence relation as in Theorem 1 for k = 3 with no applications mentioned. The remaining columns do not appear in [7].

3. The number of occurrences of various summands among all the compositions of *n* with no occurrences of *k*

First let us look at the number of occurrences of various summands among all the compositions with no restrictions. Chinn et al. [1] showed that $x(n,1) = (n+2)2^{n-3}$ for $n > 1$, and $x(n+j,i+j) = x(n,i)$. The latter formula can easily be extended to the general case where no k's are allowed, as shown in the next theorem.

Theorem 2. The number of i's among all compositions of n with no k's is the same as the number of occurrences of $i+j$ among all the compositions of $n+j$ with no k's, i.e., $x(n+j,i+j,\hat{k}) = x(n,i,\hat{k})$ for all $i \neq k,\ i+j \neq k$.

Proof: Consider any occurrence of the summand i among the compositions of n without k's. There is a corresponding occurrence of $i+j$ in a composition of $n+j$ in which the summand i has been replaced by $i+j$ and all other summands are the same, as long as $i+j \neq k$. This process is reversible, thus the correspondence is one-to-one. ■

As a result of Theorem 2, we only need to generate the number of occurrences of 1 among all the compositions of n without k's, as long as $k \neq 1$, in order to know the number of occurrences of any summand. In the case that k =1, one needs to calculate $x(n,2,\hat{k})$, which by Theorem 2 gives the number of occurrences of $i > 2$. Note that Grimaldi calculated $x(n,2,\hat{k})$ in Table 1 in [5].

Theorem 3. The number of occurrences of 1 among all compositions of n

without k's for $k > 1$ is given by

$$\begin{aligned} x(n,1,\hat{k}) &= 2 \cdot x(n-1,1,\hat{k}) - x(n-k,1,\hat{k}) + x(n-(k+1),1,\hat{k}) \\ &\quad + C(n-1,\hat{k}) - C(n-2,\hat{k}) \end{aligned} \tag{5}$$

or by

$$x(n,1,\hat{k}) = n \cdot C(n,\hat{k}) - \sum_{i=1}^{n-1} (n-i+1) \cdot x(i,1,\hat{k}) , \tag{6}$$

with initial conditions $x(n,1,\hat{k}) = x(n,1) = (n+2)2^{n-3}$ for $3 \le n \le k$, $x(1,1,\hat{k})$ $= x(1,1) = 1$ and $x(2,1,\hat{k}) = x(2,1) = 2$. Furthermore,

$$x(n,1,\hat{k}) = \sum_{i=0}^{n-1} C(i,\hat{k}) \cdot C(n-1-i,\hat{k}) , \tag{7}$$

which implies that the generating function $G_x(t)$ is given by

$$G_x(t) = t \cdot G_C(t)^2 = \frac{t(1-t)^2}{(1-2t+t^k-t^{k+1})^2} .$$

Proof: The initial conditions follow from the fact that for $n < k$, no k can occur. For $n = k$, the only composition that is excluded is the one consisting of k which does not contain any 1's.

Eq. (5) follows from the creation of the compositions of n from those of $n-1$ by either adding a 1 or by increasing the rightmost summand by 1. When adding a 1, we get all the "old" 1's, and for each composition an additional 1, altogether $x(n-1,1,\hat{k}) + C(n-1,\hat{k})$ 1's. When increasing the rightmost summand by 1, again we get all the "old" 1's (of which there are $x(n-1,1,\hat{k})$), except that we need to make the following adjustments: 1) subtract the 1's of those compositions of $n-1$ with terminal summand $k-1$, as they would result in a forbidden k; 2) subtract the terminal 1's in the compositions of $n-1$ that are lost when they turn into 2's; and 3) add the 1's for the compositions of n that end in $k+1$, which have to be created separately. The number of 1's in the compositions of $n-1$ ending in $k-1$ is identical to the number of 1's in the compositions of $(n-1)-(k-1) = n-k$, hence we subtract $x(n-k,1,\hat{k})$. We lose a 1 in every composition of $n-1$ with terminal 1, which equals the number of compositions of $n-2$, thus we subtract $C(n-2,\hat{k})$. Finally, the number of 1's in the compositions of n that end in $k+1$ is given by $x(n-(k+1),1,\hat{k})$, which we add to the total. Simplification gives the stated result.

The second formula for $x(n,1,\hat{k})$, Eq. (6), is based on a geometric argument involving all tilings of a 1-by-n board. The total area of all these tilings, given by $n \cdot C(n,\hat{k})$, has to equal the sum of the areas covered by 1-by-1, 1-by-2,…, and 1-by-n tiles. The area covered by 1-by-i tiles is given by $i \cdot x(n,i,\hat{k})$, and thus, $n \cdot C(n,\hat{k}) = \sum_{i=1}^{n} i \cdot x(n,i,\hat{k})$. Solving for $x(n,1,\hat{k})$, using Theorem 2 to express the right-hand summands in terms of $x(i,1,\hat{k})$, and then re-indexing ($j = n-i+1$) gives that

$$\begin{aligned} x(n,1,\hat{k}) &= n \cdot C(n,\hat{k}) - \sum_{i=2}^{n} i \cdot x(n,i,\hat{k}) \\ &= n \cdot C(n,\hat{k}) - \sum_{i=2}^{n} i \cdot x(n-i+1,1,\hat{k}) \\ &= n \cdot C(n,\hat{k}) - \sum_{j=1}^{n-1} (n-j+1) \cdot x(j,1,\hat{k}). \end{aligned}$$

Eq. (7) also can be seen easily in the framework of tilings. The number of 1's in all compositions of n corresponds to the number of 1-by-1 tiles in all tilings of a 1-by-n board. If a tiling of length n has a 1-by-1 tile at position i, then this tile is preceded by any tiling of length $i-1$ and followed by a tiling of length $n-i$. The number of 1-by-1 tiles at position i is thus given by $C(i-1,\hat{k}) \cdot C(n-i,\hat{k})$. Since 1-by-1 tiles can occur at positions 1 through $n-1$, the formula follows after a simple re-indexing of the summation index. This formula for $x(n,1,\hat{k})$ implies (see for example [8], Rules 1 and 3, Section 2.2) that the generating function is of the form $G_x(t) = t \cdot G_C(t)^2$, from which the result follows by Theorem 1. ■

Table 2 gives the number of occurrences of 1's among all compositions of n with no k's for $1 \le k \le 6$. We also include the number of occurrences of 1's among the compositions of n without restrictions.

	No restrictions	No 2's	No 3's	No 4's	No 5's	No 6's
$n=1$	1	1	1	1	1	1
2	2	2	2	2	2	2
3	5	**3**	5	5	5	5
4	12	6	**10**	12	12	12
5	28	13	22	**26**	28	28
6	64	26	46	58	**62**	64
7	144	50	97	126	138	**142**
8	320	96	200	270	302	314
9	704	184	410	575	654	686
10	1536	350	832	1212	1404	1486
11	3328	661	1679	2538	2995	3196

Table 2. Values of $x(n,1,\hat{k})$

Clearly, the entries in the column for "no k's" agree with the entry in the column for the number of 1's without restrictions for $n \le k$, as indicated in the derivation of the initial conditions in the proof of Theorem 3. We now look at diagonals of slope –1, for the part of the table that refers to compositions without k. These sequences are given by $\{x(k,1,\widehat{k+i})\}_{k=\max\{1,2-i\}}^{\infty}$, where i is an integer. Values of $i \ge 0$ produce the part of the table that is shaded in dark gray, and the entries in those diagonals satisfy $x(k,1,\widehat{k+i}) = x(k,1)$, as explained above.

The next diagonal of slope –1 (entries in bold) is given by $\{x(k,1,\widehat{k-1})\}_{k=3}^{\infty}$, and we have $x(k,1,\widehat{k-1}) = x(k,1) - 2$ since the only compositions that are not allowed are $k+1$ and $1+k$, resulting in a difference of two 1's. Finally, the diagonal with entries $\{x(k,1,\widehat{k-2})\}_{k=4}^{\infty}$ (entries in light gray) satisfies $x(k,1,\widehat{k-2}) = x(k,1) - 6$, since the only compositions that are not allowed are those consisting of one k and two 1's, of which there are three, for a total of six 1's.

4. The number of palindromes with no *k*'s.

The number of palindromes of n with no occurrence of k depends on the relative parity of n and k as detailed in the following theorem. For simplicity in stating the results, let $n = 2m$ or $n = 2m+1$ and $k = 2j$ or $k = 2j+1$.

Theorem 4. The number of palindromes of n with no k's is given by the following formulas.

a) $$P(n,\hat{k}) = \begin{cases} \displaystyle\sum_{i=0,\ i\ne m-j}^{m} C(i,\hat{k}) & \text{if } n \text{ and } k \text{ have the same parity} \\ \displaystyle\sum_{i=0}^{m} C(i,\hat{k}) & \text{if } n \text{ and } k \text{ have opposite parity.} \end{cases}$$

b) $$P(n,\hat{k}) = \sum_{i=1}^{k} P(n-2i,\hat{k}) + P(n-4k,\hat{k}) + \delta_{n,3k} \quad \text{for } n \ge 2k,$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, with initial conditions $P(n,\hat{k}) = 0$ for $n < 0$, $P(0,\hat{k}) = 1$, $P(n,\hat{k}) = 2^{\lfloor n/2 \rfloor}$ for $0 < n < k$, $P(n,\hat{k}) = 2^{\lfloor n/2 \rfloor} - 1$ for $n = k$, $P(n,\hat{k}) = 2^{\lfloor n/2 \rfloor} - 2^{(n-k)/2-1}$ for $k < n < 2k$, n and k having the same parity, and $P(n,\hat{k}) = 2^{\lfloor n/2 \rfloor}$ for $k < n < 2k$, n and k having opposite parity.

Proof: a) An odd length palindrome has to have an odd middle summand, and an even length palindrome has to have an even middle summand or an even split, i.e., the parity of the middle summand is the same of that of n. Each palindrome can be created by attaching a composition of length $(n-l)/2$ on the left of a middle summand of size l and the reverse of the composition on the right. Thus, if n and k have opposite parity, k will never be a middle summand. If n and k have the same parity, then k has to be excluded as a possible middle summand. Now we count the palindromes according to the compositions used to form them.

b) The palindromes of n without k's can also be created by adding the summand i on both sides of a palindrome of length $n-2i$ for $i=1,\ldots,k-1$. To create end summands that are larger than k, we can increase both end summands of the palindromes of $n-2k$ by k if those palindromes consist of more than one summand. For the single palindrome of $n-2k$ consisting of just $n-2k$ we increase this summand by $2k$, thus creating the single palindrome consisting of n. The only palindromes that will not be created in this manner are the palindromes with end summands $2k$, of which there are $P(n-4k,\hat{k})$, and the palindrome of $n=3k$ consisting of the single summand $3k$, which need to be added in separately. The initial conditions for $n<k$ follow from [3, Lemma 11], as $P(n,\hat{k})$ agrees with the number of palindromes of n without restrictions, given by $2^{\lfloor n/2 \rfloor}$. For $n=k$, we get one fewer palindrome, as we have to exclude the palindrome consisting of just k. For $k<n<2k$, we have to exclude those palindromes that contain a single k. This can only occur when n and k have the same parity, with k as the middle summand, combined with a composition of $(n-k)/2$ on either side. There are $C((n-k)/2)=2^{(n-k)/2-1}$ such palindromes, which need to be subtracted from the total. ■

Table 3 gives the number of palindromes of n with no occurrence of k for $1 \le k \le 6$. Note that none of the columns in the Table 3 appears in [7]. However, since there is a different formula for even and odd length palindromes, it makes sense to look at the subsequences consisting of every other entry in each column.

For $k=2$, we get interleaved Fibonacci sequences. If we look at the subsequences with $k \ge 3$ for which n and k have opposite parity, then the sequences initially agree with the k-generalized Fibonacci numbers [2] (sequence A000073 for $k=3$, A000078 for $k=4$, A001591 for $k=5$, and A001592 for $k=6$), which have a recursion of the form $F(n) = \sum_{i=1}^{k} F(n-i)$. This can be seen to agree with the formula given in

Theorem 3, part b), as long as $n \le 4k$, since $\delta_{n,3k} = 0$ whenever n and k have opposite parity. The first term that differs from the respective k-generalized Fibonacci sequence is displayed in bold in Table 3.

	$P(n,\hat{1})$	$P(n,\hat{2})$	$P(n,\hat{3})$	$P(n,\hat{4})$	$P(n,\hat{5})$	$P(n,\hat{6})$
$n=0$	1	1	1	1	1	1
1	0	1	1	1	1	1
2	1	1	2	2	2	2
3	1	2	1	2	2	2
4	2	2	4	3	4	4
5	1	3	3	4	3	4
6	3	4	7	7	8	7
7	2	5	5	8	7	8
8	5	7	13	13	16	15
9	3	**9**	10	15	14	16
10	8	12	24	25	31	30
11	5	16	18	29	27	32
12	13	21	**45**	49	61	59
13	8	28	34	56	53	63
14	21	37	84	94	120	117
15	13	49	63	108	105	125
16	34	65	157	182	236	232
17	21	86	118	**209**	206	248
18	54	114	293	352	464	461
19	34	151	220	404	405	492
20	88	200	547	680	**913**	914

Table 3. Values of $P(n,\hat{k})$ for $k \le 6$

However, two of the odd or even subsequences do agree with sequences in [7]. For $k = 2$, the subsequence for even n agrees with A005314, as was shown in [4]. For $k = 3$, the subsequence for even n agrees with all the terms given for A059633 in [7].

5. Some results on the number of parts in compositions with no *k*'s

When compositions are viewed as tilings, it is quite natural to sort tilings by the number of tiles used. This corresponds to the number of parts in compositions. In the current study of compositions with no occurrence of a particular summand, the number of tiles (parts in the composition) depends not only on n but also on k, the number omitted as a summand. Thus a single table cannot show the number of compositions with a given number of parts with variable forbidden summands. We will state a general result for the number of compositions with no k's with a given number of parts, and then focus our attention on some special cases with $k \le 7$. The case $k = 2$ was thoroughly investigated in [4].

Theorem 5. The number of compositions of n with exactly j parts for $n \ge 1, j \ge 2, k \ge 2$ is given by either of these two formulas:

a) $C_j(n,\hat{k}) = \sum_{i=1}^{n-1} C_{j-1}(n-i,\hat{k}) - C_{j-1}(n-k,\hat{k})$,

b) $C_j(n,\hat{k}) = C_{j-1}(n-1,\hat{k}) + C_j(n-1,\hat{k}) - C_{j-1}(n-k,\hat{k}) + C_{j-1}(n-k-1,\hat{k})$,

with initial conditions $C_j(n,\hat{k}) = 0$ for $n \le 0$, $C_1(n,\hat{k}) = 1$ for $n \ne k$, and $C_1(k,\hat{k}) = 0$.

Proof: a) For any composition of $n-i$ having $j-1$ parts, we can form a composition of n having j parts by adding the summand i to the end of the shorter composition, except for $i = k$. This increases the number of parts by one as required. The initial conditions follow easily, as the only composition of n with one part is n itself.

b) A composition of n with j parts can either be created from a composition of $n-1$ with $j-1$ parts by adding a 1, or from a composition of $n-1$ having j parts by increasing the final summand by 1. The latter count needs to be modified to exclude those compositions that would end in a k if increased, and by adding in those compositions that end in $k+1$, which would not be created in the extension process. The compositions of n with j parts that end in k can be thought of as compositions of $n-k$ with $j-1$ parts, followed by a k, so there are $C_{j-1}(n-k,\hat{k})$ compositions that need to be subtracted. A similar argument shows that there are $C_{j-1}(n-(k+1),\hat{k})$ compositions of n with j parts that end in $k+1$, which need to be added to the total. ■

To understand some of the patterns for values of $C_j(n,\hat{k})$, let us first look at Table 4 which contains the number of compositions of n with j parts when there are no restrictions on the summands. For notational convenience, we will use $\text{bin}(n,k)$ to denote $\binom{n}{k}$.

Note that each row in Table 4 agrees with the corresponding row of Pascal's triangle. To understand why the binomial coefficients appear in this table it is once again convenient to think of a composition as a tiling. Note that any tiling of a 1-by-n rectangle with j parts can be formed by selecting $j-1$ positions to separate the whole rectangle into shorter tiles. This can be accomplished in $\text{bin}(n-1, j-1)$ ways.

	$j=1$	2	3	4	5	6	7	8	9	10
$n=1$	1									
2	1	1								
3	1	2	1							
4	1	3	3	1						
5	1	4	6	4	1					
6	1	5	10	10	5	1				
7	1	6	15	20	15	6	1			
8	1	7	21	35	35	21	7	1		
9	1	8	28	56	70	56	28	8	1	
10	1	9	36	84	126	126	84	36	9	1
11	1	10	45	120	210	252	210	120	45	10
12	1	11	55	165	330	462	462	330	165	55
13	1	12	66	220	495	792	924	792	495	220
14	1	13	78	286	715	1287	1716	1716	1287	715
15	1	14	91	364	1001	2002	3003	3432	3003	2002

Table 4. Values of $C_j(n)$

We now look at tables of values of $C_j(n, \hat{k})$ for $k = 3, \dots, 7$ to illustrate the patterns that hold across the tables. We first look at the columns, then at diagonals of slope –1. For the entry in row n and column j of the m^{th} diagonal, we have $n - j = m - 1$, and thus the entries in m^{th} diagonal are given by $C_{n-m+1}(n, \hat{k})$. The column for $j = 1$ follows directly from the initial conditions. The next theorem states results for the second column and the first $k-1$ diagonals. No obvious uniform pattern exists for the other columns.

Theorem 6. a) The entries in the second column in Tables 5 to 9 are given by

$$C_2(n, \hat{k}) = \begin{cases} n-1 & n < k \\ n-2 & n = 2k \\ n-3 & \text{otherwise.} \end{cases}$$

b) For $k \geq 2$, the first $k-1$ diagonals in the respective table agree with the corresponding diagonals in Table 4, i.e., $C_j(n, \hat{k}) = C_j(n) = \text{bin}(n-1, j-1)$ for $n + 1 - k < j \leq n$.

Proof: a) The values in column 2 can be explained combinatorially by looking at tilings. If no k is allowed, then no cut can be made in the 1-by-n rectangle at position k or at position $n - k$. For $n < k$, this cannot happen, and thus the single cut can be made at any of the $n-1$ cutting positions. For $n = 2k$, the positions k and $n - k$ are identical, thus there is only one forbidden position, hence $C_2(2k, \hat{k}) = n-2$. In all other cases, two of the $n-1$ cutting positions are forbidden, thus $C_2(n, \hat{k}) = n-3$ (for $n > k, n \neq 2k$).

b) Note that for $m < k$, the number of parts $j = n - m + 1 > n - k + 1$, thus k cannot occur as a part. ■

Since the 1st through $(k-1)$st diagonals agree with the values in Table 4, they also appear as diagonals within Pascal's triangle. These entries are shown in bold in Tables 5 through 9. Note that any such diagonal that agrees with Pascal's triangle for a given value of k will also occur in the tables where the forbidden summand is bigger than the given value of k. We will give combinatorial interpretations for the diagonal sequences that also occur as diagonal sequences in Pascal's triangle, and also for the entries of the k^{th} diagonals, which do not reappear in the tables for larger values of k.

To explain the combinatorial interpretations, it is convenient to create the compositions of n having j parts as follows: we start with j 1's (as there are to be j parts), and then distribute the difference $n - j$ across these j parts, adding to the 1's that are already there. In order to count all possibilities, we will find the partitions of $n - j$, then count how many associated compositions without k exist. We illustrate the procedure for the compositions of $n = 4$ without 3's having $j = 2$ parts. First create two 1's, resulting in the composition 1+1. Next distribute the difference $n - j = 2$, i.e., consider all the partitions of 2, namely $\{2\}$ and $\{1,1\}$. Using the first partition leads to 3+1 (the first 1 is increased by 2) or 1+3 (the second 1 is increased by 2), and the second partition creates 2+2 (both 1's are increased by 1). The first two compositions are not allowed as they contain a 3, so we have to disregard all the partitions of $n - j$ that contain a 2, and in general, all the partitions of $n - j$ that contain $k-1$. We will refer to this procedure as the *distributive creation method.*

Table 5 through Table 9 contain the values of $C_j(n, \hat{k})$ for $3 \leq k \leq 7$. We begin by giving a derivation of the formula for the k^{th} diagonals in these tables (shown in gray), and in the case k = 3, also for the 4th diagonal, which is a known sequence.

The third (m = 3) diagonal of $n-2$ parts in Table 5 corresponds to a composition with two 2's and $n-4$ 1's for a total of $\text{bin}(n-2, 2)$ compositions, i.e., a triangle number of them. There is only one additional known sequence that occurs in Table 5, namely the diagonal of

$j = n-3$ parts (m = 4, entries in italic). To count these compositions, we use the distributive creation method described above. The partitions of $n-j=3$ without 2's are {1,1,1} and {3}, which result in the following partitions of n: either three 2's and $j-3$ 1's or one 4 and $j-1$ 1's for a total of $\text{bin}(j,3)+j=(j^3-3j^2+8j)/6$ compositions. This sequence occurs as A000125 in [7], the cake number, which gives the maximal number of pieces resulting from i planar cuts through a cube (or cake), and is given by $a(i)=(i^3+5i+6)/6$. Basic algebra shows that $C_j(j+3,\hat{3}) = a(j-1)$.

	$j=1$	2	3	4	5	6	7	8	9	10
$n=1$	**1**									
2	**1**	**1**								
3	0	**2**	**1**							
4	*1*	1	**3**	**1**						
5	1	*2*	3	**4**	**1**					
6	1	4	*4*	6	**5**	**1**				
7	1	4	9	*8*	10	**6**	**1**			
8	1	5	12	17	*15*	15	**7**	**1**		
9	1	6	15	28	30	*26*	21	**8**	**1**	
10	1	7	21	38	56	51	*42*	28	**9**	**1**
11	1	8	27	56	85	102	84	*64*	36	**10**
12	1	9	34	80	130	172	175	134	*93*	45
13	1	10	42	108	200	276	322	288	207	*130*
14	1	11	51	144	290	447	547	568	459	310
15	1	12	61	188	410	692	924	1024	957	712

Table 5. Values for $C_j(n,\hat{3})$

We now look at the case k = 4. Table 6 shows the number of compositions of n with no 4's having j parts. The diagonal of $j=n-3$ parts (m = 4) occurs in [7] as A005581 and is given by the formula $a(i)=(i-1)\cdot i\cdot(i+4)/6$, which can be derived as follows. The partitions of $n-j=3$ without 3's are {1,1,1} and {2,1}, which result in these partitions of n: three 2's and $j-3$ 1's, or one 3, one 2, and $j-2$ 1's, for a total of $\text{bin}(j,3)+2\cdot\text{bin}(j,2)=j(j-1)(j+4)/6$ compositions, thus $C_j(j+3,\hat{4})=a(j)$.

Next we look at the case k = 5. Table 7 shows the number of compositions of n with no 5's having j parts.

	$j=1$	2	3	4	5	6	7	8	9
$n=1$	**1**								
2	**1**	**1**							
3	**1**	**2**	**1**						
4	0	**3**	**3**	**1**					
5	1	2	**6**	**4**	**1**				
6	1	3	7	**10**	**5**	**1**			
7	1	4	9	16	**15**	**6**	**1**		
8	1	6	12	23	30	**21**	**7**	**1**	
9	1	6	19	32	50	50	**28**	**8**	**1**
10	1	7	24	50	76	96	77	**36**	**9**
11	1	8	30	72	120	162	168	112	**45**
12	1	9	36	99	185	267	315	274	156
13	1	10	45	128	275	432	553	568	423
14	1	11	54	168	385	681	939	1072	963
15	1	12	64	216	531	1022	1554	1920	1959

Table 6. Values for $C_j(n,\hat{4})$

	$j=1$	2	3	4	5	6	7	8	9
$n=1$	**1**								
2	**1**	**1**							
3	**1**	**2**	**1**						
4	**1**	**3**	**3**	**1**					
5	0	**4**	**6**	**4**	**1**				
6	1	3	**10**	**10**	**5**	**1**			
7	1	4	12	**20**	**15**	**6**	**1**		
8	1	5	15	31	**35**	**21**	**7**	**1**	
9	1	6	19	44	65	**56**	**28**	**8**	**1**
10	1	8	24	60	106	120	**84**	**36**	**9**
11	1	8	33	80	160	222	203	**120**	**45**
12	1	9	40	111	230	372	420	322	**165**
13	1	10	48	148	330	582	777	736	486
14	1	11	57	192	465	882	1324	1492	1215

Table 7. Values for $C_j(n,\hat{5})$

The diagonal of $j=n-4$ parts (m = 5) occurs in [7] as A005718, the quadrinomial coefficients, and is given by $a(i)=\text{bin}(i,2)\cdot(i^2+7i+18)/12$, where $a(j)=C_j(j+4,\hat{5})$. We can show the equivalence of the two sequences as follows: The partitions of $n-j=4$ without 4's are {1,1,1,1}, {2,2}, {2,1,1}, and {3,1} which result in these partitions of *n*: four 2's and $j-4$ 1's, of which there are $\operatorname{bin}(j,4)$; two 3's and $j-2$ 1's, of which there are $\operatorname{bin}(j,2)$; one 3, two 2's and $j-3$ 1's, of which there are $j\cdot\operatorname{bin}(j-1,2)$; one 4, one 2, and $j-2$ 1's, of which there are $j(j-1)$. Thus, $C_j(j+4,\hat{5})$ = $\operatorname{bin}(j,4)+$ $\operatorname{bin}(j,2)$ + $j\cdot\operatorname{bin}(j-1,2)$ + $j(j-1)$ = $\operatorname{bin}(j,2)\cdot(j^2+7j+18)/12$.

Next we look at the case k = 6. Table 8 shows the number of compositions of *n* with no 6's having j parts.

	$j=1$	2	3	4	5	6	7	8	9
$n=1$	**1**								
2	**1**	**1**							
3	**1**	**2**	**1**						
4	**1**	**3**	**3**	**1**					
5	**1**	**4**	**6**	**4**	**1**				
6	0	**5**	**10**	**10**	**5**	**1**			
7	1	4	**15**	**20**	**15**	**6**	**1**		
8	1	5	18	**35**	**35**	**21**	**7**	**1**	
9	1	6	22	52	**70**	**56**	**28**	**8**	**1**
10	1	7	27	72	121	**126**	**84**	**36**	**9**
11	1	8	33	96	190	246	**210**	**120**	**45**
12	1	10	40	125	280	432	455	**330**	**165**
13	1	10	51	160	395	702	882	784	**495**
14	1	11	60	208	540	1077	1569	1660	1278
15	1	12	70	264	731	1582	2611	3208	2931
16	1	13	81	329	975	2262	4123	5763	6111
17	1	14	93	404	1280	3168	6265	9760	11790

Table 8. Values for $C_j(n,\hat{6})$

The diagonal of $j=n-5$ parts (*m* = 6) occurs in [7] as A027659, the sixth column of the quintinomial coefficients, and is given by $a(i)=\operatorname{bin}(i,2)+$ $\operatorname{bin}(i+1,3)+\operatorname{bin}(i+2,4)+\operatorname{bin}(i+3,5)$. We can show the equivalence of the two sequences as follows: The partitions of $n-j=5$ without 5's are {1,1,1,1,1}, {2,2,1}, {2,1,1,1}, {3,2}, {3,1,1}, and {4,1} which result in these partitions of *n*: five 2's and $j-5$ 1's, of which there are $\operatorname{bin}(j,5)$; two 3's, one 2 and $j-3$ 1's, of which there are $j\cdot\operatorname{bin}(j-1,2)$; one 3 , three 2's and $j-4$ 1's, of which there are $j\cdot\operatorname{bin}(j-1,3)$; one 4, one 3 and $j-2$ 1's, of which there are $j(j-1)$; one 4, two 2's and $j-4$ 1's, of which there are

$j \cdot \text{bin}(j-1,2)$; and one 5, one 2 and $j-2$ 1's, of which there are $j(j-1)$. Thus there are a total of $\text{bin}(j,5) + j \cdot \text{bin}(j-1,3) +\ 2 \cdot j \cdot \text{bin}(j-1,2)$ $+ 2 \cdot j \cdot (j-1)$ compositions, which can be shown to agree with the formula given for sequence A027659, with $C_j(j+5,\hat{6}) = a(j-2)$.

Finally, we look at the case $k = 7$. Table 9 shows the number of compositions of n with no 7's having j parts.

	$j=1$	2	3	4	5	6	7	8	9
$n=1$	**1**								
2	**1**	**1**							
3	**1**	**2**	**1**						
4	**1**	**3**	**3**	**1**					
5	**1**	**4**	**6**	**4**	**1**				
6	**1**	**5**	**10**	**10**	**5**	**1**			
7	0	**6**	**15**	**20**	**15**	**6**	**1**		
8	1	5	**21**	**35**	**35**	**21**	**7**	**1**	
9	1	6	25	**56**	**70**	**56**	**28**	**8**	**1**
10	1	7	30	80	**126**	**126**	**84**	**36**	**9**
11	1	8	36	108	205	**252**	**210**	**120**	**45**
12	1	9	43	141	310	456	**462**	**330**	**165**
13	1	10	51	180	445	762	917	**792**	**495**
14	1	12	60	226	615	1197	1674	1708	**1287**
15	1	12	73	280	826	1792	2856	3376	2994
16	1	13	84	349	1085	2583	4613	6211	6363

Table 9. Values for $C_j(n,\hat{7})$

The diagonal of $j = n-6$ parts (m = 7) occurs in [7] as A062989, the 7th column of the generalized Catalan Array $\text{FS}[5;i,6]$ and is given by $a(i) =$, $(i+1)(i+2)(i^4 + 24i^3 + 221i^2 + 954i + 1800)/6!$,where $C_j(j+6,\hat{7}) = a(j-2)$. We can show the equivalence of the two sequences as follows: The partitions of $n-j=6$ without 6's are {1,1,1,1,1,1}, {2,2,2}, {2,2,1,1}, {2,1,1,1,1}, {3,3}, {3,2,1}, {3,1,1,1}, {4,2}, {4,1,1}, and {5,1} which result in these partitions of n: six 2's and $j-6$ 1's, of which there are $\text{bin}(j,6)$; three 3's and $j-3$ 1's, of which there are $\text{bin}(j,3)$, two 3's and two 2's and $j-6$ 1's, of which there are $\text{bin}(j,4) \cdot \text{bin}(4,2)$, one 3, four 2's and $j-5$ 1's, of which there are $j \cdot \text{bin}(j-1,4)$; two 4's and $j-2$ 1's, of which there are $\text{bin}(j,2)$; one 4, one 3, one 2 and $j-3$ 1's, of which there are $j(j-1)(j-2)$; one 4, three 2's and $j-4$ 1's, of which there are $j \cdot \text{bin}(j-1,3)$; one 5, one 3, and $j-2$ 1's, of which there are $j(j-1)$; one 5, two 2's and $j-3$ 1's, of which there are $j \cdot \mathrm{bin}(j-1,2)$; and one 6, one 2, and $j-2$ 1's, of which there are $j(j-1)$. Summing these terms and simplifying shows agreement of the number of compositions with the formula given for sequence A062989.

Similar derivations can be made for larger values of *k*; however, the number of partitions increases quite rapidly, and so far no pattern has emerged that would allow for easier counting of these quantities. Likewise, in each table, formulas for the diagonals for $m > k$ can be derived in the same manner. We have checked some of these diagonals, and none (except for $k = 3$, $m = 4$) appear in [7].

We now give derivations for the first six diagonals in Table 9 that occur as diagonals in Pascal's triangle. The first ($m = 1$) diagonal of *n* parts consists of all 1's, since there is only one composition of *n* with *n* parts. The diagonal of $n-1$ parts ($m = 2$) corresponds to a composition with one 2 and $n-2$ 1's for a total of $n-1$ compositions. The diagonal of $j = n-2$ parts ($m = 3$) consists of the triangle numbers, as the only compositions with $n-2$ parts are the $\mathrm{bin}(j,2)$ compositions with two 2's and $j-2$ 1's. The diagonal of $j = n-3$ parts ($m = 4$) occurs as A000292 in [7], the tetrahedral or pyramidal numbers, and is given by $a(i) = (i+1)(i+2)(i+3)/6$, where $C_j(j+3,\hat{k}) = a(j-1)$. The partitions of $n-j=3$ are {1,1,1}, {2,1}, and {3}, which result in these partitions of *n*: three 2's and $j-3$ 1's; one 3, one 2, and $j-2$ 1's; or one 4 and $j-1$ 1's, for a total of $\mathrm{bin}(j,3)+2\cdot\mathrm{bin}(j,2)+j$ compositions. Algebraic simplification shows the equivalence of the 4^{th} diagonal and sequence A000292. The diagonal of $j = n-4$ parts ($m = 5$) appears as A000332 in [7], with $a(i) = (i^4-6i^3+11i^2-6i)/24$, which has several interpretations, for example the number of intersection points of the diagonals of a convex *i*-gon. Arguments similar to the ones above show that $C_j(j+4,\hat{k}) = a(j+3)$. Finally, the 6^{th} diagonal appears as A000389 in [7], with $a(i) = \mathrm{bin}(i,5)$. Using the distributive creation method once more, it can be shown that $C_j(j+5,\hat{k}) = a(j+4)$.

For higher values of *k*, more diagonals from Pascal's triangle will occur, and in each case their formulas can be derived and shown to be equivalent to the known sequences using the distributive creation method.

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