

# Notes on Dynamics

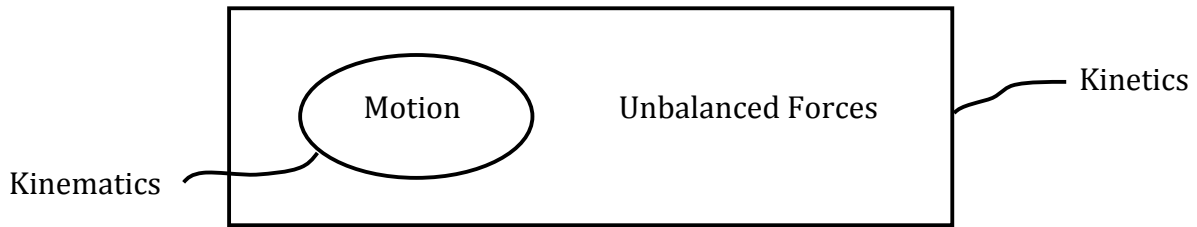
by

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These notes are a supplement to *FE Reference Handbook*, 10.2 Edition, for the Computer-Based Exam, NCEES, July 2022, pp. 114-129.

These notes were prepared for the FE/EIT Exam Review Course class meeting held on July 30, 2022, 9:00 a.m. to 4:00 p.m.

## Dynamics

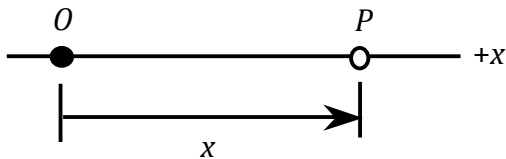


Kinematics – deals with motion alone apart from considerations of force and mass.

Kinetics – relates unbalanced forces with changes in motion.

### Kinematics of Particles

#### Rectilinear Motion of a Particle



Position coordinate

(Rectilinear displacement):  $x = f(t) \rightarrow x = x(t)$

$$\text{Velocity: } v = \frac{dx}{dt} = \dot{x}$$

$$\text{Acceleration: } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x}$$

Suppose  $v = v(x)$ ; apply “Chain Rule”:

$$\frac{dv}{dt} = a = \frac{dv}{dx} \frac{dx}{dt} \rightarrow a = \frac{dv}{dx} v$$

#### Determination of Motion of a Particle

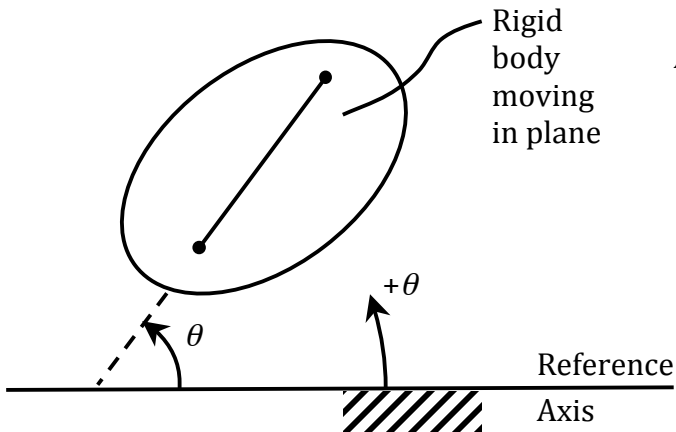
Integrate differential relations:

$$dx = v dt$$

$$dv = a dt$$

$$v dv = a dx$$

## Angular Motion of a Line



Angular position coordinate  
(Angular displacement):  $\theta = f(t)$

$$\text{Angular velocity: } \omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\text{Angular acceleration: } \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Differential relations:

$$d\theta = \omega dt$$

$$d\omega = \alpha dt$$

$$\omega d\omega = \alpha d\theta$$

Note analogy with rectilinear motion.

Two common cases:

1. Acceleration  $a = \text{constant}$ , or  $\alpha = \text{constant}$
2. Acceleration  $a = f(t)$ , or  $\alpha = f(t)$

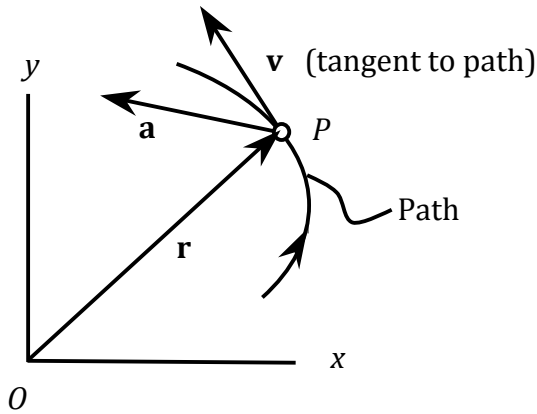
See motion equations in the 10.2 Handbook on pp. 117-118.

## Curvilinear Motion of a Particle

Vectors will be denoted by upright boldface letters, e.g.,  $\mathbf{r}$ .

Vectors will be denoted by underlined letters in handwriting, e.g.,  $\underline{r}$ .

Scalar component of vector  $\mathbf{r}$  will be denoted by *italic r*.

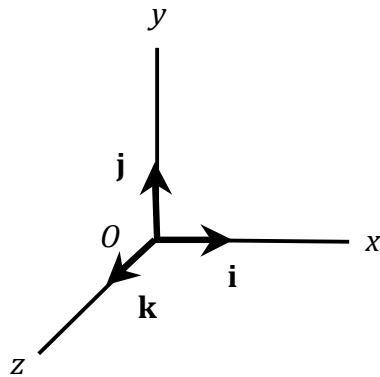


Position vector:  $\mathbf{r} = \mathbf{r}(t)$   
(Vector function)

$$\text{Velocity: } \mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

$$\text{Acceleration: } \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$$

### Rectangular Components



Position vector:  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

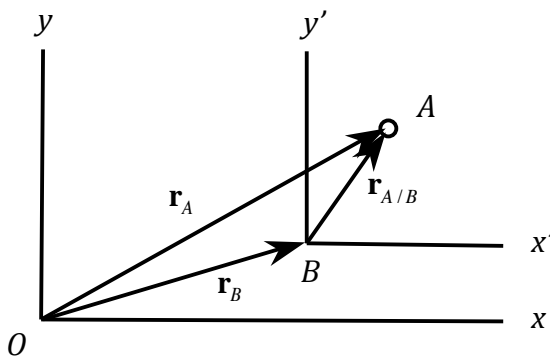
Velocity:  $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$

Acceleration:  $\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}$

We write:  $v_x = \dot{x}$ , etc.  
 $a_x = \ddot{x}$ , etc.

Application: See projectile motion in the 10.2 Handbook on p. 118.

### Motion Relative to Translating Reference Axes



“Translating” means  $x' - y'$  axes move but remain parallel to  $x - y$  axes.

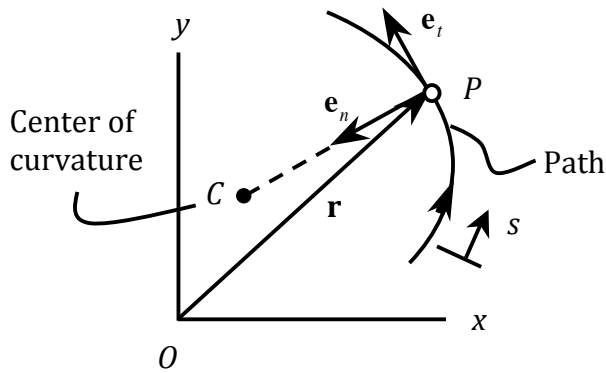
$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{A/B}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

## Tangential and Normal Components



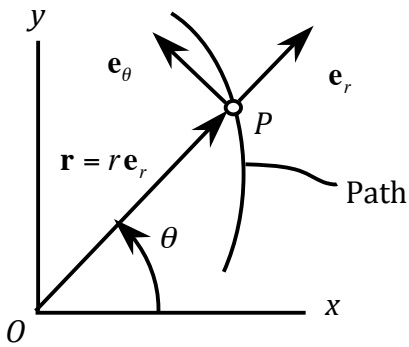
$|CP| = \rho = \text{radius of curvature}$   
 $\mathbf{e}_t = \text{unit vector tangent to path}$   
 $\mathbf{e}_n = \text{unit vector normal to path}$   
 pointing to  $C$   
 $s = \text{directed distance along path}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \mathbf{e}_t = v \mathbf{e}_t$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2s}{dt^2} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n = \frac{dv}{dt} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n$$

$$= a_t \mathbf{e}_t + a_n \mathbf{e}_n = \mathbf{a}_t + \mathbf{a}_n$$

## Radial and Transverse Components



Polar coordinates of  $P$ :  $(r, \theta)$

$\mathbf{e}_r = \text{unit vector in } \mathbf{r} \text{ direction}$

$\mathbf{e}_\theta = \text{unit vector perpendicular to } \mathbf{r}$   
in direction of increasing  $\theta$

$$\mathbf{r} = r \mathbf{e}_r$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{e}_\theta$$

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

If path is a circle, then  $r = \text{constant}$ ,  $\dot{r} = \ddot{r} = 0$ ,

$$\mathbf{v} = r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta$$

### Kinetics of Particles: Newton's Second Law

$$\sum \mathbf{F} = m\mathbf{a} \quad (\text{Equation of motion})$$

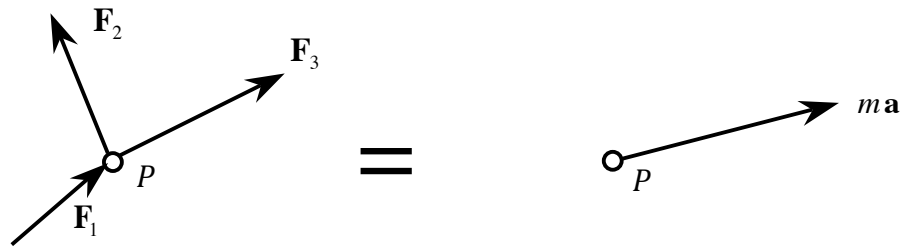
where

$\sum \mathbf{F}$  = resultant force

$m$  = mass of particle

$\mathbf{a}$  = absolute acceleration, measured in a newtonian frame of reference (inertial system)

### Graphical Representation of Newton's 2nd Law



Free-body diagram  
(FBD)

Kinetic diagram (KD)  
(Mass-acceleration diagram)

### Units

Quantity \ System	Length	Time	Mass	Force
SI	m	s	kg	$\text{N} = \text{kg} \cdot \frac{\text{m}}{\text{s}^2}$
USCS	ft	s	$\text{slug} = \text{lb} \cdot \frac{\text{s}^2}{\text{ft}}$	lb

In either system,  $W = mg$ , where

$W$  = weight

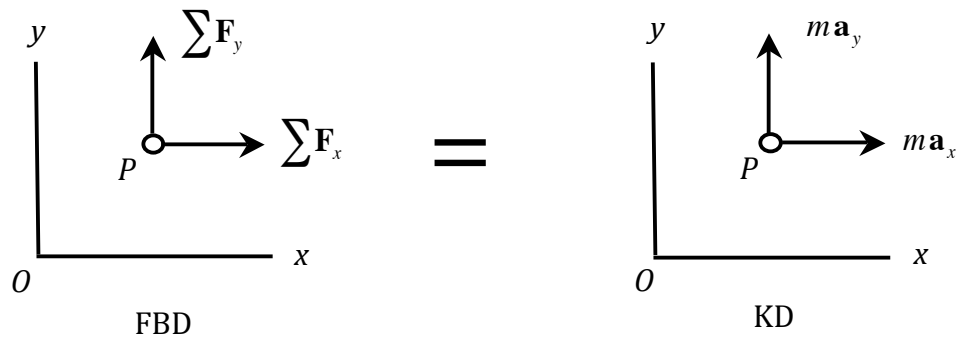
$g$  = acceleration due to gravity

At surface of earth: (SI)  $g = 9.807 \text{ m/s}^2$

(USCS)  $g = 32.174 \text{ ft/s}^2$

AVOID: lbf, lbm

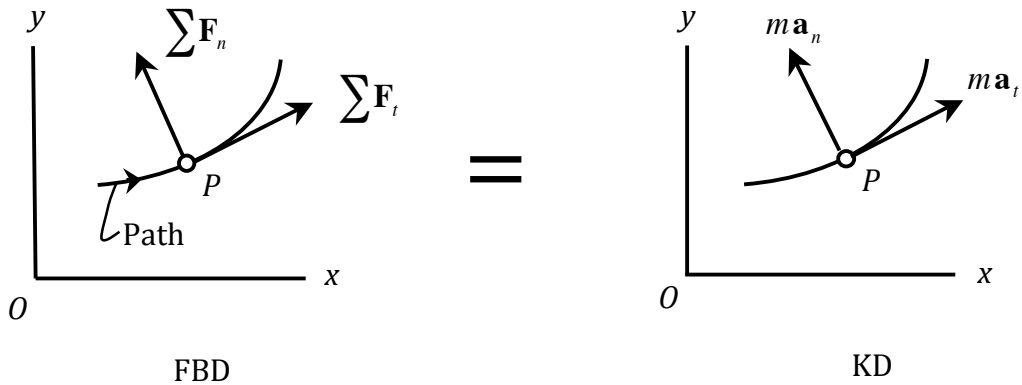
Equations of Motion: Rectangular Components



$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

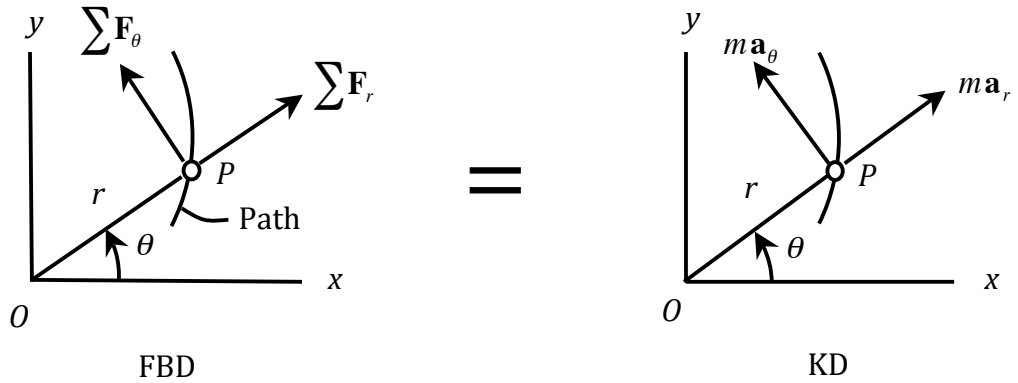
Equations of Motion: Tangential and Normal Components



$$\sum F_t = ma_t$$

$$\sum F_n = ma_n$$

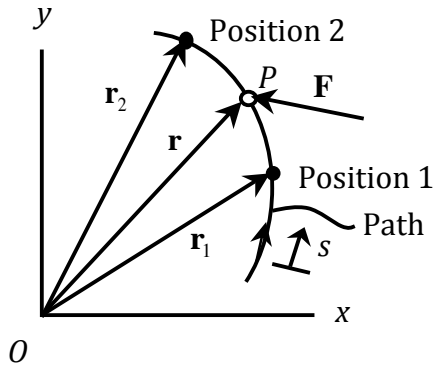
Equations of Motion: Radial and Transverse Components



$$\sum F_r = m a_r$$

$$\sum F_\theta = m a_\theta$$

**Kinetics of Particles: Energy Methods**



The work done by  $\mathbf{F}$  on the particle during a finite movement of the particle along a curved path from position 1 to position 2 is  $U_{1 \rightarrow 2}$  :

$$U_{1 \rightarrow 2} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} \quad (\text{Line integral})$$

It can be shown:

$$U_{1 \rightarrow 2} = \int_{s_1}^{s_2} F_t ds$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

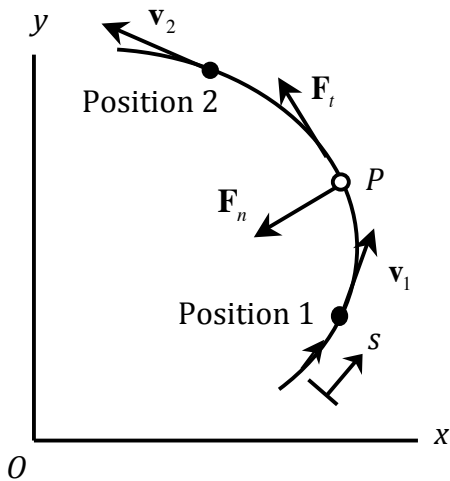
Let  $T = \frac{1}{2} m v^2 = \text{kinetic energy of particle}$

Then,

$$U_{1 \rightarrow 2} = T_2 - T_1$$

$$= \Delta T$$

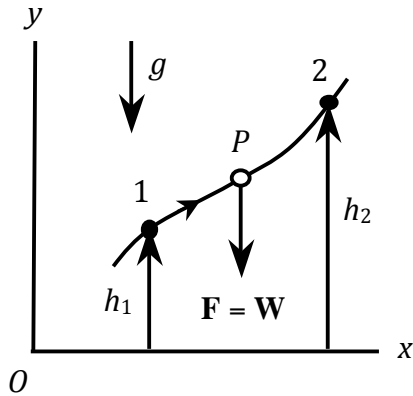
or  $T_2 = T_1 + U_{1 \rightarrow 2}$



Above result is the *principle of work and energy*. Units: (SI) N·m = J; (USCS) ft·lb



Work Done on Particle by Gravitational Force



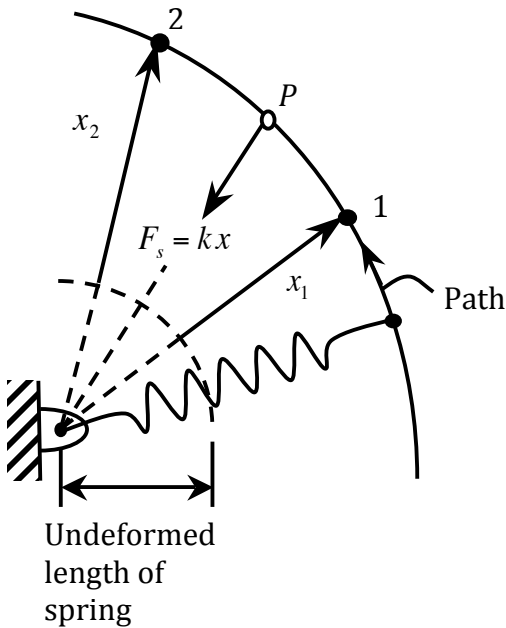
$$U_{1 \rightarrow 2} = - \int_{h_1}^{h_2} W \, dy = -(Wh_2 - Wh_1)$$

Let  $V_g = Wy = mgy =$  *gravitational potential energy of particle*

$$\begin{aligned} \text{Then, } U_{1 \rightarrow 2} &= -[(V_g)_2 - (V_g)_1] \\ &= -\Delta V_g \end{aligned}$$

Note  $U_{1 \rightarrow 2}$  is independent of path from 1 to 2. For this reason  $\mathbf{W}$  is called a *conservative force*.

Work Done on Particle by a Linearly-Elastic Spring Force



Let  $k =$  spring constant  
 $x =$  spring elongation  
 $F_s = kx =$  spring force

Then,

$$\begin{aligned} U_{1 \rightarrow 2} &= - \int_{x_1}^{x_2} F_s \, dx = - \int_{x_1}^{x_2} kx \, dx \\ &= - \left( \frac{1}{2} kx_2^2 - \frac{1}{2} kx_1^2 \right) \end{aligned}$$

Let  $V_e = \frac{1}{2} kx^2 =$  *elastic potential energy of particle*

$$\begin{aligned} \text{Then, } U_{1 \rightarrow 2} &= -[(V_e)_2 - (V_e)_1] \\ &= -\Delta V_e \end{aligned}$$

Note  $U_{1 \rightarrow 2}$  is independent of path from 1 to 2. For this reason  $F_s$  is called a *conservative force*.

## Summary

The work-energy equation can now be written as:

$$U_{1 \rightarrow 2} = \Delta T + \Delta V_g + \Delta V_e$$

where  $U_{1 \rightarrow 2}$  is the work done on the particle by forces other than gravitational and spring forces.

If  $U_{1 \rightarrow 2}$  above is zero, then:

$$T_2 + (V_g)_2 + (V_e)_2 = T_1 + (V_g)_1 + (V_e)_1$$

This is the *law of conservation of total mechanical energy*.

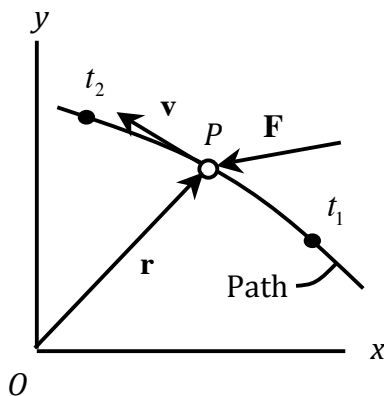
## Power and Efficiency

Power is the time rate of doing work by a force on a particle.

$$\text{Power} = \mathbf{F} \cdot \mathbf{v} \quad \text{Units: (SI) } \text{N} \cdot \text{m/s} = \text{J/s} = \text{W}; \text{ (USCS) } \text{hp} = 550 \frac{\text{ft} \cdot \text{lb}}{\text{s}}$$

$$\eta = \frac{\text{power output}}{\text{power input}} = \text{mechanical efficiency}$$

## **Kinetics of Particles: Momentum Methods**



Recall Newton's 2<sup>nd</sup> law:

$$\mathbf{F} = m\mathbf{a} = \frac{d}{dt}(m\mathbf{v})$$

where  $\mathbf{F}$  = resultant force

$m\mathbf{v}$  = linear momentum of particle

Define *angular momentum*  $\mathbf{H}_O$  of particle about  $O$ :

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

$$\text{Then, } \dot{\mathbf{H}}_O = \mathbf{r} \times m \mathbf{a} = \mathbf{r} \times \mathbf{F} = \mathbf{M}_O$$

$$\text{or } \mathbf{M}_O = \dot{\mathbf{H}}_O$$

where  $\mathbf{M}_O$  = sum of the moments about  $O$  of  
all forces acting on particle

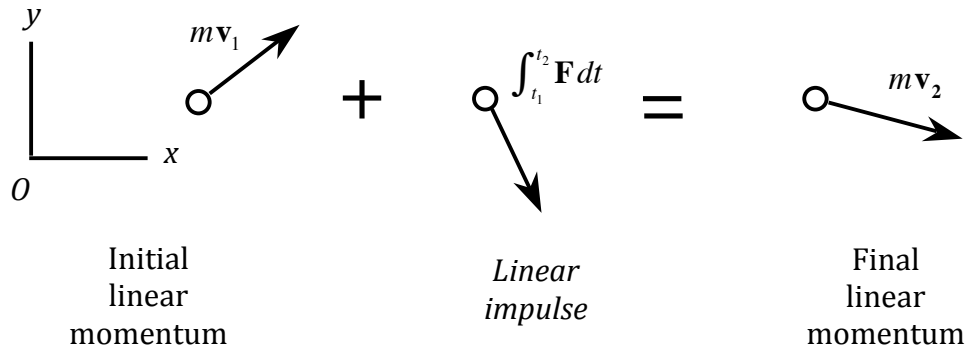
### Equations of Impulse and Momentum

What is the cumulative effect of integrating  $\mathbf{F}$  and  $\mathbf{M}_O$  with respect to time over an interval from  $t_1$  to  $t_2$ ?

$$\int_{t_1}^{t_2} \mathbf{F} dt = \int_{m\mathbf{v}_1}^{m\mathbf{v}_2} d(m\mathbf{v}) = m\mathbf{v}_2 - m\mathbf{v}_1$$

$$\text{or } m\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{F} dt = m\mathbf{v}_2$$

Graphical interpretation:



$$\text{or } (mv_x)_1 + \int_{t_1}^{t_2} F_x dt = (mv_x)_2$$

$$(mv_y)_1 + \int_{t_1}^{t_2} F_y dt = (mv_y)_2$$

$$\text{Units: (SI) } \text{kg} \cdot \frac{\text{m}}{\text{s}} = \text{N} \cdot \text{s}; \text{ (USCS) } \text{lb} \cdot \text{s}$$

$$\text{Recall } \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt}$$

$$\int_{t_1}^{t_2} \mathbf{M}_O dt = \int_{(\mathbf{H}_O)_1}^{(\mathbf{H}_O)_2} d\mathbf{H}_O = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

$$\text{or } (\mathbf{H}_O)_1 + \int_{t_1}^{t_2} \mathbf{M}_O dt = (\mathbf{H}_O)_2$$

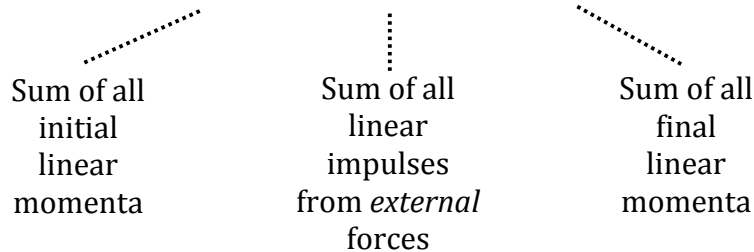
$$\begin{array}{ccccc} \text{Initial} & & & & \text{Final} \\ \text{angular} & + & \text{Angular} & = & \text{angular} \\ \text{momentum} & & \text{impulse} & & \text{momentum} \end{array}$$

$$\text{Units: (SI) } \text{kg} \cdot \frac{\text{m}^2}{\text{s}} = \text{N} \cdot \text{m} \cdot \text{s}; \text{ (USCS) } \text{lb} \cdot \text{ft} \cdot \text{s}$$

Extension to System of  $n$  Particles

$$\text{Let } \sum m\mathbf{v} = \sum_{i=1}^n m_i \mathbf{v}_i$$

$$\sum m\mathbf{v}_1 + \sum \int_{t_1}^{t_2} \mathbf{F} dt = \sum m\mathbf{v}_2$$



Note: Linear impulses from *internal* forces of action and reaction cancel.

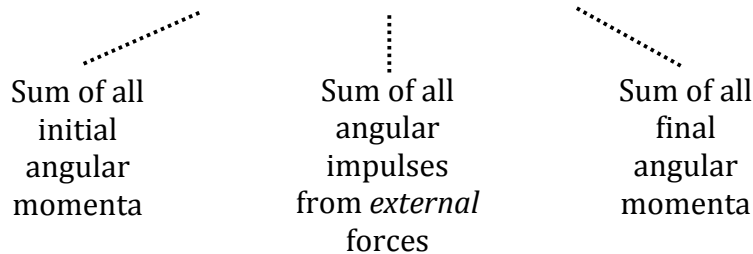
If no external forces act from time  $t_1$  to  $t_2$ , then

$$\sum m\mathbf{v}_1 = \sum m\mathbf{v}_2$$

and the *total linear momentum* of the particles is *conserved*.

$$\text{Let } \sum \mathbf{H}_O = \sum_{i=1}^n \mathbf{r}_i \times m_i \mathbf{v}_i$$

$$\sum (\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O dt = \sum (\mathbf{H}_O)_2$$



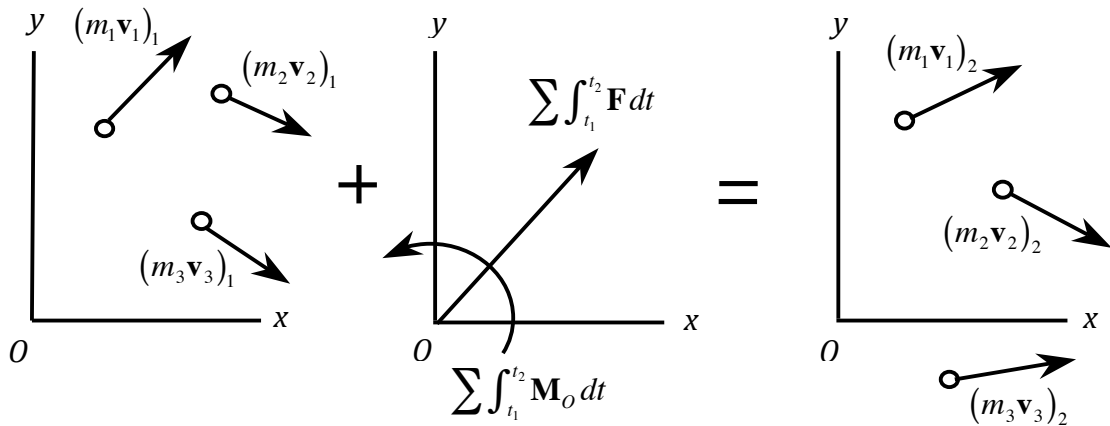
Note: Angular impulses from *internal* forces of action and reaction cancel.

If no external forces act from time  $t_1$  to  $t_2$ , then

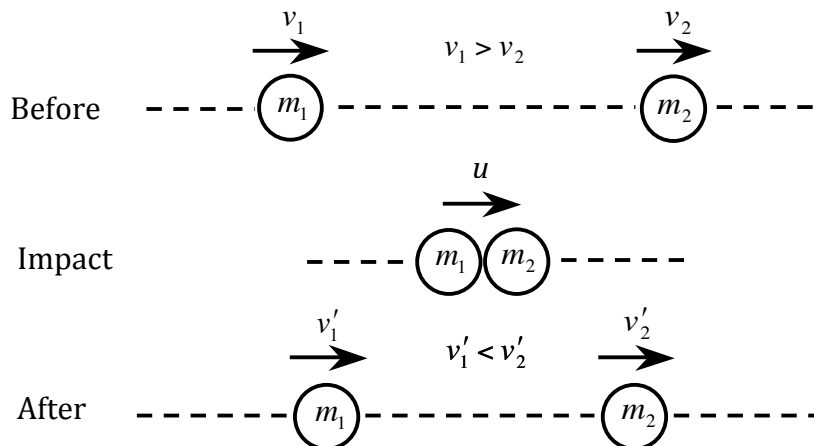
$$\sum(\mathbf{H}_O)_1 = \sum(\mathbf{H}_O)_2$$

and the *total angular momentum* of the particles is *conserved*.

Graphical interpretation:



### Direct Central Impact



Total linear momentum is conserved during impact:

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2$$

$$\text{Coefficient of restitution: } e = \frac{\text{velocity of separation}}{\text{velocity of approach}} = \frac{v'_2 - v'_1}{v_1 - v_2}$$

If total kinetic energy is conserved, impact is said to be *perfectly elastic* and  $e = 1$ .

If particles stick together after impact,  $v'_1 = v'_2$ , impact is said to be *perfectly plastic*, and  $e = 0$ .

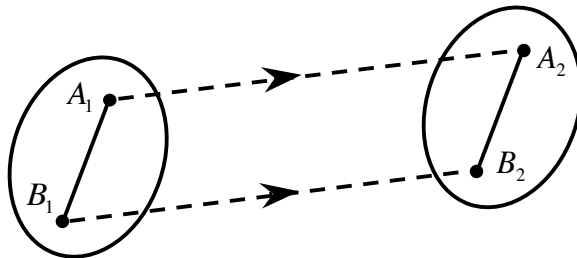
For all other impact cases,  $0 \leq e \leq 1$ .

A *special case* occurs when  $m_1 = m_2$ , collision is *elastic*,  $v_1 > 0$ , and  $v_2 = 0$ . Then,  $v'_1 = 0$  and  $v'_2 = v_1$ .

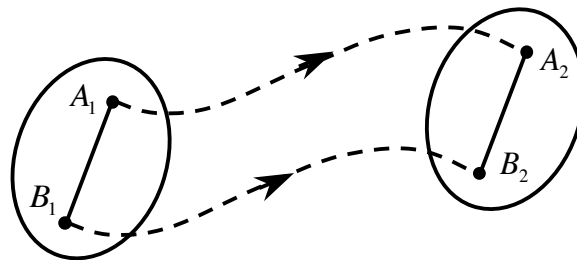
### Kinematics of Rigid Bodies

Types of plane motion:

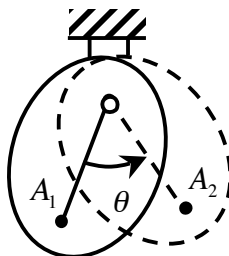
Rectilinear translation



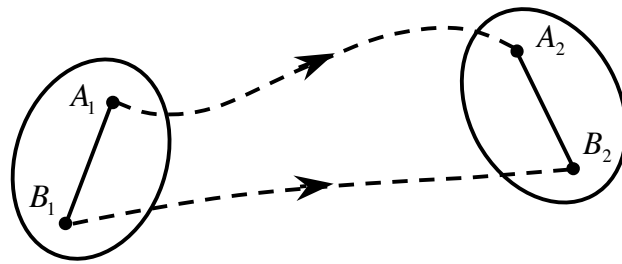
Curvilinear translation



Fixed-axis rotation



General plane motion



Combination of translation and rotation

### Translation

Recall analysis of “Motion Relative to Translating Reference Axes”:

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

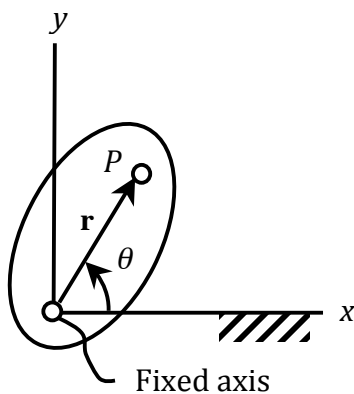
Now  $A$  and  $B$  are any two particles in the translating rigid body. Therefore,  $\mathbf{r}_{A/B} = \mathbf{constant\ vector}$ , and

$$\mathbf{v}_A = \mathbf{v}_B$$

$$\mathbf{a}_A = \mathbf{a}_B$$

### Rotation About a Fixed Axis

Recall analysis of “Angular Motion of a Line”:



$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Define angular velocity vector  $\boldsymbol{\omega}$  and angular acceleration vector  $\boldsymbol{\alpha}$  as follows:

$$\boldsymbol{\omega} = \omega \mathbf{k}$$

$$\boldsymbol{\alpha} = \alpha \mathbf{k}$$

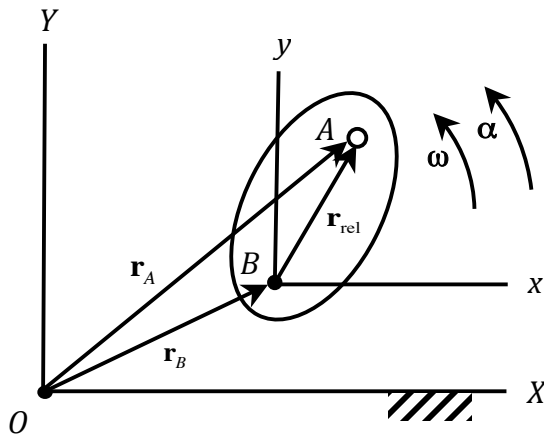
Then, the velocity of particle  $P$  is  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$  and the acceleration is

$$\begin{aligned} \mathbf{a} &= \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r} \end{aligned}$$

Note: In  $r - \theta$  coordinates,  $v_r = 0$        $v_\theta = \omega r$   
 $a_r = -\omega^2 r$        $a_\theta = \alpha r$

In  $t - n$  axes,  $v = \omega r$   
 $a_n = \omega^2 r$        $a_t = \alpha r$

General Plane Motion – Absolute and Relative Velocity and Acceleration



Axes  $x - y$  translate with their origin attached to particle  $B$ .

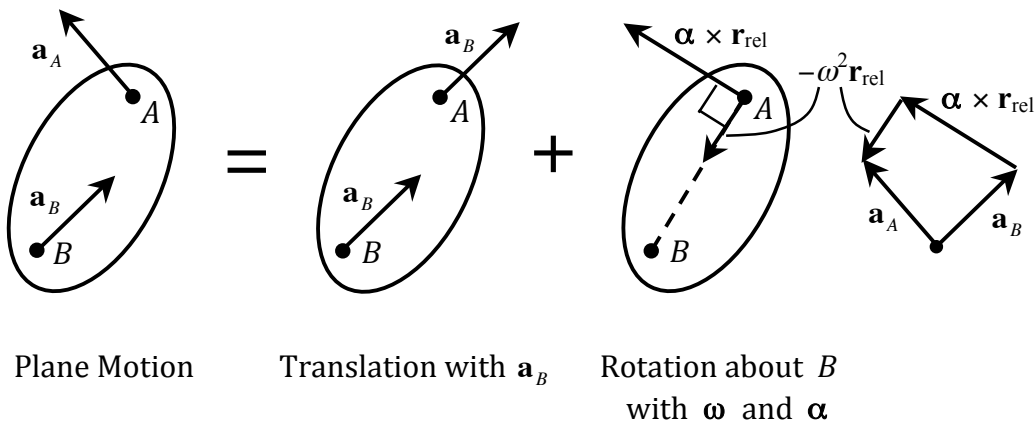
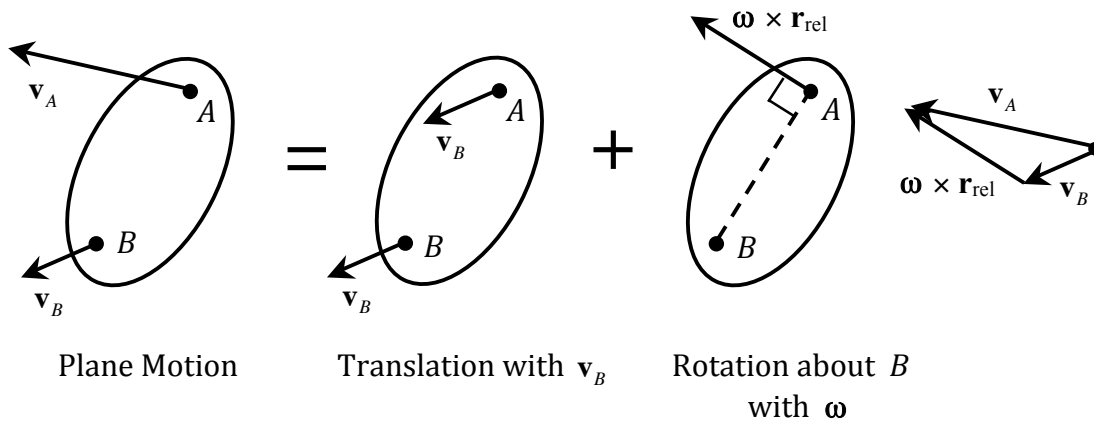
$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{rel}$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{rel}$$

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel})$$

$$= \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{rel} - \omega^2 \mathbf{r}_{rel}$$

Graphical interpretation:



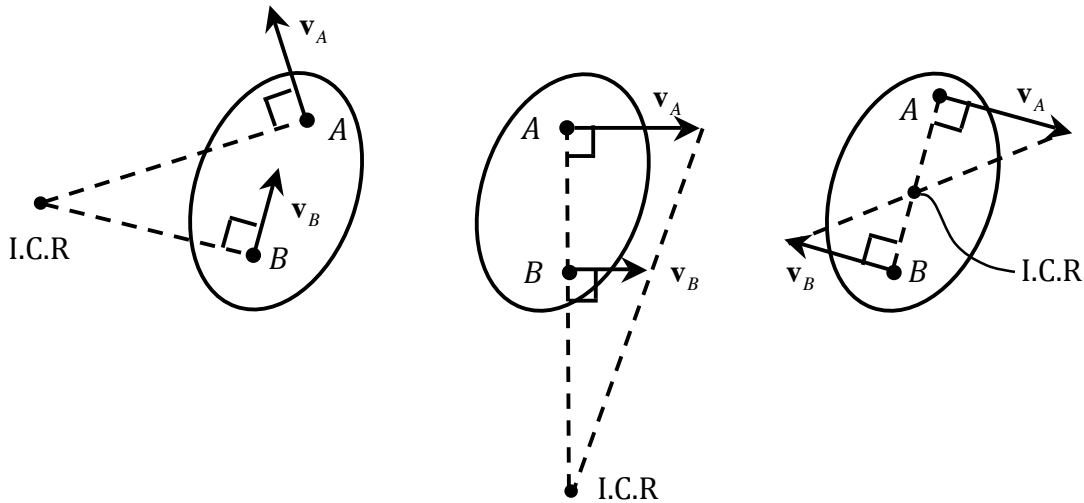


## Instantaneous Center of Rotation in Plane Motion

Suppose  $\mathbf{v}_B = \mathbf{0}$  in the previous analysis. Then,

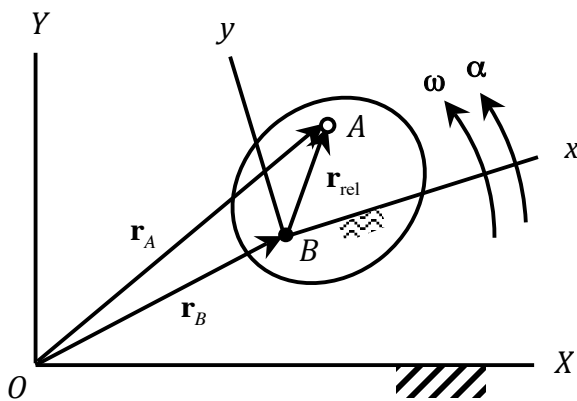
$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{rel}.$$

This result implies the body is rotating for an *instant* about point  $B$ . Such a point is called an *instantaneous center of rotation* (I.C.R.). Such a point can be determined, as follows, if the velocities of two different particles in a body are known.



Note: The location of the I.C.R. changes with time in general. Hence,  $\mathbf{a}_{ICR} \neq \mathbf{0}$  in general!

## Plane Motion of a Particle Relative to a Rotating Frame



Axes  $x - y$  are body-fixed axes, which have angular velocity  $\boldsymbol{\omega}$  and angular acceleration  $\boldsymbol{\alpha}$ .

Particle  $A$  moves relative to the body-fixed axes  $x - y$ . The relative position vector of  $A$  referenced to the  $x - y$  axes is

$$\mathbf{r}_{rel} = x\mathbf{i} + y\mathbf{j}$$



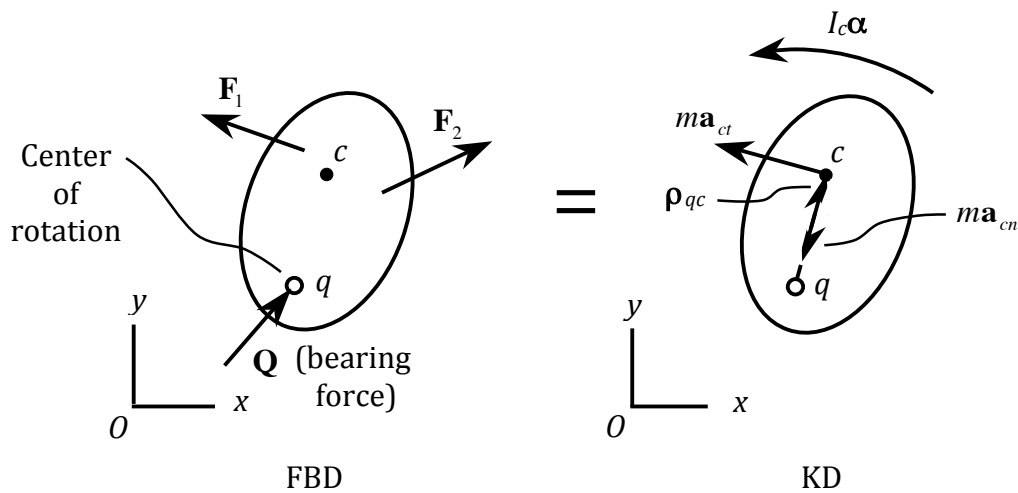
where:  $m$  = total mass  
 $c$  = center of mass  
 $I_c$  = mass moment of inertia about axis through  $c$  parallel to  $z$ -axis  
 $p$  = any moment center in  $x - y$  plane

In component form:  $\sum F_x = ma_{cx}$

$$\sum F_y = ma_{cy}$$

$$\curvearrowright \sum M_c = I_c \alpha$$

### Noncentroidal Rotation

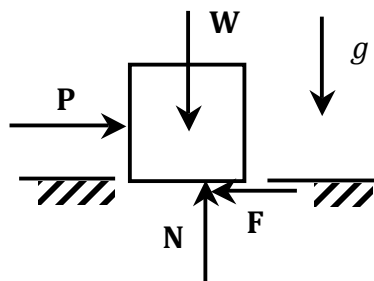


$$\sum M_q = I_c \alpha + \rho_{qc} \times ma_{ct}$$

$$= I_q \alpha$$

where  $I_q$  = mass moment of inertia about axis through  $q$  parallel to  $z$ -axis.

### Laws of Friction



Block is initially at rest when force  $\mathbf{P}$  is applied and its magnitude is progressively increased from zero. As long as

$$P = F < \mu_s N,$$

the block will *not* slide.

$\mu_s$  = coefficient of static friction.

When  $P = F = \mu_s N$ , the block starts to slide, and  $F$  becomes:

$$F = \mu_k N$$

where  $\mu_k =$  coefficient of kinetic friction,  
 $\mu_k < \mu_s$ .

### Kinetics of Rigid Bodies: Energy Methods

For a body in plane motion, the work done on the body by all external forces  $\mathbf{F}_i$  is

$$U_{1 \rightarrow 2} = \sum \int_{(\mathbf{r}_i)_1}^{(\mathbf{r}_i)_2} \mathbf{F}_i \cdot d\mathbf{r}_i$$

when the body is displaced from position 1 to position 2.

For a body in plane motion, the kinetic energy is

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$$

For a body in plane motion, the work done on the body by a couple  $M$  is

$$U_{1 \rightarrow 2} = \int_{\theta_1}^{\theta_2} M d\theta$$

when the body is displaced from position 1 to position 2.

$$\text{In general, } U_{1 \rightarrow 2} = \frac{1}{2} m (v_c)_2^2 + \frac{1}{2} I_c \omega_2^2 - \left[ \frac{1}{2} m (v_c)_1^2 + \frac{1}{2} I_c \omega_1^2 \right]$$

$$= T_2 - T_1$$

$$= \Delta T$$

$$\text{or } T_2 = T_1 + U_{1 \rightarrow 2}$$

If a gravitational force  $\mathbf{W}$  acts on the body, and/or a linearly-elastic spring force, then the work-energy equation can be written as:

$$U_{1 \rightarrow 2} = \Delta T + \Delta V_g + \Delta V_e$$

where  $U_{1 \rightarrow 2}$  now excludes gravitational and spring forces. If  $U_{1 \rightarrow 2}$  above is zero, *total mechanical energy is conserved*.

## Noncentroidal Rotation

$$T = \frac{1}{2} I_q \omega^2 \quad \text{where } q \text{ is the center of rotation.}$$

## Power developed by a couple $M$

$$\text{Power} = M\omega$$

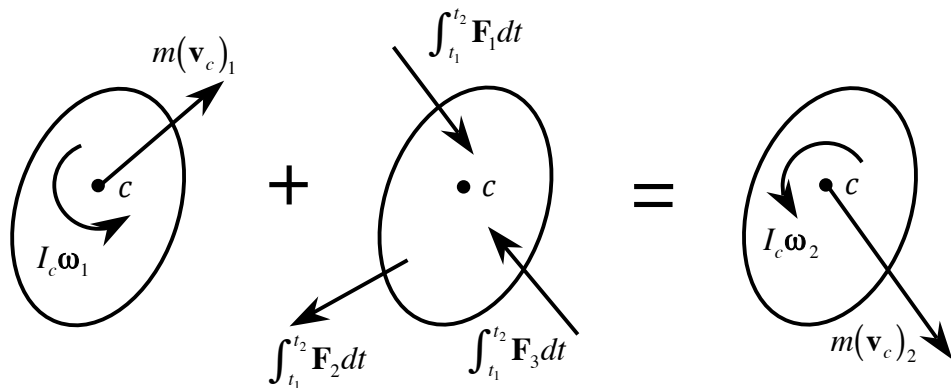
## **Kinetics of Rigid Bodies: Momentum Methods**

For a body in plane motion, the equations of impulse and momentum are;

$$m(\mathbf{v}_c)_1 + \sum \int_{t_1}^{t_2} \mathbf{F}_i dt = m(\mathbf{v}_c)_2$$

$$I_c \omega_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_c dt = I_c \omega_2$$

Graphical interpretation



Initial linear and angular momenta

Sum of all linear and angular impulses (about  $c$ ) from external forces

Final linear and angular momenta

If  $\sum \int_{t_1}^{t_2} \mathbf{F}_i dt = \mathbf{0}$ , then  $m(\mathbf{v}_c)_1 = m(\mathbf{v}_c)_2$  and we say *linear momentum is conserved*.

If  $\sum \int_{t_1}^{t_2} \mathbf{M}_p dt = \mathbf{0}$  about some point  $p$ , then

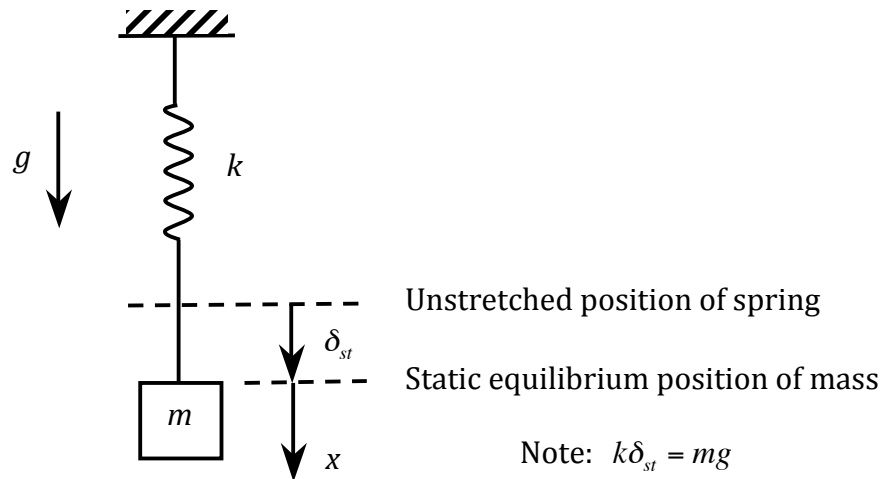
$$I_c \omega_1 + \rho_{pc} \times m(\mathbf{v}_c)_1 = I_c \omega_2 + \rho_{pc} \times m(\mathbf{v}_c)_2$$

and we say *total angular momentum about point  $p$  is conserved*.

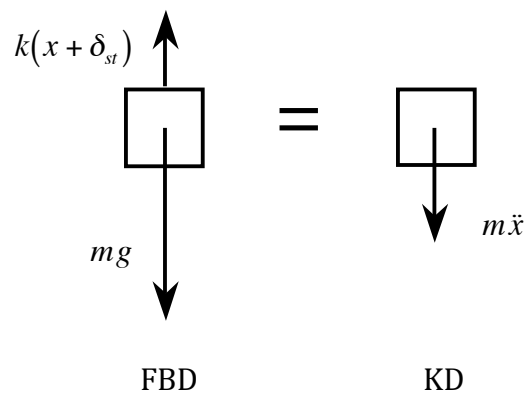
## Vibrations

When a mass moves back and forth about an equilibrium position, the motion is described as *vibration*.

A simple example of vibration is the motion of a mass  $m$  connected to a massless spring with spring constant  $k$ .



Mass  $m$  displaced from its equilibrium position



$$\sum F_x = m\ddot{x}$$

$$-k(x + \cancel{\delta_{st}}) + \cancel{mg} = m\ddot{x}$$

$$m\ddot{x} + kx = 0 \quad (1) \quad \text{(Equation of motion)}$$

Let  $\omega_n^2 = \frac{k}{m}$ , then Eq. (1) becomes:

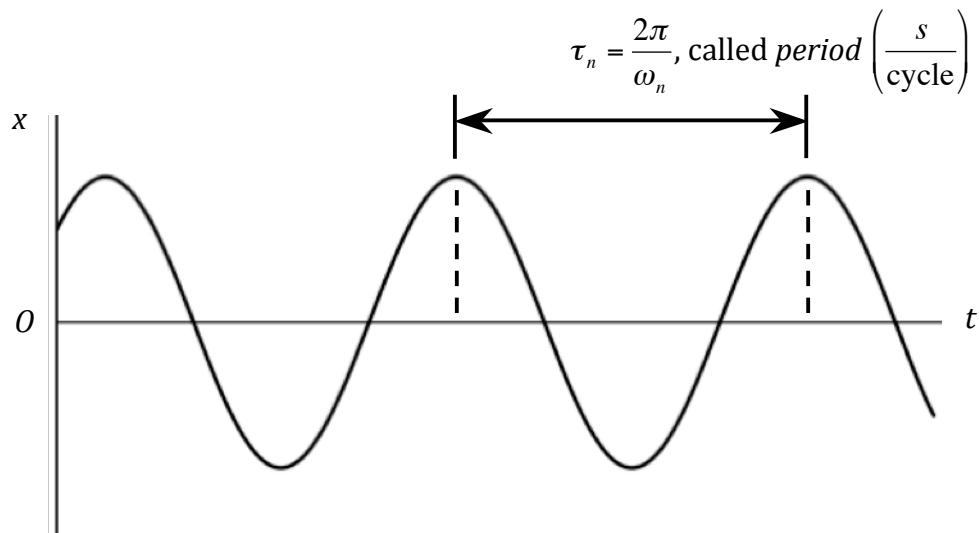
$$\ddot{x} + \omega_n^2 x = 0 \quad (2)$$

The solution of Eq. (2) is

$$x = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t)$$

where  $x(0)$  and  $\dot{x}(0)$  are initial conditions.

The motion is called *simple harmonic motion*, and  $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$  is known as the *natural (circular) frequency* (rad/s).



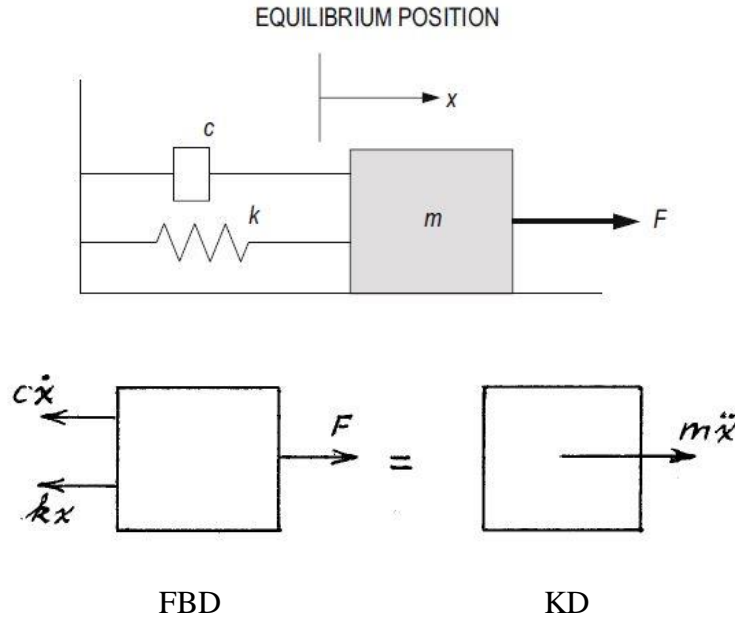
Since no forcing function appears in the equation of motion (1), the vibration above is called *free vibration*.

For an example of torsional vibration, see p. 127 in the 10.2 Handbook.

The following material is adapted from *FE Reference Handbook* 10.2

### Free and Forced Vibration

A single degree-of-freedom vibration system, containing a mass  $m$ , a spring  $k$ , a viscous damper  $c$ , and an external applied force  $F$  can be diagrammed as shown:



The equation of motion for the displacement of  $x$  is:

$$m\ddot{x} = -kx - c\dot{x} + F$$

or in terms of  $x$ ,

$$m\ddot{x} + c\dot{x} + kx = F$$

One can define

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$K = \frac{1}{k}$$

Then:

$$\frac{1}{\omega_n^2}\ddot{x} + \frac{2\zeta}{\omega_n}\dot{x} + x = KF$$

If the externally applied force is 0, this is a free vibration, and the motion of  $x$  is solved as the solution to a homogeneous ordinary differential equation.

In a forced vibration system, the externally applied force  $F$  is typically periodic (for example,  $F = F_0 \sin \omega t$ ). The solution is the sum of the homogeneous solution and a particular solution.

For forced vibrations, one is typically interested in the steady state behavior (i.e. a long time after the system has started), which is the particular solution.

For  $F = F_0 \sin \omega t$ , the particular solution is:

$$x(t) = X_0 \sin(\omega t + \phi)$$

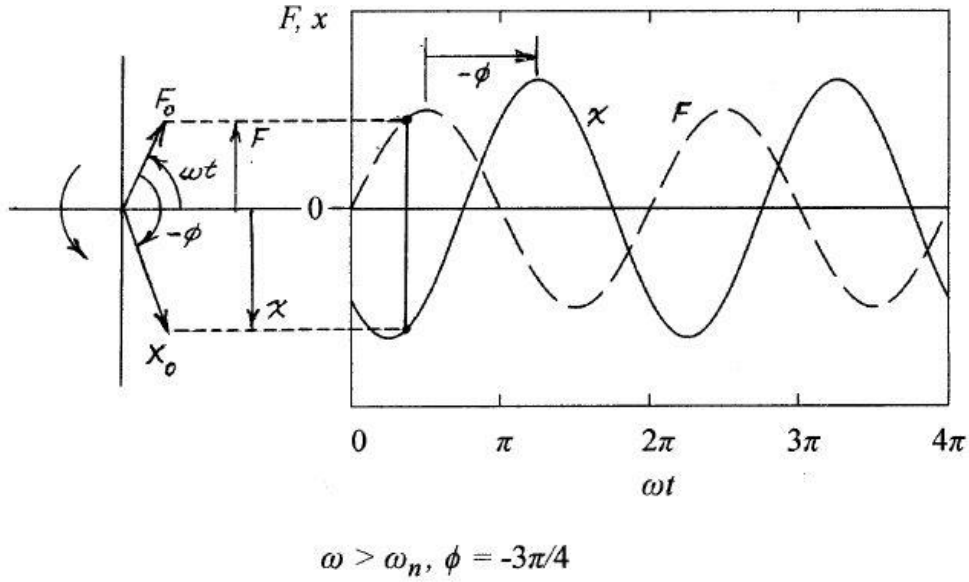
where

$$X_0 = \frac{KF_0}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

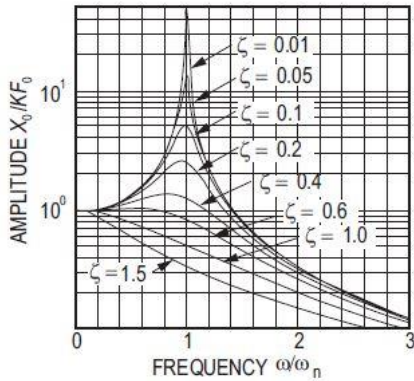


$$\phi = \tan^{-1} \frac{2\zeta\omega}{1 - \frac{\omega^2}{\omega_n^2}}$$

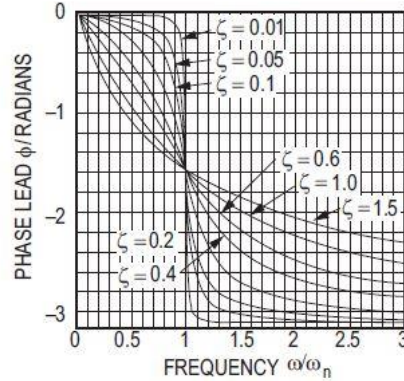
An example of a particular solution is:



The following figures provide illustrative plots of relative amplitude and phase, depending on  $\omega$  and  $\omega_n$ .



(a)



(b)

Steady state vibration of a force spring-mass system (a) amplitude (b) phase.

From Brown University School of Engineering, Introduction to Dynamics and Vibrations, as posted on [www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations\\_forced/vibrations\\_forced.htm](http://www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations_forced/vibrations_forced.htm), April 2019.