**Department of Mathematics**
**California State University Los Angeles**

**Master's Degree Comprehensive Examination in**

**NUMERICAL ANALYSIS**
**SPRING 2021**

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**Instructions:**

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, internet or cell phones are allowed.

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**PART A:** Do only **TWO** problems

1. Given a $3 \times 3$ matrix

$$A = \begin{pmatrix} 3 & 0 & k \\ 0 & 2 & k \\ k & k & 1 \end{pmatrix} \quad \text{with } k \in \mathbb{R}.$$

   (a) Determine *all* values $k$ for which $A$ is

      i. positive definite. [3 points]

      ii. strictly diagonally dominant. [3 points]

      iii. orthogonal. [3 points]

   (b) Suppose that the Jacobi's method is used for solving $A\mathbf{x} = \mathbf{b}$. Can the Gerschgorin Circle Theorem be used to determine all values of $k$ for which the Jacobi's iteration converges? Explain your answer. [8 points]

   (c) Find the spectral radius of the Gauss-Seidel iteration matrix for solving $A\mathbf{x} = \mathbf{b}$. Determine *all* values $k$ for which the Gauss-Seidel iteration converges. [8 points]

2. Consider a linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 11 & 9 \\ 6 & 9 & 26 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -4 \\ -17 \\ 2 \end{pmatrix}.$$

(a) Use Gaussian elimination to solve the above linear system. Identify the unit lower triangular matrix $L$ and the upper triangular matrix $U$ such that $A = LU$. [9 points]

(b) Briefly describe the partial and complete pivoting strategies, and give two reasons why pivoting is needed in practice. [5 points]

(c) Use your $LU$ decomposition from (a) to show that matrix $A$ has a decomposition $A = LDL^T$, where $D$ is a diagonal matrix. [4 points]

(d) Find a decomposition of the form $A = R^T R$, where $R$ is upper triangular with $r_{ii} > 0$ for $i = 1, 2, 3$.

  i. Is the decomposition unique? Explain your answer. [3 points]

  ii. If $r_{ii}$ is allowed to be negative, is the decomposition unique? If so, explain your answer. If not, find another upper triangular $R$ such that $A = R^T R$. [4 points]

3. Consider the $3 \times 3$ matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -1$.

(a) Find the eigenvectors associated with each eigenvalue. Is matrix $A$ semisimple? Explain your answer. [8 points]

(b) Show that the Power Method converges when applied to $A$ with the initial vector $\mathbf{q}_0 = (2, 0, 2)^T$. Identify the order of convergence and the converge ratio. [8 points]

(c) Will the Power Method converge to a dominant eigenvector of $A$ with any initial guess $\mathbf{q}_0$? If so, explain your reasoning. If not, give an example and explain. [4 points]

(d) Describe one step of Rayleigh Quotient iteration and explain how it improves convergence for power method. Give one advantage and one disadvantage of the Rayleigh quotient iteration in comparison to the Power Method. [5 points]

**PART B:** Do only **TWO** problems

1. Given the initial boundary value problem (IBVP):

$$\begin{aligned} U_t &= aU_{xx} - bU, && \text{for } 0 < x < 1,\ t > 0, \\ U(x,0) &= x(1-x) && \text{for } 0 \le x \le 1, \\ U(0,t) &= t, U(1,t) = 0 && \text{for } t > 0, \end{aligned}$$

where $a > 0$ and $b > 0$.

(a) Letting $h = \Delta x, k = \Delta t$ and $r = k/h^2$, write down the explicit scheme for the above PDE. Solve for $u_{i,j+1}$ and simplify as much as possible. [4 points]

(b) Construct the set of equations needed to advance the solution by one time step and express them in the matrix form $\mathbf{u}_{j+1} = A\mathbf{u}_j + \mathbf{f}_j$. Be sure to include the initial condition $\mathbf{u}_0$. [7 points]

(c) By computing the eigenvalues of the matrix $A$ from part (b), derive a condition on $r$ that guarantees the stability of the scheme. [7 points]

(d) Assuming the stability condition derived in part (c) holds, does the explicit method converge? Explain your answer. [3 points]

(e) Write down (no proof) a consistent, unconditionally stable finite difference scheme for the above PDE that is $O(h^2, k^2)$, if there is any. If there is none, explain why. [4 points]

2. Consider the boundary value problem:

$$\begin{aligned} &2U_{xx} + U_{yy} = xy, && \text{for } (x,y) \in (0,3) \times (0,4), \\ &U(x,y) = y(y-3), && \text{for } (x,y) \in \{0,3\} \times [0,4] \\ &U(x,0) = x(3-x), U(x,4) = 4, && \text{for } x \in (0,3) \end{aligned}$$

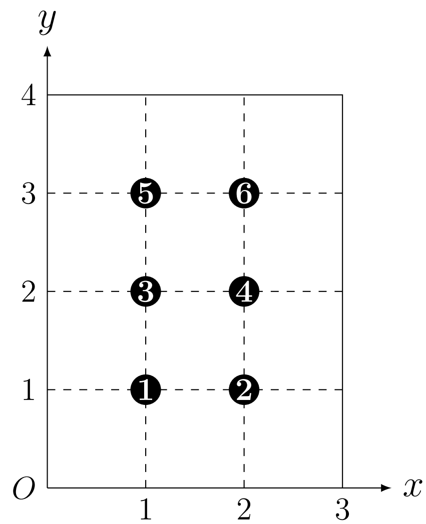
(a) Find the region in which this PDE is elliptic. [3 points]

(b) Write down the 5-point scheme to discretize the above PDE. What is the order of accuracy of this approximation? [4 points]

(c) Using the given ordering on the grid below with $h = k = 1$, write the system of linear equations that can be solved to approximate the solution to the above boundary value problem. Express your answer in the matrix equation $A\mathbf{u} = \mathbf{b}$. [9 points]

(d) Without performing any computation, discuss whether the system obtained in part (c) has a unique solution. [3 points]

(e) Suppose the boundary condition $U(x,4) = 4$ is replaced by $U_y(x,4) = 4$ for $x \in (0,3)$. Give two consistent finite difference approximations for this derivative boundary condition and write down the corresponding equations (in terms of the other nodes) to approximate the solution at the point $(2,4)$. [6 points]




3. (a) Determine all value(s) of $x$ for which the following PDE is hyperbolic: [4 points]

$$U_{xx} - 2xU_{xy} - (3x^2 - 2)U_{yy} = 0. \tag{1}$$

(b) Consider the PDE

$$\begin{aligned} &U_{xx} - 2xU_{xy} - 3x^2U_{yy} = 0, \ -\infty < x < \infty, y \geq 0 \\ &U(x,0) = x, \ -\infty < x < \infty \\ &U_y(x,0) = 0, -\infty < x < \infty. \end{aligned} \tag{2}$$

i. Suppose the characteristic curves that pass through the points $A(-1,0)$ and $B(-2,0)$ intersect at a point $P(x_P, y_P)$. Find the exact value of $x_P$ and $y_P$. [8 points]

ii. Letting $h = \Delta x$ and $k = \Delta y$ and using the central differences, write down a consistent scheme for the PDE (2). You need not prove consistency. [5 points]

iii. Construct a consistent finite difference approximation for $U_y$ that uses the values $u_{i,j}, u_{i,j+1}$, and $u_{i,j+2}$. You need not prove consistency. From the principal error term, determine the order of accuracy of this approximation. [8 points]