Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2020

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, internet or cell phones are allowed.

PART A: Do only **TWO** problems

1. (a) Let $A = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}$, where α, β and γ are real numbers with $\alpha > 0$ and $\beta > 0$.

    i. Give the conditions on α, β and γ under which A is strictly diagonally dominant. [2 points]

    ii. Find the eigenvalues (in terms of α, β and γ) of the Jacobi iteration matrix when applied to solve the system $A\mathbf{x} = \mathbf{b}$ for some vector $\mathbf{b}$. [6 points]

    iii. Under what conditions on α, β and γ will the Jacobi iteration converge? [3 points]

   (b) Given the system of linear equations $B\mathbf{x} = \mathbf{b}$, where B is a strictly diagonally dominant $n \times n$ matrix and $\mathbf{b}$ is an arbitrary n-vector. Prove that the Jacobi iteration matrix G_J for this system satisfies $\|G_J\|_\infty < 1$. [9 points]

   (c) Determine whether the following statement is true or false:
   If a square matrix C is positive definite, then it is diagonally dominant.
   If it is true, prove the statement. If it is false, give a counter example. [5 points]

2. (a) The Power Method and the QR Method are techniques for finding approximations to the eigenvalues of a square matrix A.

    i. State the sufficient conditions for the convergence of the Power Method. [4 points]

    ii. Briefly describe the QR algorithm for finding the eigenvalues of A. [5 points]

    iii. Let $A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$. Perform one iteration of the QR Method to approximate the eigenvalues of A by letting $A_0 = A$. [8 points]

iv. Give one advantage of the Power Method over the QR Method. [2 points]

(b) Find the 3×3 matrix B that has eigenvalues $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2$ and the corresponding orthogonal eigenvectors $\mathbf{v}_1 = (1/\sqrt{2}, 0, 1/\sqrt{2})^T, \mathbf{v}_2 = (0, 1, 0)^T, \mathbf{v}_3 = (-1/\sqrt{2}, 0, 1/\sqrt{2})^T$. [6 points]

3. (a) Let $A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix}$.

i. Use the Gaussian elimination with partial pivoting to write A in the form $PA = LU$, where P is a permutation matrix, L is a unit lower triangular, and U is an upper triangular matrix. [6 points]

ii. Use the result from part (i) to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (1, 0, 5)^T$. [6 points]

(b) Let B be an invertible $n \times n$ matrix. Show that if B can be factored as $B = LU$, where L is a unit lower triangular and U is an upper-triangular, then this factorization is unique. [7 points]

(c) Compare (do not calculate) the flop-counts of Gaussian elimination with no pivoting, partial pivoting and complete pivoting for a general $n \times n$ system of linear equations. [6 points]

PART B: Do only **TWO** problems

1. Consider the elliptic partial differential equation (PDE)

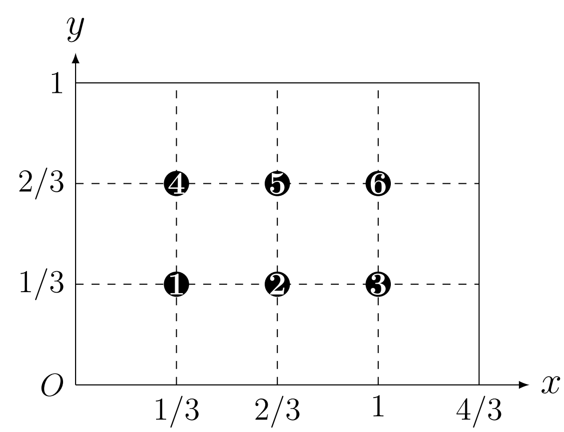
$$\begin{aligned} U_{xx} + U_{xy} + U_{yy} &= f(x, y), &&\text{for } (x, y) \in (0, 4/3) \times (0, 1), \\ U(x, y) &= 0, &&\text{for } (x, y) \in \{0, 4/3\} \times [0, 1] \cup [0, 4/3] \times \{0, 1\}. \end{aligned}$$

   (a) On a regular grid with $\Delta x = \Delta y = h$, find a consistent finite difference approximation to U_{xy} by applying central difference in the two coordinate directions. You need not prove that the approximation is consistent. [5 points]

   (b) Use the standard 5-point stencil for $U_{xx} + U_{yy}$ and the approximation in (a) to discretize the elliptic PDE on the grid below with the given ordering. First, write 6 linear equations by discretizing the PDE at each interior grid point. Then, express the scheme in the form of a linear system $A\mathbf{u} = \mathbf{f}$. [9 points]

   (c) Explain the process of finding the local truncation error of the finite difference scheme in (b). You need not find the expression of the local truncation error. [3 points]

   (d) Use the scheme developed in (b) to explain the concepts of *consistency* and *convergence*. Does consistency imply convergence? Explain your answer. [8 points]



2. Given the parabolic PDE

$$\begin{aligned} U_t - U_{xx} &= 0 &&\text{for } 0 < x < 1,\ t > 0, \\ U(x, t) &= 0 &&\text{for } x \in \{0, 1\},\ t > 0, \\ U(x, 0) &= x(1 - x) &&\text{for } 0 \le x \le 1. \end{aligned}$$

   (a) Write a consistent finite difference approximation to the above PDE including the initial and boundary conditions. No need to show that the scheme is consistent. [5 points]

(b) Use your scheme in (a) to explain the following concepts:

i. consistency [2 points]

ii. stability [2 points]

iii. convergence [2 points]

(c) Determine the stability of your scheme in (a). If your scheme is stable, determine whether it is conditionally stable or unconditionally stable. If your scheme is unstable, write a stable scheme. [8 points]

(d) Can a scheme be consistent but not stable? If so, give an example. If not, explain why not. [3 points]

(e) Can a consistent scheme be stable but not convergent? If so, give an example. If not, explain why not. [3 points]

3. (a) Consider the wave equation

$$U_{tt} = 4\,U_{xx}, \quad x \in \mathbb{R},\ t > 0,\ a \in \mathbb{R}.$$

i. Determine the two characteristic directions. Calculate and sketch the two characteristic curves passing through the point $(2, 2)$ in the xt-plane. [5 points]

ii. Suppose the following two schemes are derived for approximating the equation

$$\begin{aligned} \text{Scheme (A):} \quad & u_j^{n+1} = a u_{j+1}^n + b u_j^n + c u_{j-1}^n + d u_j^{n-1}; \\ \text{Scheme (B):} \quad & u_j^{n+1} = \alpha u_{j+1}^{n+1} + \beta u_j^n + \gamma u_{j-1}^{n+1} + \delta u_j^{n-1}, \end{aligned}$$

where u_j^n is the approximation to $U(x_j, t_n)$. Consider a fixed ratio $k/h = 1$, where $h = \Delta x$ and $k = \Delta t$. Find the numerical domain of dependence of the grid point (x_j, t_{n+1}) for both schemes. What can be said about the convergence of schemes (A) and (B)? Justify your answer. [8 points]

(b) Consider the PDE

$$U_t = 2\,U_x, \quad 0 < x < 1,\ t > 0$$

with periodic boundary condition, that is, $U(0, t) = U(1, t)$.

i. Consider the finite difference scheme

$$\frac{u_j^{n+1} - u_j^n}{k} = 2\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

for approximating the PDE. Here u_j^n approximates $U(x_j, t_n)$. Show that the scheme is unstable when k is chosen proportional to h. [7 points]

ii. Write a stable scheme when k is chosen proportional to h. No need to show that the scheme is stable. [5 points]