Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination

NUMERICAL ANALYSIS
FALL 2010

Instructions: Do **2** problems from Part **A** AND **2** problems from Part **B**

PART A (Do two problems)

A-1 Consider the following elliptic boundary-value problem in the region $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$:

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < 1, 0 < y < 1$$
$$u(0, y) = -y^2, \; u(1, y) = 1 - y^2, \qquad 0 \le y \le 1$$
$$u(x, 0) = x^2, \; u(x, 1) = x^2 - 1, \qquad 0 \le x \le 1$$

a. Show that $u(x, y) = x^2 - y^2$ is an exact solution of this boundary-value problem. [5%]

b. What are the maximum and minimum values achieved by the solution, u, to the given boundary-value problem in the region D? At what points (x, y) do they occur? [4%]

c. With $\Delta x = \Delta y = 1/3$, use the usual five-point difference scheme for approximating the given PDE to obtain a system of linear equations for solving this problem. Express this system in the form $A\mathbf{u} = \mathbf{b}$, where A is a 4×4 matrix. [12%]

d. Explain why the solution to your difference approximation in *part* ***c*** is unique. [4%]

A-2 Consider the following difference approximation to

$$u_t = c\,u_{xx} \quad (\text{where } c > 0) \qquad 0 < x < 1,\ t > 0$$
$$u(x, 0) = x(1 - x) \qquad 0 < x < 1$$
$$u(0, t) = 0,\ u(1, t) = 0 \qquad t > 0$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = c\left[\frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right], \quad \text{where } u_{i,j} = u(ih, jk)$$

a. Is this an explicit or implicit scheme? [2%]

b. If this scheme is written as $B\mathbf{u}_{j+1} = C\mathbf{u}_j$, where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})$, taking $r = k/h^2$, determine the matrices B and C. [8%]

c. Use the Neumann (Fourier) method to determine all values of $r = k/h^2$ for which this scheme is stable. [12%]

d. Assume that this scheme is consistent with the given PDE. Is the scheme convergent? Why or why not? [3%]

A-3 Given the hyperbolic initial value problem

$$\begin{cases} u_{tt} - 9u_{xx} = 0, & (-\infty \le x \le \infty,\ t > 0) \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & (-\infty \le x \le \infty) \end{cases}$$

where f(x) and g(x) are given continuous functions.

a. Derive an explicit finite difference scheme with $u_{i,j} = u(ih, jk)$ ($h = \Delta x$, $k = \Delta t$, and taking $r = k/h$), solved for $u_{i,j+1}$, for obtaining approximate solutions to this problem. Explain how to use this scheme to compute values along the "first row"; that is, when $t = k$. [12%]

b. Find the characteristic curves of the given PDE through the point (0, ½). [6%]

c. State the Courant-Friedrichs-Levy (C.F.L) condition. What values of $r = k/h$ will ensure that the C.F.L condition will be satisfied for your scheme? If r is less than this value, what conclusion can you draw concerning your scheme? [7%]

PART B (Do two problems)

B-1 Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, where a, b, c are real numbers with $a > 0,\ c > 0$.

a. Find the spectral radius of the Jacobi iteration matrix for A. [8%]

b. Using the results of part **a**, give conditions on a, b, and c that ensure that Jacobi iteration will converge for the linear system A**x** = **b** (**b** arbitrary). [3%]

c. Give necessary and sufficient conditions on a, b, and c that ensure that the matrix A is diagonally dominant. [3%]

d. Show that A is positive definite if and only if $ac - b^2 > 0$. [5%]

e. Is each statement *true* or *false* for this matrix A? [3% each]

i. If A is diagonally dominant, then it is positive definite.
ii. If A is positive definite, then it is diagonally dominant.

B-2 **a.** Let $A = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 7 & 7 \\ -2 & -7 & 5 \end{bmatrix}$.

Find the LU decomposition of A, A = LU, where L is unit lower-triangular and U is upper-triangular. [8%]

b. Use your result from part **a** to find the LDU factorization of A, where L is unit lower-triangular, D is diagonal, and U is unit upper-triangular. [3%]

c. Let B be an $n \times n$ matrix and suppose we have obtained the LU factorization of B. Determine the number of multiplications / divisions it takes to solve U**x** = **c** by backward substitution, where **c** is an arbitrary n-vector. [6%]

d. Let B be an $n \times n$ matrix. Show that if B can be factored as B = LU, where L is unit lower-triangular and U is upper-triangular, then this factorization is unique. [8%]

B-3 The matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues 2 and 4 and corresponding eigenvectors $[s\ \ -s]^T$ and $[s\ \ s]^T$, respectively, where $s \neq 0$.

a. Apply *two* iterations of the Power Method to the matrix A with initial vector $\mathbf{x}^{(0)} = [1, 0]^T$ to obtain $\mathbf{x}^{(2)}$, an approximation to the eigenvector of A corresponding to eigenvalue 4. [6%]

b. Will the Power Method converge in this case? Explain why or why not. [4%]

c. Give an example of an initial vector for which the Power Method will *not* converge. [3%]

d. Obtain the QR factorization of the matrix A. [6%]

e. Obtain the first iterate in the QR method for A. [6%]