Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2014

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.

- No calculators and no cell phones.

- Closed books and closed notes.

PART A: Do only **TWO** problems

1.

a. [6 pts] Use Gaussian elimination with partial pivoting (GEPP) to find matrices L and U such that U is upper triangular and L is lower triangular with $|l_{ij}| \leq 1 \; \forall \; i > j$ and $LU = \hat{A}$ where $\hat{A}$ can be obtained from A by making row interchanges.

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}$$

b. [6 pts] Use the LU decomposition found in (a) to solve the system $Ax = b$.

c. [4 pts] Show that under the general conditions stated in 1(a), $A = P^T LU$, where P is a suitable permutation matrix. (i.e., Prove the general result and not the one for the given matrix A)

d. [4 pts] State and prove the corresponding general result for Gaussian elimination with complete pivoting.

e. [5 pts] Compute the flops for GEPP applied on an n x n matrix.

2.

a. [6 pts] Find the eigenvalues and eigenvectors of A.

$$A = \begin{bmatrix} 0.99 & 0 \\ 0 & 1 \end{bmatrix}$$

b. [6 pts] Perform direct power iteration (two iterations) on A starting with $q_0 = [1 \quad 1]^T$. Derive a general expression for q_j.

c. [4 pts] How many iterations are required to obtain $\|q_j - v_1\|_\infty / \|v_1\|_\infty < 10^{-6}$ where v_1 is the eigenvector associated with the dominant eigenvalue.

d. [6 pts] Show that the power method fails to converge for B starting with $q_0 = [a \quad b]^T where\ a \geq 0, b \geq 0\ and\ a \neq b.$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

e. [3 pts] Explain why the sequence obtained in 2 (d) fails to converge.

3. Consider the system $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a. [8 pts] Show that the Jacobi iteration scheme for this system converges.

b. [4 pts] Find k such that $\|\mathbf{x}^{(k)} - \mathbf{x}\|_2 \leq 10^{-3} \|\mathbf{x}^{(0)} - \mathbf{x}\|_2$, where $x^{(k)}$ is the k-th iterate of the Jacobi iteration.

c. [2 pts] Is it possible to tell if Gauss-Seidel iteration scheme for this system converges from the result in a? Explain.

d. [8 pts] Show that the Gauss-Seidel iteration scheme for this system converges.

e. [3 pts] Find k such that $\|\mathbf{x}^{(k)} - \mathbf{x}\|_2 \leq 10^{-3} \|\mathbf{x}^{(0)} - \mathbf{x}\|_2$, where $x^{(k)}$ is the k-th iterate of the Gauss-Seidel iteration.

PART B: Do only TWO problems

1. (a) [6 pts] Find and sketch the regions of hyperbolicity, ellipticity and parabolicity for the PDE:

$$u_{xx} + 3xu_{xy} + (x + y)u_{yy} = u. \tag{1}$$

   (b) [5 pts] Derive a consistent finite difference approximation for the term $3xu_{xy}$ that is second order accurate. You need not prove that your approximation is consistent.

   (c) [5 pts] Find the partial derivative which is approximated by the finite difference

$$\frac{u_{i,j+2} + 2u_{i,j+1} - 2u_{i,j-1} - u_{i,j-2}}{8(\Delta t)}$$

   and find the associated local truncation error.

   (d) Given the hyperbolic PDE:

$$\begin{aligned} &u_{xx} - 4x^2 u_{yy} = 0. \\ &u(x, 0) = x^2, \quad -\infty \leq x \leq \infty \\ &u_y(x, 0) = 0, \quad -\infty \leq x \leq \infty. \end{aligned}$$

      i. [3 pts] Find the slope of the characteristic curves of this PDE.

      ii. [6 pts] Suppose the characteristic curves that pass through the points $A(0.3, 0)$ and $B(0.4, 0)$ intersect at a point $R(x_R, y_R)$. Find the exact values of x_R and y_R.

2. Given the following PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u \tag{2}$$

with initial and boundary conditions

$$\begin{aligned} u(0, x) &= f(x), \text{ for } 0 < x < 1 \\ u(t, 0) &= u(t, 1) = 0, \text{ for } t > 0. \end{aligned}$$

   (a) [4 pts] Write down the Crank-Nicolson approximation for the above PDE.

   (b) [6 pts] Express the set of equations needed to advance the solution by one time step in the form

$$A_1 u_{j+1} = A_2 u_j.$$

   Find the matrices A_1 and A_2.

   (c) [6 pts] Show that A_1 and A_2 commute, that is, $A_1 A_2 = A_2 A_1$.

   (d) [9 pts] Using A_1 and A_2 you found in part (b), show that the method is unconditionally stable.

3. Consider the following boundary-value problem:

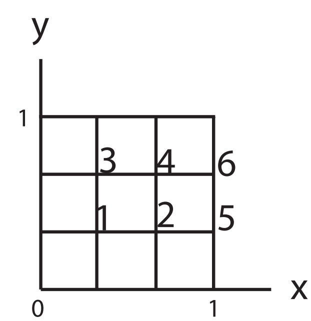
$$u_{xx} + 3u_{yy} = 24, \quad \text{on } [0,1]\times[0,1]$$
$$u(x,0) = 9x^2, \quad u(0,y) = y^2$$
$$u(x,1) = (3x-1)^2, \quad u(1,y) = (3-y)^2$$

(a) [4 pts] Show that $u = (3x-y)^2$ is a solution to the above boundary value problem.

(b) [4 pts] Find the maximum value of u on $[0,1]\times[0,1]$. At what points (x,y) do they occur?

(c) [4 pts] Write down a consistent 5-point finite difference approximation for the above PDE.

For parts (d) and (e), use the following labeling for the nodes:



(d) [6 pts] Using $\Delta x = \Delta y = 1/3$, write down the finite difference equation to approximate the solution u at the point $(1/3, 1/3)$ in terms of the other nodal values. Simplify your answer as much as possible.

(e) [7 pts] Suppose one of the boundary conditions is changed from $u(1,y) = (3-y)^2$ to $u_x(1,y) = 1$. Write down the finite difference equation to approximate the solution u at the point $(1, 2/3)$ in terms of the other nodal values. Simplify your answer as much as possible.