Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2017

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, or cell phones may be used during this exam.

PART A: Do only **TWO** problems

1. (a) Let $A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 8 & 0 \\ -2 & 0 & 24 \end{pmatrix}$.

   i. [6 points] Find the LU factorization of A, where L is a unit lower triangular and U is an upper triangular matrix.

   ii. [5 points] Find the LDL^T factorization of A, where L is a unit lower triangular and D is a diagonal matrix.

   iii. [6 points] Find the $R^T R$ factorization of A, where R is an upper triangular matrix with positive diagonal entries. What is this factorization called?

   (b) [8 points] Show that an arbitrary $n \times n$ symmetric matrix S is positive definite if and only if it can be factored into $R^T R$, where R is an upper triangular matrix with positive diagonal entries.

2. (a) Let

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Note that B is symmetric and nonsingular ($\det(B){=}{-1}$).

i. In one sentence each, give a reason for your answer to the following questions: [3 points each]

(i.1.) Is B diagonalizable?

(i.2.) Is B positive definite?

(i.3.) Is B an orthogonal matrix?

ii. [10 points] By finding the spectral radius of its iteration matrix, determine whether or not Gauss-Seidel iteration converges for the linear system $B\mathbf{x} = \mathbf{b}$, where B is given above and $\mathbf{b}$ is an arbitrary vector.

(b) [6 points] Show that if an *arbitrary* 3×3 matrix C with positive entries is strictly diagonally dominant, then Jacobi iteration converges for the linear system $C\mathbf{x} = \mathbf{d}$, for all vectors $\mathbf{d}$. (Hint: Use Gershgorin's circle theorem on the Jacobi iteration matrix.)

3. (a) [5 points] Let Q be an orthogonal matrix. Fill in each of the following blanks:

i. $Q^T Q = \ldots\ldots$

ii. $\det(Q) = \ldots\ldots$

iii. Condition number $\kappa(Q) = \ldots\ldots$

iv. $\|Q\|_\infty = \ldots\ldots$

v. If $\mathbf{c}_1$ and $\mathbf{c}_2$ are columns of Q, then the inner product $\mathbf{c}_1^T \mathbf{c}_2 = \ldots\ldots$

(b) Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i. [7 points] Find the QR decomposition of A; that is, find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

ii. [4 points] Obtain the first iterate of the QR method for finding the eigenvalues of A.

iii. [6 points] Perform two iterations of the Power Method to the matrix A with initial vector $\mathbf{x}^{(0)} = (1, 0, 0)^T$ to obtain $\mathbf{x}^{(2)}$.

(c) [3 points] Give one advantage and one disadvantage of Power Method over QR method when applied to solve the eigenvalue problem.

PART B: Do only **TWO** problems

1. Consider the elliptic boundary value problem (BVP):

$$U_{xx} + U_{yy} = 6 \quad \text{for } 0 < x < 3, 0 < y < 3$$
$$U(x,0) = 3x^2,\; U(x,3) = 6x + 3x^2 \quad \text{for } 0 \le x \le 3$$
$$U(0,y) = 0,\; U(3,y) = 6y + 27 \quad \text{for } 0 \le y \le 3$$

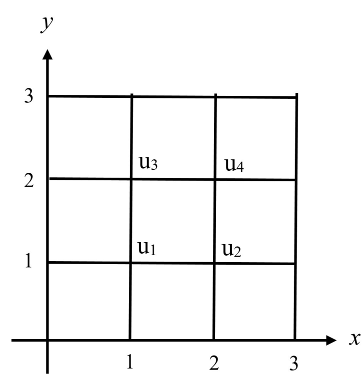
   (a) [4 points] An exact solution of this BVP has the form $U(x, y) = Axy + Bx^2$, where A and B are constants. Find A and B.

   (b) [3 points] Is there more than one correct answer to Part (a)? Briefly explain why or why not.

   (c) [10 points] Use the usual 5-point approximation to $U_{xx} + U_{yy}$ to get a scheme that approximates the given partial differential equation. Then, find the system of 4 linear equations in the 4 unknowns $u_1 = u(1, 1), u_2 = u(2, 1), u_3 = u(1, 2), u_4 = u(2, 2)$ that results from applying this scheme to the given BVP on the grid shown below. (Note that $h = 1$).

   (d) [3 points] Explain why the solution to the system of equations in Part (c) is unique.

   (e) [5 points] Now suppose that for $x = 0$ $(0 < y < 3)$, we replace the given boundary values $(U(0, y) = 0)$ with the boundary condition $U_x(0, y) = 0$. Approximating U_x by central differences, express the approximate solution $u(0, 1)$ in terms of $u(0, 2)$ and $u(1, 1)$.



2. Consider the following difference approximation to the parabolic partial differential equation (PDE) $U_t = cU_{xx}$ $(0 < x < 1, t > 0)$, where $c > 0$ is a constant.

$$-cru_{i-1,j+1} + (1 + 2cr)u_{i,j+1} - cru_{i+1,j+1} = u_{i,j}. \quad (1)$$

Here, $u_{i,j} = u(i\Delta x, j\Delta t) = u(ih, jk)$ and $r = k/h^2$.

(a) [2 points] Is this difference scheme *explicit* or *implicit*?

(b) [5 points] Suppose that for $x = 0$ and $x = 1$, $U(x,t) = 0$ for all $t > 0$. Letting

$$\mathbf{v}_j = [u_{1,j}, u_{2,j}, \ldots, u_{N-1,j}]^T, \quad (2)$$

find matrices B and C (of order $N - 1$) so that the scheme takes the form

$$B\mathbf{v}_{j+1} = C\mathbf{v}_j. \quad (3)$$

(c) [10 points] Use EITHER the von Neumann (Fourier) method OR the matrix (eigenvalue) method to determine the values of $r = k/h^2$ for which this scheme is stable.

(d) [3 points] The truncation error for this scheme is $T(h,k) = O(k) + O(h^2)$. Explain why this fact implies that the scheme is consistent with the given PDE.

(e) [5 points] Define the notion of *convergence* of a difference approximation for a PDE and, with the help of previous parts of this problem, explain why the given scheme (together with appropriate initial and boundary conditions) converges for the values of r found in part (c).

3. Consider the PDE

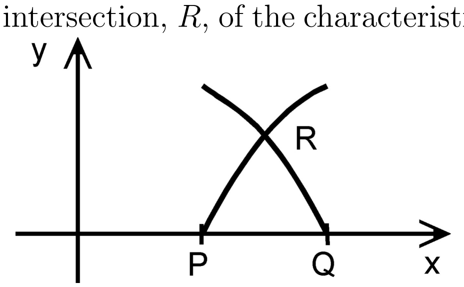
$$U_{xx} + xU_{xy} - 2x^2U_{yy} = 0$$

with initial data given on $y = 0$.

(a) [4 points] Determine all values of x for which the given PDE is hyperbolic.

(b) [5 points] Determine the two characteristic directions (slopes), dy/dx, for the given PDE at a general point (x, y).

(c) [6 points] Using the result of part (b), find the **exact values** of the coordinates of the point of intersection, R, of the characteristic curves through the points $P(1, 0)$ and $Q(2, 0)$.



(d) [2 points] Give the interval of dependence for $U(x,y)$ at the point R of part (c).

(e) [4 points] Derive a consistent finite difference approximation for the xU_{xy} term. (You need **not** show that your approximation is consistent.)

(f) [4 points] Suppose we approximate the given PDE by a consistent explicit difference scheme with $h = k = 1$. Referring to the CFL condition, explain why or why not this scheme converges at the point R of part (c).