Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
Fall 2018

Do exactly 2 problems from part I AND exactly two problems from part II.
No notes AND no calculators are allowed.

Part I: (Do EXACTLY two problems)

1. a. Use Gaussian elimination with partial pivoting to write the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$$

in the form $PA = LU$, where P is a permutation matrix, L is a unit lower triangular with $|l_{ij}| \leq 1$ for $i \neq j$, and U is an upper triangular matrix. (8 pts)

b. Use the result from part (a) to solve the system $Ax = b$ where b= $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$. (6 pts)

c. Give one advantage of Gaussian elimination with partial pivoting vs. Gaussian elimination without row interchanges. (3 pts)

d. State the condition(s) under which a triangular matrix is non-singular. (3 pts)

e. Prove that the product of two lower triangular matrices is lower triangular. (5 pts)

2 a. Let A and B be two $n \times n$ orthogonal real matrices.

i. Must the sum $A + B$ be orthogonal? Prove or give a counter example. (5 pts)

ii. Must the product AB be orthogonal? Prove or give a counter example. (5 pts)

b. Perform one iteration of the QR method for finding the eigenvalue of matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. (8 pts)

c. Let $\{A_k\}$ be the sequence of matrices generated by QR algorithm. Show that A_k and A_{k+1} have the same eigenvalue. (7 pts)

3 Let

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}.$$

(a) In one sentence each, give a reason for your answer to the following questions (3 pts each)
   i. Is A singular?
   ii. Is A positive definite?
   iii. Is A diagonalizable?

(b) Perform one iteration of the Gauss-Seidel method for solving the system Ax=b, with starting vector $x^{(0)}=(1,1,1)^T$ and $b=(1,2,-1)^T$, to find $x^{(1)}$. (8 pts)

(c) Show that if an arbitrary 3x3 matrix B with positive entries is strictly diagonally dominant, then the Jacobi iteration converges for the linear system Bx=c, for any vector c. (Hint: use Gershgorin's circle theorem on the Jacobi iteration matrix). (8 pts)

Part II: (Do EXACTLY two problems)

1. Consider the initial boundary value problem

$$u_t = a u_{xx} \text{ for } x \in (0,1), t > 0,$$
$$u(0,t) = u(1,t) = 0 \text{ for } t > 0,$$
$$u(x,0) = u_0(x) \text{ for } x \in [0,1],$$

where $a > 0$ is a constant. This problem can be solved using the scheme

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = a\,\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}$$

where $U_i^n \approx u(x_i, t_n) = u(i\Delta x, n\Delta t)$.

a. Determine if this scheme is implicit or explicit. Explain your answer. (5 pts)

b. Show that the scheme is consistent, and use this to find the order of accuracy of this scheme. (9 pts)

c. Use Von Neumann analysis to determine the values of $r = \frac{\Delta t}{\Delta x^2}$ for which the scheme is stable. Does this imply convergence? Explain why or why not. (11 pts)

2. Consider the 2D Poisson equation

$$-\Delta u = f(x,y) \quad \text{in } \Omega = (0,1)^2.$$

a. If the Dirichlet boundary conditions $u(x,y) = g(x,y)$ are prescribed for the above Poisson equation on the boundary $\partial\Omega$. Use the standard 5-point finite difference approximation to obtain a scheme for solving the boundary value problem on a rectangular grid with $\Delta x = \Delta y = \frac{1}{3}$. Write the linear system in the matrix form $A\,\vec{u} = \vec{b}$. (10 pts)

b. Show that the system obtained from part (1) has a unique solution. (5 pts)

c. Assume that the Neumann boundary condition $u_x(x,y) = p(x,y)$ is prescribed along part of the boundary $\Gamma = (0,1)\times\{0\} \subset \partial\Omega$. Use the second order central difference approximation for discretizing the Neumann boundary conditions and write the linear system in the matrix form. How does the resulting linear system change compared to part (a)? (10 pts)

3. Consider the advection equation

$$u_t = 3\,u_x \text{ for } x \in \mathbb{R}, t > 0$$

with initial condition $u(x,0) = \sin(\pi x)$.

a. Find the exact value of u at the point P that lies on the characteristic curve through $\left(-\frac{1}{3}, 0\right)$. (3 pts)

b. Find the exact solution to the problem by using the method of characteristics. (5 pts)

c. Write down a scheme that uses first order finite difference approximation in both space and time to discretize the equation. Find the CFL condition for this scheme. Does this CFL condition imply stability? Explain why or why not. (9 pts)

d. If a scheme uses forward Euler in time and the second order central difference approximation in space. Is this scheme stable? Explain your answer. (8 pts)