Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2010

No Calculators

Do exactly 2 problems from part I AND 2 problems from part II.

Part I: (Do two problems)

1. **a.** [9 points] Let

$$B = \begin{bmatrix} 2 & 4 & -4 \\ 3 & 3 & 3 \\ 10 & 10 & 5 \end{bmatrix}$$

By computing the eigenvalues of the iteration matrix, B_J, for Gauss-Seidel iteration for solving $A\mathbf{x} = \mathbf{b}$, show that the method converges.

b. [7 points] Prove that a matrix T is convergent (i.e. $(T^k)_{ij}$, for all i, j, or equivalently $\|T^k\| \to 0$, for some norm) if and only if $\rho(T) < 1$.

c. [9 points] Use (b) to show that the iterative method $\mathbf{x}^{(k)} = B\mathbf{x}^{(k-1)} + \mathbf{c}$ converges to a solution of $\mathbf{x} = B\mathbf{x} + \mathbf{c}$ if and only if $\rho(B) < 1$.

2. **a.** [9 points] Using the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad \text{and the initial vector } \mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

prove that the power method converges. (You must do more than just compute $\mathbf{x}^{(n)}$.)

(b) [9 points] Do one iteration of the QR method using the matrix A in (a).

(c) [7 points] Do one iteration of the Rayleigh Quotient method using the matrix A and vector $\mathbf{x}^{(0)}$.

3. Consider the matrix

$$A = \begin{bmatrix} \xi & 2 & 0 \\ 1 & \xi & 1 \\ 0 & 1 & \xi \end{bmatrix}.$$

(a) [7 points] Determine all values of ξ for which A fails to have an LU factorization.

(b) [6 points] Suppose $\xi > 3$, determine the LU factors for A.

(c) [5 points] Use the LU factors to determine A^{-1}

(d) [7 points] Suppose $\xi > 3$, show that A is positive.

Part II: (Do two problems)

1. (a) [7 points] Consider the boundary value problem on the unit square $D = \{(x, y)\colon 0 \le x \le 1, 0 \le y \le 1\}$

$$u_{xx} + u_{yy} = 0 \qquad \text{in } D$$
$$u = g \qquad \text{on the boundary of } D$$

Show that if there is a solution it is unique.

(b) [9 points] Write a consistent finite difference scheme for the differential equation in (a). Letting $\Delta x = \Delta y = h$ show the scheme is consistent.

(c) [9 points] Consider the boundary value problem

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 \qquad 0 < x < 1, 0 < y < 1\\
&u(0, y) = 0 \qquad 0 \le y \le 1\\
&u(1, y) = 1 - 3y^2 \qquad 0 \le y \le 1\\
&u(x, 0) = x^3 \qquad 0 \le x \le 1\\
&u(x, 1) = x^3 - 3x \qquad 0 \le x \le 1.
\end{aligned}$$

Letting $\Delta x = \Delta y = 1/3$ write out the equations you get for the values at the nodes using your scheme in (b). Put them in the form $A\mathbf{x} = \mathbf{c}$

2. (a) [9 points] Given $u(x, y)$ satisfies the initial value problem

$$\begin{aligned}
&u_x + 3xu_y = xu, \qquad 0 < x < \infty, y > 0\\
&u(x, 0) = x^2, \qquad 0 < x < \infty
\end{aligned}$$

(i) Find the characteristic curves.

(ii) Use the method of characteristics to find the first approximation to the solution and to the value of y at the point $(2, y)$, for $y > 0$ on the numerical characteristic curve through $(1, 0)$.

(b) [16 points] Given

$$\begin{aligned}
&ku_{xx} = u_{tt}, \qquad t > 0, \quad k > 0\\
&u(x, 0) = f(x), \qquad -\infty < x < \infty\\
&\frac{\partial u}{\partial t}(x, 0) = g(x), \qquad -\infty < x < \infty
\end{aligned}$$

(i) Find the characteristic curves and the interval of dependence for a point P in the domain.

(ii) Using the usual central difference approximation to approximate the partial derivatives, find the resulting finite-difference equation in terms of $r = \dfrac{\Delta t}{\Delta x}$. Simplify your answer solving for $U_{i,j+1}$, where U is the numerical solution (show your work).

(iii) State the Courant-Friedrichs-Lewy (C.F.L) condition for convergence of the numerical solution to the exact solution.

(iv) For $k = 1$ and $r = 1$, prove that if the forward-difference formula is used to approximate the initial conditions, then $|e_{i,1}| \le \frac{1}{2}h^2 M$, where $e = u - U$, the $h = \Delta t = \Delta x$ and M is a constant. (Assume all partial derivatives are bounded in the domain.)

3. Consider the Schrodinger equation

$$u_t = i u_{xx}, \qquad i = \sqrt{-1} \tag{1}$$

on the interval [0,1] with smooth data and periodic boundary conditions.

(a) [10 points] Using forward differencing in time and central differencing in space construct a scheme for this equation. Prove that it is consistent and that it is first order accurate in time.

(b) [10 points] Analyze the stability of the method in (a).

(c) [5 points] Construct a second order in time unconditionally stable method for the equation (1).