Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
Spring 2012

Do exactly 2 problems from part I AND exactly two problems from part II.
No notes or books; No calculators

Part I (Do two problems)

1. Consider the following nonsingular matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}$$

(a) [9 pts] Show that A cannot be directly factored as A = LU (that is, show that this equation has no solution), where L is unit lower-triangular and U is upper-triangular.

(b) [9 pts] Apply Gaussian elimination with row interchanges on A to obtain a permutation matrix P, a unit lower-triangular matrix L, and an upper-triangular matrix U such that PA = LU.

(c) Partial-pivoting is a technique that is commonly used in conjunction with the Gaussian elimination method when solving a system of linear equations.

(i) [4 pts] Briefly explain what is meant by "partial-pivoting."
(ii) [3 pts] Briefly explain the purpose of using partial-pivoting in solving large linear systems by Gaussian elimination.

2. Let $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$, where a and c are arbitrary positive real numbers.

(a) [10 pts] Find the iteration matrices for both Jacobi and Gauss-Seidel iteration in solving a linear system with the coefficient matrix A.

(b) [7 pts] By finding the eigenvalues of the iteration matrices of part (a), show that for the given matrix A, Jacobi iteration converges if and only if Gauss-Seidel iteration converges.

(c) [3 pts] When both methods converge for the matrix A, which converges faster? Why?

(d) [5 pts] For a *general* n × n linear system, B**x** = **b**, give one advantage of an iteration method such as Gauss-Seidel iteration over the method of Gaussian elimination.

3. (a) [7 pts] Factor $A = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ into A = QR, where Q is an orthogonal matrix and R is an upper-triangular matrix.

(b) [6 pts] Let *B* be an n x n symmetric matrix. Briefly describe the QR algorithm for finding the eigenvalues of *B*. Does this algorithm find the eigenvectors of B as well? Explain.

(c) [6 pts] Let B_k be the *k*th iterate of the QR algorithm, Show that B_k is similar to *B*.

(d) [6 pts] Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $x^{(1)}$ and $x^{(2)}$ using the Power method for approximating the dominant eigenvalue of the matrix *A* in part (a).

Part II (Do two problems)

1. Suppose that the function $U(x, y)$ defined on a unit square [0,1]x[0,1] satisfies

$$xU_{xx} + yU_{yy} = 3 \qquad (1)$$

with boundary conditions

$$U(x,0) = x, \quad U(x,1) = 2, \quad U(0,y) = 1, \quad U(1,y) = 3y$$

(a) [3 pts] Show that (1) is elliptic in the given domain 0< x, y <1.
(b) [10 pts] Determine the system of linear equations that results from solving this boundary-value problem using the usual 5-point scheme with $\Delta x = \Delta y = 1/3$. Write your system in the matrix form $A\boldsymbol{u} = \boldsymbol{b}$.
(c) [5 pts] Verify that the system found in part (b) has a unique solution.
(d) [7 pts] For any function $v(x, y)$, find the constants *A, B,* and *C* such that

$$\left| \frac{\partial v}{\partial x}(x_0, y_0) - \frac{1}{h}[Av(x_0, y_0) + Bv(x_0 - h, y_0) + Cv(x_0 - 2h, y_0)] \right| < Kh^2,$$

where *K* is a constant independent of *h*. You may assume that $v(x, y)$ has continuous partial derivatives of all orders.

2. Consider the PDE

$$U_{xx} - 2xU_{xt} - 3x^2U_{tt} = 0, -\infty < x < \infty, t \geq 0$$
$$U(x,0) = x, -\infty < x < \infty$$
$$U_t(x,0) = 1, -\infty < x < \infty$$

(a) [6 pts] Find the slope of the characteristic curves of this PDE.

(b) [6 pts] Suppose the characteristic curves that pass through the points $P(1,0)$ and $Q(2,0)$ intersect at a point $R(x_R, t_R)$. Find the exact values of x_R and t_R.

(c) [8 pts] Letting $\Delta x = h$ and $\Delta t = k$ and using central differences, write down a consistent scheme for the above PDE. You need not prove consistency.

(d) [5 pts] By computing the local truncation error, determine whether or not the scheme

$$\frac{u_{i,j-2} - 2u_{i,j} + u_{i,j+2}}{(\Delta t)^2}$$

gives a consistent approximation to U_{tt} at the gridpoint (i,j).

3. Given the initial-boundary value problem

$U_t = 4U_{xx}$	for $0 < x < 1,\ t > 0$
$U(x,0) = x(1-x)$	for $0 \leq x \leq 1$
$U(0,t) = 0,\ U(1,t) = 0$	for $t > 0$

(a) [3 pts] Is the given partial differential equation parabolic, elliptic, or hyperbolic? Why?

(b) [9 pts] Explain (in one sentence each) what it means for a finite difference approximation to the given initial-boundary value problem to be

(i) consistent (ii) stable (iii) convergent

(c) [3 pts] Is it possible for a finite difference approximation to the given initial-boundary value problem to be consistent but not stable? If so, give an example; if not, explain why not.

(d) [3 pts] Is it possible for a consistent finite difference approximation to the given initial-boundary value problem to be stable but not convergent? If so, give an example; if not, explain why not.

(e) [5 pts] Construct a finite difference approximation to the given initial-boundary value problem that is consistent, stable, and convergent. (You need not show that it is!)

(f) [2 pts] Is your scheme of part **e** explicit or implicit?