Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2014

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators and no cell phones.
- Closed books and closed notes.

PART A: Do only **TWO** problems

1. (a) [8 points] Let

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}.$$

Solve $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination (without partial pivoting).

(b) [2 points] If partial pivoting were used in part (a), what would be the *first* row operation performed on the augmented matrix for the system $A\mathbf{x} = \mathbf{b}$?

(c) Given a matrix B with the LU factorization

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \text{ and a vector } \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

i. [2 points] In the LDU factorization of B, where L and U are unit lower and unit upper triangular matrices, respectively, what is the diagonal matrix D?

ii. [8 points] Solve $B\mathbf{x} = \mathbf{c}$ using the LU factorization (forward/backward substitution) method.

(d) [5 points] Let P be an arbitrary $p \times n$ matrix and let Q be an arbitrary $n \times q$ matrix. Determine how many *multiplications* are needed to compute the product PQ. (Show your work)

2. The matrix

$$A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -1, \lambda_2 = 3$ with corresponding eigenvectors $\mathbf{x}_1 = (3, -1)^T$ and $\mathbf{x}_2 = (1, 1)^T$.

(a) [5 points] Perform *two* iterations of the Power Method with scaling on $\mathbf{u}_0 = (4, 1)^T$ to find an approximation $\mathbf{u}_2$ to the eigenvector $\mathbf{x}_2$ of A.

(b) [3 points] Give a nonzero initial vector for which the Power Method applied to the matrix A fails to converge to $\mathbf{x}_2$. Explain, in one sentence, why you believe that convergence fails in this case.

(c) [4 points] Use the vector $\mathbf{u}_2$ found in part (a) to approximate the eigenvalue λ_2.

(d) [6 points] Prove the theorem: If B and C are $n \times n$ matrices and there exists an invertible $n \times n$ matrix P such that $C = P^{-1}BP$, then B and C have the same eigenvalues.

(e) Let M be an invertible symmetric $n \times n$ matrix.

i. [4 points] Outline the QR method for finding the eigenvalues of M.

ii. [3 points] Explain why the success of the QR method depends on the theorem stated in part (d).

3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Note that A is symmetric and nonsingular ($\det(A) = -1$).

(a) In one sentence each, give a reason for your answer to the following questions: [3 points each]

i. Is A diagonalizable?

ii. Is A positive definite?

iii. Is A an orthogonal matrix?

(b) [10 points] By funding the spectral radius of its iteration matrix, determine whether or not Gauss-Seidel iteration converges for the linear system $A\mathbf{x} = \mathbf{b}$, where A is given above and $\mathbf{b}$ is arbitraty.

(c) [6 points] Show that if an *arbitrary* 3×3 matrix B with positive entries is strictly diagonally dominant, then Jacobi iteration converges for the linear system $B\mathbf{x} = \mathbf{c}$, for all vectors $\mathbf{c}$. (Hint: Use Gershgorin's circle theorem on the Jacobi iteration matrix).

PART B: Do only **TWO** problems

1. Consider

$$\begin{cases} U_t = U_{xx}, & 0 \le x \le 1, t > 0; \\ U(x,0) = x, & 0 \le x \le 1; \\ U(0,t) = U(1,t) = 0, & t > 0. \end{cases}$$

Suppose we approximate the above PDE by the finite difference scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

where $h = \Delta x, k = \Delta t$ are the given grid sizes.

(a) [3 points] Is the above scheme explicit or implicit? Explain your answer in one sentence.

(b) [10 points] Show that the scheme is consistent.

(c) [2 points] Based on your answer to part (b), what is the order of accuracy of the scheme?

(d) [10 points] Using the matrix of Von Neumann (Fourier) method, determine the values of $r = k/h^2$ for which the scheme converges.

2. Consider the PDE

$$\begin{cases} U_x + 2xU_y = xU & 0 < x < \infty, y > 0; \\ U(x,0) = x^2, & 0 < x < \infty. \end{cases}$$

(a) [15 points] Find the characteristic curves and the solution U along the characteristic curve through the point $P(x_p, 0)$ for any $x_p \in (0, \infty)$.

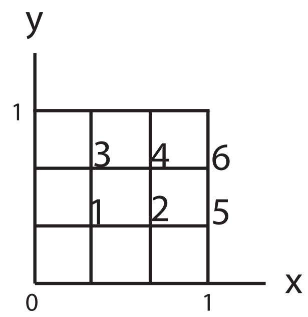
(b) [10 points] Find the first approximation to the solution and to the value of y at the point $(2, y)$ for $y > 0$ on the numerical characteristic curve through $(1, 0)$.

3. Consider the PDE

$$\begin{cases} U_{xx} + U_{yy} = 3, & 0 < x < 1, 0 < y < 1; \\ U(x,0) = x, \quad U(x,1) = 2, & 0 \le x \le 1; \\ U(0,y) = 1, \quad U_x(1,y) = 0, & 0 < y < 1. \end{cases}$$

(a) [5 points] Write down the 5-point scheme to approximate the PDE. ($h = \Delta x = \Delta y$.)

(b) [5 points] Use the central difference scheme to approximate the boundary condition at $x = 1$.



(c) [15 points] Use the schemes in (a) and (b) with $h = 1/3$ to write the equations for the nodes in the form of $A\mathbf{u} = \mathbf{b}$, where the vector $\mathbf{u} = (u_1, u_2, \ldots, u_6)$ consists of the unknown nodal values shown in the figure above.