Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
Spring 2015

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators, no cell phones and no electronic devices.
- Closed books and closed notes.

Part A

1. In this problem, A is the 3 x 3 matrix given by

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{bmatrix}$$

a. [6 points] Find the LU decomposition of A; that is, find a unit lower triangular matrix L and an upper triangular matrix U such that A = LU.

b. [3 points] **Use the result of part *a*** to find *det* (A).

c. [3 points] If Gaussian elimination **with partial pivoting** were used to solve A**x** = **b**, where **b** is an arbitrary 3-vector, what would be the first elementary row operation performed?

d. [4 points] Give an example of a nonsingular 2 x 2 matrix B for which an LU decomposition does not exist, AND explain why it does not exist.

e. [9 points] For a general n x n system of linear equations:
   (i) Give one advantage of Gaussian elimination over an iterative method (such as Jacobi iteration) for solving it.
   (ii) When solving it by Gaussian elimination, give one advantage of employing partial pivoting over *not* using partial pivoting.
   (iii) Compare (do not calculate) the flop-counts of complete pivoting with that of partial pivoting.

2. a. [9 points] Suppose U is an arbitrary nonsingular upper triangular 3 x 3 matrix; that is, $u_{ij} = 0$ if $i > j$. Let G be the Gauss-Seidel iteration matrix for solving U**x** = **b** (where **b** is an arbitrary 3-vector).

   (i) Show that Gauss-Seidel iteration converges for this system by finding the spectral radius of G.

   (ii) Find the rate of convergence of Gauss-Seidel iteration method on this system.

   b. [8+4 points] Now let U be the following specific 3 x 3 upper triangular matrix and let **b** be the following specific 3-vector:

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

   Note that the system U**x** = **b** has the solution $[1\ \ 2\ \ 3]^T$.

   (i) Using the initial vector $\mathbf{x}_0 = [1\ \ 1\ \ 1]^T$, compute the third approximation to the solution, $\mathbf{x}_3$, by the Gauss-Seidel method.

   (ii) Based on your answer to part ***b(i)***, make a conjecture concerning what it means for an iterative method to have an infinite rate of convergence.

   c. [4 points] If we were to use Jacobi iteration to solve the system of part ***b***, would we get the exact same iterates? Explain why or why not. (Do **not** actually solve that system using Jacobi iteration.)

3. Suppose that A is a symmetric nonsingular *n* x *n* matrix with dominant eigenvalue λ_1 ($|\lambda_1| > |\lambda_k|$ for $k = 2, 3, ..., n$) and corresponding eigenvector $\mathbf{v}_1$.

   a. [6 points] Describe the Power Method algorithm for approximating the eigenvector $\mathbf{v}_1$. In what key way does this method differ from the Inverse Power Method?

   b. [5 points] Explain how we use the approximation to $\mathbf{v}_1$ of part ***a***, call it $\mathbf{x}^{(k)}$, to obtain an approximation to the eigenvalue λ_1.

   c. [5 points] Give the restriction on the initial vector $\mathbf{x}^{(0)}$ that would ensure convergence of the Power Method under the conditions stated for this problem.

   d. [6 points] Assuming that all conditions for convergence are satisfied, prove that the sequence of Power Method approximations $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, ...,$ given by $\mathbf{x}^{(k)} = A\mathbf{x}^{(k-1)},$ converges to the eigenvector $\mathbf{v}_1$.

   e. [3 points] Provide one application of numerical linear algebra in which we would want to approximate the dominant eigenvalue of a matrix, but not be interested in the other eigenvalues of that matrix.

Part B

1. Consider the boundary value problem defined on a closed bounded domain *D*:

$$2U_{xx} + U_{yy} = 0 \text{ in } D \qquad (1)$$
$$U(x,y) = f(x,y) \text{ on the boundary of } D$$

a. [6 points] Show that if there is a solution to (1), it is unique.

For parts b-d, let $D = \{(x,y) | 0 \leq x \leq 2, 0 \leq y \leq 3\}$ with boundary conditions

$$U(x,0) = 0, \quad U(x,3) = 3, \qquad (2)$$
$$U_x(0,y) = U, \quad U(2,y) = y$$

b. [4 points] Write down the usual 5-point finite difference approximation to the PDE in (1).

c. [10 points] Determine the system of linear equations that results from solving the PDE (1) with boundary conditions (2) using the usual 5-point scheme from part (b) with $\Delta x = \Delta y = 1$. Write your system in the matrix form $A\boldsymbol{u} = \boldsymbol{b}$.

d. [5 points] Explain why the solution to the difference approximation in part (c) is unique.

2. a. [2 points each] Explain briefly (or give the definition) what it means for a finite difference approximation to a given initial-boundary value problem to be:
   (i) consistent
   (ii) stable
   (iii) convergent

b. [5 points] Determine whether the difference scheme

$$\frac{u_{i+1,j-1} - u_{i,j-1} - u_{i+1,j} + u_{i,j}}{\Delta x \ \Delta t}$$

is a consistent approximation to U_{xt}. Justify your answer.

c. Consider the following second-order PDE:

$$U_{xx} = x^2 U_{tt} + U, -\infty < x < \infty, t \geq 0 \qquad (3)$$
$$U(x,0) = x, -\infty < x < \infty$$
$$U_t(x,0) = 1, -\infty < x < \infty$$

(i) [3 points] Find the value(s) (x,t) for which the PDE (3) is hyperbolic.

(ii) [7 points] Suppose the characteristic curves that passes through the points *P*(1,0) and *Q*(2,0) intersect at a point $R(x_R, t_R)$. Find the exact values of x_R and t_R.

(iii) [4 points] Write down an explicit finite difference scheme that is consistent with the PDE (3). You need not prove it is consistent.

3. Consider the initial boundary value problem (IBVP):

$$U_t = U_{xx}, 0 \leq x \leq 1, t \geq 0 \qquad (4)$$

$$U(x,0) = x(1-x), 0 \le x \le 1$$
$$U(0,t) = U(1,t) = 0, t > 0$$

a. [8 points] Suppose we use the explicit scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

to approximate the PDE (4) above. Use von Neumann analysis to derive the stability condition for the scheme.

b. [4 points] Write down (do not prove) a consistent and unconditionally stable finite difference method that is of order $O(h^2, k^2)$ if there is any. If there is none, explain why.

c. Suppose an approximation to the PDE (4) with $r = k/h^2$ has the matrix form

$$A\boldsymbol{u}_{j+1} = I\boldsymbol{u}_j, \quad (5)$$

where $\boldsymbol{u}_j = \left[u_{1,j}, u_{2,j}, \dots, u_{N,j}\right]^T$, I is the identity matrix and A is the tridiagonal matrix of order N-1:

$$A = \begin{pmatrix} 1+2r & -r & 0 & 0 \\ -r & 1+2r & -r & 0 \\ 0 & -r & 1+2r & -r \\ 0 & 0 & -r & \ddots \end{pmatrix}$$

(i) [5 points] Write down the finite difference scheme given by the matrix equation (5). Is it an explicit or implicit scheme?

(ii) [8 points] By computing the eigenvalues of the appropriate matrix, derive the condition for stability for the scheme.