Department of Mathematics
California State University Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2016

Instructions

- Do exactly two problems from Part A AND two problems from Part B. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators.
- Closed books and closed notes.

PART A: Do only TWO problems

1. a. The QR method is a technique for finding approximations to the eigenvalues of a square matrix A.

   (i) [4 pts] Write down and briefly explain the QR algorithm. Be sure to briefly discuss the convergence aspect of the QR algorithm.

   (ii) [6 pts] Perform one iteration of the QR algorithm when applied to the matrix $A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$.

   (iii) [4 pts] Give two possible drawbacks of QR algorithm and suggest remedies to overcome these drawbacks.

   b. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{pmatrix}$.

   (i) [7 pts] Factor A into $A = QR$, where R is an upper triangular matrix and Q is a matrix whose columns are orthonormal.

   (ii) [4 pts] Use the factorization from part (i) to solve the least square problem $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2. Consider a system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

a. [7 pts] Perform one iteration of the Gauss-Seidel method from the starting point $\boldsymbol{x}^{(0)} = (2,2,2)^T$.

b. [4 pts] By analyzing the eigenvalues of its iteration matrix, determine whether the Gauss-Seidel method converges when applied to the above system.

c. [7 pts] Given a linear system $A\boldsymbol{x} = \boldsymbol{b}$ where A is an $n \times n$ matrix and $\boldsymbol{b}$ an n-vector, write $A = M - N$ where M is nonsingular splitting matrix. Consider the iterative scheme

$$\boldsymbol{x}^{(k+1)} = G\boldsymbol{x}^{(k)} + \boldsymbol{c},\ \ k = 1, 2,3, \dots,$$

where $G = M^{-1}N$ and $\boldsymbol{c} = M^{-1}\boldsymbol{b}$.
Show that $\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\| \le \|G\|^k \, \|\boldsymbol{x}^{(0)} - \boldsymbol{x}\|$.

d. [7 pts] Suppose an iterative method for solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$ reduces the initial error $\boldsymbol{e}_0$ by a factor of 10^{-4} in 50 iterations. How many iterations will it need to reduce $\boldsymbol{e}_0$ by a factor of 10^{-5}?

3. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix}$ and $c = \begin{bmatrix} 3 \\ 10 \\ 11 \end{bmatrix}$

a. [9 pts] Solve $A\boldsymbol{x} = \boldsymbol{c}$ if possible, using Gaussian elimination with partial pivoting.

b. [6 pts] Give flop counts when solving $A\boldsymbol{x} = \boldsymbol{c}$ via GE, for the following three cases: (i) without pivoting, with (ii) partial pivoting and (iii) complete pivoting (when A is a general n x n matrix) with brief supporting reason.

c. [4 pts] Write PA = LU for A in part (a) above, where P is the appropriate permutation matrix. Is this decomposition unique? Why or why not?

d. [6 pts] Write the Cholesky factors of $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, if possible. From your result, conclude whether B is positive definite (provide reason).

PART B: Do only TWO problems

1. Consider the boundary value problem defined on a closed bounded domain D:

$$u_{xx} + u_{yy} = 0 \text{ in } D \qquad (1)$$
$$u(x,y) = f(x,y) \text{ on the boundary of } D$$

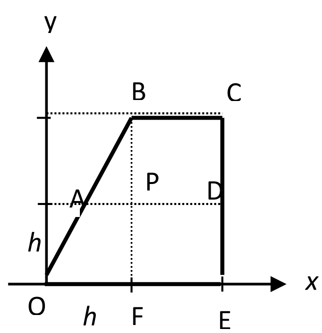
For parts (a) and (b), let $D = \{(x,y)|0 \le x \le 1, 0 \le y \le 1\}$ with boundary conditions

$$u(x,0) = x^2, \ \ u(x,1) = x^2 - 1,$$
$$u(0,y) = -y^2, \ \ u(1,y) = 1 - y^2$$

a. [12 pts] Using the standard 5-point difference scheme for approximating the PDE, determine the system of linear equations that is obtained when $\Delta x = \Delta y = 1/3$. Simplify and write the system in the matrix form $A\boldsymbol{u} = \boldsymbol{b}$.

b. [5 pts] For arbitrary Δx and Δy, explain how you know the equations in (a) have a unique solution.

c. [8 pts] Consider the PDE (1) in the trapezoidal region shown below ($\Delta x = \Delta y = h = 1$) with boundary condition $u(x,y) = 2x$ along the boundary. Use the weighted 5-point approximation to find an approximation to u at point *P.*



2. a. [10 pts] Show that the explicit method

$$\frac{u_{i,j+1} - u_{i,j}}{k} = a\left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}\right) + bu_{i,j}$$

Where $h = \Delta x$ and $k = \Delta t$ for approximating $u_t = au_{xx} + bu$ (a, b constants, a>0) is consistent.

b. [5 pts] Write an explicit consistent scheme for approximating $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{1}{x^2+1}\frac{\partial u}{\partial x}\right]$

(Do not simplify your answer)

c. [10pts] Using the 4-point explicit method for approximating the differential equation and central-differences for approximating the boundary conditions, with $\Delta x = 0.1$ and $\Delta t = 0.0025$, find and simplify the system of finite-difference equations for approximating the initial boundary value problem below.

$$\begin{cases} u_t = u_{xx}, \\ u = 0, \ \ when \quad t = 0, 0 \le x \le 1 \\ u_x = u, \, x = 0, t > 0 \\ u_x = -u, \, x = 1, t > 0 \end{cases}$$

3. The unidirectional wave equation

$$\begin{cases} u_t + 3u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

can be solved by using an upwind scheme $\frac{u_{i,j+1} - u_{i,j}}{k} + 3\left(\frac{u_{i,j} - u_{i-1,j}}{h}\right) = 0$ where $h = \Delta x$ and $k = \Delta t$ are the given grid sizes.

a. [5 pts] Find the CFL condition for the above scheme.

b. [10 pts] Perform the von Neumann analysis to show that the scheme is stable under the CFL condition found in part (a).

c. [10pts] Prove that under the same CFL condition, the scheme satisfies a local maximum-minimum principle, i.e.

$$\min(u_{i-1,j}, u_{i,j}) \le u_{i,j+1} \le \max(u_{i-1,j}, u_{i,j})$$