Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2019

Do exactly 2 problems from part I AND exactly two problems from part II.
No notes AND no calculators are allowed.

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1. Answer the following questions:

a. [4 pts] Find the eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2 of $A = \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix}$.

b. [5 pts] Carry out the power iteration with initial guess $q_0 = (1, 1)^T$ and derive a general expression for the ith iteration q_i, the approximation to the eigenvector associated to the dominant eigenvalue.

c. [5 pts] Explain why the above method is convergent. Verify that it is linearly convergent and find the convergence ratio.

d. [5 pts] How many iterations are required to obtain $\frac{\|q_i - v\|_\infty}{\|v\|_\infty} < 10^{-8}$? Here v is the eigenvector associated to the dominant eigenvalue.

e. [6 pts] Show that the power method fails for matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with initial guess $q_0 = (a, b)^T$, where $a \geq 0, b \geq 0, a \neq b$. Explain why.

2. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & -2 \\ -1 & 5 & 2 \end{pmatrix}$.

a. [4 pts] Does A have LU factorization of the form $A = LU$? Here L is a unit lower triangular matrix and U is an upper triangular matrix. Explain your answer.

b. [7 pts] Use Gaussian elimination with partial pivoting to write A in the form $PA = LU$, where P is a permutation matrix, L is unit lower triangular with $|l_{ij}| < 1$ for $i > j$, and U is upper triangular.

c. [4 pts] Use part (b) above to solve $Ax = b$, where $b = (17, 9, 15)^T$.

d. [3 pts] Give two pertinent reasons as to why partial pivoting is preferable in practice.

e. [7 pts] Show that if an $n \times n$ matrix M has Cholesky decomposition of the form $M = LL^T$ where L is lower triangular with $l_{ii} > 0$ for $i = 0, \ldots, n$, then M is positive definite.

3. Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

a. [7 pts] Determine if Jacobi iteration converges in solving the above system. If it converges, perform one step of Jacobi iteration with the initial guess $x^{(0)} = (1, 1, 1)^T$.

b. [7 pts] Determine if Gauss-Seidel iteration converges in solving the above system. If so, perform one step of Gauss-Seidel iteration with initial guess $x^{(0)} = (1, 1, 1)^T$.

c. [3 pts] Give one advantage of iterative methods (such as Jacobi and Gauss-Seidel) for solving $Ax = b$ over direct methods (such as Gaussian elimination).

d. [8 pts] Consider a general iterative method for solving $Ax = b$:

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b$$

where $N = M - A$ and M is nonsingular. Show that the approximation error from the above general iterative procedure satisfies

$$\left\|x^{(k)} - x^*\right\| \leq \|G\|^k \left\|x^{(0)} - x^*\right\|$$

where $G = M^{-1}N$ and $x^{(0)}$ is the initial guess.

Part II: (Do EXACTLY two problems)

1. Consider the following heat equation:

$$U_t = U_{xx}, 0 < x < 1, t > 0$$
$$U(0,t) = t, U_x(1,t) = 1, t > 0$$

Suppose the following scheme is used to approximate the PDE $U_t = U_{xx}$:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \qquad (1)$$

where $h = \Delta x, k = \Delta t$.

a. [12 pts] Suppose we use the given scheme (1) to solve the PDE with $h = 1/4$ and central difference approximation for the boundary condition at $x = 1$. Construct the resulting system of equations $A\boldsymbol{u}_{j+1} = B\boldsymbol{u}_j + f_j$, where $\boldsymbol{u}_j = [u_{1,j}, u_{2,j}, \dots, u_{N,j}]^T$. Identify the matrices A, B, and their dimensions, and the vector f_j.

b. [6 pts] Is the scheme (1) unconditionally stable? If not, what is the condition that guarantees its stability? Justify your answer.

c. [7 pts] If we change the right hand side of (1) to

$$\frac{u_{i-2,j} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i+2,j}}{2h^2},$$

will this still be consistent with U_{xx}? If yes, what is the order of accuracy? Explain your reasoning.

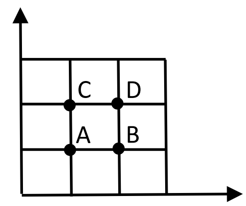
2. Consider the boundary value problem:

$$\begin{aligned} &u_{xx} + 2u_{xy} + 3u_{yy} = 0, && 0 < x < 1, 0 < y < 1 \\ &u(0,y) = u(1,y) = y, && 0 \le y \le 1 \\ &u(x,0) = x(x-1), && 0 \le x \le 1 \\ &u(x,1) = 1, && 0 \le x \le 1 \end{aligned}$$

a. [3 pts] Identify the type of PDE in this problem (i.e., hyperbolic, elliptic, or parabolic). Justify your answer.

b. [6 pts] By applying forward difference approximation twice, write a consistent finite difference approximation for u_{xy}. (You need not prove it is consistent).

c. [12 pts] Determine the system of equations that results from solving the above boundary value problem using the usual 5-point scheme for $u_{xx} + u_{yy}$ and using part (b) for u_{xy} term. Take $\Delta x = \Delta y = 1/3$, and use the following node labeling given below. Simplify your answer.



d. [4 pts] Does the system in part (c) have a unique solution? Explain your reasoning without actually solving the system.

3. a. Consider the first-order PDE

$$2t\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 2tU, -\infty < x < \infty, t > 0$$
$$U(x,0) = x^2.$$

i. [5 pts] Find and sketch the equation of the characteristic curve that passes through the point $Q(2,2)$.

ii. [5 pts] Compute the exact solution U at the point $Q(2,2)$ using the method of characteristics.

b. Given the second-order PDE

$$U_{xx} - 4U_{yy} = 0, \qquad -\infty < x < \infty, t > 0$$

i. [5 pts] Find the two characteristic curves of the given PDE that pass through the origin and the point $P(2,0)$.

ii. [3 pts] Find the intersection point R of the characteristic curves you found in part b(i).

iii. [3 pts] Determine the interval of dependence for $U(x,y)$ at the point R of part b(ii).

iv. [4 pts] Suppose we approximate the given PDE by a consistent explicit finite difference scheme with $h = k = 1$. Referring to the CFL condition, explain why or why not this scheme converges at the point R of part b(ii).