Avoidance of Partially Ordered Patterns in Compositions

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Background

- **Permutations** avoiding a **permutation pattern**
- **Permutations** avoiding more **general patterns** or **set of patterns**
- **Words** avoiding more **general patterns** or **set of patterns**
- **Compositions** enumerated according to **rises, levels and drops** (= 2-letter patterns)
- **Compositions** avoiding **3-letter patterns**
- **Compositions** enumerated according to **segmented partially ordered (generalized) patterns** = POPs

$\Rightarrow$ **Compositions** avoiding **POPs**

Things to come ...

- Definitions
- Recursion for generating function of POP-avoiding compositions
- Results for shuffle patterns and **multi-patterns**
- Result on maximum number of non-overlapping POPs in a composition

Notation and Definitions

- $\mathbb{N} =$ set of all positive integers
- $\mathbf{A} = \{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_k}\}$ ordered subset of $\mathbb{N}$
- $\sigma = \sigma_1\sigma_2\ldots\sigma_m =$ **composition of n** $\in \mathbb{N}$ with m parts where $\sum_{\mathbf{i=1}}^{\mathbf{m}} \sigma_{\mathbf{i}} = \mathbf{n}$
- $[k] = \{1, 2, \ldots, k\}$; $[k]^n =$
  set of all **words of length n** over $[k]$
- **Generalized pattern** $\tau =$ word in $[\ell]^k$ that contains each letter from $[\ell]$, possibly with repetitions and dashes
- **Classical pattern** = pattern with no adjacency requirement
- **Consecutive** or **segmented pattern** = pattern with no dashes

1234 **1-23-4** **1-2-3-4**

Notation and Definitions

- $\mathbf{C_n^A}$ $(\mathbf{C_{n;m}^A})$ = the set of all compositions of n with parts in A (m parts in A)
- $\sigma \in C_n^A$ $(C_{n;m}^A)$ **contains** τ if σ contains a subsequence isomorphic to τ. Otherwise, σ **avoids** τ and we write $\sigma \in \mathbf{C_n^A}(\tau)$ $(\sigma \in \mathbf{C_{n;m}^A}(\tau))$

  241874 contains five occurrences of **1-32**

  241874 avoids **312**

- A **POP** τ is a word consisting of letters from a partially ordered alphabet $\mathcal{T}$
- If letters a and b are incomparable in a POP τ, then the relative size of the letters in σ corresponding to a and b is unimportant in an occurrence of τ in σ.

Note that comparable letters have the same number of primes.

Example

- Let $\mathcal{T} = \{1', 1'', 2''\}$ with the only relation $1'' < 2''$. Then **113425** contains three occurrences of $\mathbf{1'1''2''}$ and seven occurrences of $\mathbf{1'\text{-}1''2''}$
  - **113425** , **113425**, **113425**
  - **11 3 425**, **1134 25**, **1 134 25**, **11 34 25**

More Definitions and Notation

- A composition σ **quasi-avoids** a consecutive pattern τ if σ has exactly **one** occurrence of τ and the occurrence consists of the $|\tau|$ **rightmost parts** in σ

  4112234 quasi-avoids **1123**

  5223411 and **1123346** do **not** quasi-avoid **1123**

- Generating functions
  - $\mathbf{C}_\tau^{\mathbf{A}}(\mathbf{x}) = \sum_{\mathbf{n}\geq \mathbf{0}} |\mathbf{C}_{\mathbf{n}}^{\mathbf{A}}(\tau)|\mathbf{x}^{\mathbf{n}}$
  - $\mathbf{C}_\tau^{\mathbf{A}}(\mathbf{x};\mathbf{m}) = \sum_{n\geq 0} |C_{n;m}^A(\tau)|x^n$
  - $\mathbf{C}_\tau^{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \sum_{m\geq 0} C_\tau^A(x;m)y^m = \sum_{n,m\geq 0} |C_{n;m}^A(\tau)|x^n y^m$
  - $\mathbf{D}_\tau^{\mathbf{A}}(\mathbf{x},\mathbf{y})$ = gf for the number of compositions in $C_{n;m}^A$ that quasi-avoid τ

General Results

Lemma 1: Let τ be a consecutive pattern. Then

$$\mathbf{D}_{\tau}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \mathbf{1} + \mathbf{C}_{\tau}^{\mathbf{A}}(\mathbf{x},\mathbf{y})\left(\mathbf{y}\sum_{\mathbf{a}\in\mathbf{A}}\mathbf{x}^{\mathbf{a}} - \mathbf{1}\right).$$

Theorem 2: Suppose $\tau = \tau_0$-ϕ, where ϕ is an arbitrary POP, and the letters of τ_0 are incomparable to the letters of ϕ. Then for all $k \geq 1$, we have

$$\mathbf{C}_{\tau}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \mathbf{C}_{\tau_0}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) + \mathbf{D}_{\tau_0}^{\mathbf{A}}(\mathbf{x},\mathbf{y})\mathbf{C}_{\phi}^{\mathbf{A}}(\mathbf{x},\mathbf{y}).$$

We will apply this results for two types of patterns: shuffle patterns and **multi-patterns**.

Proof of Theorem 2: To show:

$$\mathbf{C}_{\tau}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \mathbf{C}_{\tau_0}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) + \mathbf{D}_{\tau_0}^{\mathbf{A}}(\mathbf{x},\mathbf{y})\mathbf{C}_{\phi}^{\mathbf{A}}(\mathbf{x},\mathbf{y}).$$

Two possible cases:

- σ avoids $\tau_0 \Rightarrow C_{\tau_0}^{A}(x,y)$
- σ does not avoid $\tau_0 \Rightarrow \sigma = \sigma_1\sigma_2\sigma_3$ where
  - $\sigma_1\sigma_2$ quasi-avoids the pattern τ_0
  - σ_2 is order isomorphic to τ_0
  - σ_3 must avoid ϕ

  $\Rightarrow D_{\tau_0}^{A}(x,y)C_{\phi}^{A}(x,y)$ ■

Multi-patterns

Let $\{\tau_0, \tau_1, \ldots, \tau_s\}$ be a set of consecutive patterns.

- $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ is a **multi-pattern** if each letter of τ_i is incomparable with any letter of τ_j for $i \neq j$
- Simplest non-trivial multi-pattern is $\Phi = 1' - 1''2''$.

In this case we can derive the generating function directly:

- First letter can be any of the k letters in A
- All other letters have to be in non-increasing order

$$\begin{aligned} C^{A}_{1'\text{-}1''2''}(x,y) &= 1 + \left(y \sum_{a\in A} x^a\right) \prod_{a \in A} \left(\sum_{i\geq 0} (x^a y)^i\right) \\ &= 1 + \frac{y\sum_{a\in A} x^a}{\prod_{a\in A}(1-x^a y)}. \end{aligned}$$

General Results for Multi-Patterns

Theorem 3: let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern. Then

$$\mathbf{C}_\tau^{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{s}} \mathbf{C}_{\tau_{\mathbf{j}}}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) \prod_{\mathbf{i}=\mathbf{1}}^{\mathbf{j}-\mathbf{1}} \left[\left(\mathbf{y} \sum_{\mathbf{a}\in\mathbf{A}} \mathbf{x}^{\mathbf{a}} - \mathbf{1} \right) \mathbf{C}_{\tau_{\mathbf{i}}}^{\mathbf{A}}(\mathbf{x},\mathbf{y}) + \mathbf{1} \right].$$

Example:

Let $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ be a multi-pattern such that τ_j is equal to either 12 or 21, for $j = 1, 2, \ldots, s$. Since $C_{12}^A(x,y) = C_{21}^A(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)}$, we get

$$\mathbf{C}_\tau^{\mathbf{A}}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{1} - \left(\mathbf{1} + \frac{\mathbf{y}\sum_{\mathbf{a}\in\mathbf{A}} \mathbf{x}^{\mathbf{a}} - \mathbf{1}}{\prod_{\mathbf{a}\in\mathbf{A}}(\mathbf{1}-\mathbf{x}^{\mathbf{a}}\mathbf{y})} \right)^{\mathbf{s}}}{\mathbf{1} - \mathbf{y}\sum_{\mathbf{a}\in\mathbf{A}} \mathbf{x}^{\mathbf{a}}}.$$

Equivalence of Patterns

- **Reversal map** $R(\sigma) = R(\sigma_1\sigma_2\ldots\sigma_k) = \sigma_k\sigma_{k-1}\ldots\sigma_1$
- Reversal map R and identity map I are called **trivial** bijections of $C^A_{n;m}$ to itself
- τ_1 and τ_2 are **equivalent**, denoted by $\boldsymbol{\tau_1 \equiv \tau_2}$, if $|C^A_{n;m}(\tau_1)| = |C^A_{n;m}(\tau_2)|$ for all A, m and n.
- $\tau \equiv R(\tau)$ for any pattern τ
- $\{\tau, R(\tau)\} =$ **symmetry class of τ**

Results for Families of Multi-Patterns

Theorem 4: Let $\tau = \tau_0\text{-}\tau_1$ and $\phi = f_1(\tau_0)\text{-}f_2(\tau_1)$, where f_1 and f_2 are any of the trivial bijections. Then $\tau \equiv \phi$.

Theorem 5: Suppose we have multi-patterns $\tau = \tau_1\text{-}\tau_2\text{-}\cdots\text{-}\tau_s$ and $\phi = \phi_1\text{-}\phi_2\text{-}\cdots\text{-}\phi_s$, where $\tau_1\tau_2\ldots\tau_s$ is a permutation of $\phi_1\phi_2\ldots\phi_s$. Then $\tau \equiv \phi$.

Results for Families of Multi-Patterns

Proof of Theorem 4: Show that $\tau = \tau_0\text{-}\tau_1 \equiv \tau_0\text{-}f(\tau_1)$. If σ avoids τ, then either

- σ has no occurrence of τ_0, so σ also avoids $\tau_0\text{-}f(\tau_1)$
- σ can be written as $\sigma = \sigma_1\sigma_2\sigma_3$, where $\sigma_1\sigma_2$ has exactly one occurrence of τ_0, namely σ_2. Then σ_3 must avoid τ_1, so $f(\sigma_3)$ avoids $f(\tau_1)$ and $\sigma_f = \sigma_1\sigma_2 f(\sigma_3)$ avoids $\tau_0\text{-}f(\tau_1)$.
- Converse also true $\Rightarrow$ bijection between class of compositions avoiding τ and those avoiding $\tau_0\text{-}f(\tau_1)$.
- This result and properties of trivial bijections finish proof.

Proof of Theorem 5: By induction.

Non-Overlapping Occurrences of POPs

- Two occurrences of a pattern τ **overlap** if they contain any of the same parts of σ
- **τ-nlap(σ)** = maximum number of non-overlapping occurrences of a consecutive pattern τ
- **descent** = 21 occurs at position i if $\sigma_i > \sigma_{i+1}$
- Two descents at positions i and j overlap if $j = i + 1$
- **MND** = maximum number of non-overlapping descents

  MND(333 211) $= 1$

  MND(133 21111 43 211) $= 3$

- Results on statistic τ-nlap(σ) exist for permutations and words

Non-Overlapping Occurrences of POPs

Theorem 6: Let τ be a consecutive pattern. Then

$$\sum_{\mathbf{n},\mathbf{m}\geq 0}\sum_{\sigma\in \mathbf{C}^{\mathbf{A}}_{\mathbf{n};\mathbf{m}}} \mathbf{t}^{\tau\text{-nlap}(\sigma)}\mathbf{x}^{\mathbf{n}}\mathbf{y}^{\mathbf{m}} = \frac{\mathbf{C}^{\mathbf{A}}_{\tau}(\mathbf{x},\mathbf{y})}{\mathbf{1}-\mathbf{t}\left[\left(\mathbf{y}\sum_{\mathbf{a}\in\mathbf{A}}\mathbf{x}^{\mathbf{a}}-\mathbf{1}\right)\mathbf{C}^{\mathbf{A}}_{\tau}(\mathbf{x},\mathbf{y})+\mathbf{1}\right]},$$

where τ-nlap(σ) is the maximum number of non-overlapping occurrences of τ in σ.

Remark: We only need to know the gf for the number of compositions avoiding τ.

Proof: Fix s and let $\Phi_s = \tau\text{-}\tau\text{-}\cdots\text{-}\tau$ with s copies of τ

- σ avoids $\Phi_s \Rightarrow \sigma$ has at most $s-1$ non-overlapping occurrences of τ
- Compute $C^A_{\Phi_{s+1}}(x, y)$ from general theorem for multi patterns
- gf for number of compositions with exactly s non-overlapping copies of τ is given by $C^A_{\Phi_{s+1}}(x, y) - C^A_{\Phi_s}(x, y)$
- Sum over s

Example:

- Apply theorem to descent pattern
- $C_{12}^{A}(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)}$
- distribution of *MND* is given by

$$\sum_{\mathbf{n,m\geq 0}} \sum_{\sigma\in \mathbf{C^{A}_{n;m}}} \mathbf{t}^{\mathbf{12}\text{-nlap}(\sigma)} \mathbf{x^n y^m}$$

$$= \frac{\mathbf{1}}{\prod_{\mathbf{a\in A}}(\mathbf{1}-\mathbf{x^a y}) + \mathbf{t}\left(\mathbf{1}-\mathbf{y}\sum_{\mathbf{a\in A}}\mathbf{x^a} - \prod_{\mathbf{a\in A}}(\mathbf{1}-\mathbf{x^a y})\right)}.$$

- For $A = \{1,2\}$, distribution of *MND* on the set of compositions of n with parts in A is given by

$$\frac{\mathbf{1}}{(\mathbf{1}-\mathbf{x})(\mathbf{1}-\mathbf{x^2}) - \mathbf{x^3 t}} = \sum_{\mathbf{s\geq 0}} \frac{\mathbf{x^{3s}}}{(\mathbf{1}-\mathbf{x})^{\mathbf{2s+2}}(\mathbf{1}+\mathbf{x})^{\mathbf{s+1}}}\mathbf{t^s}.$$

Preprint available from my web site at
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also at ArXiv (http://www.arxiv.org/pdf/math.CO/0610030)

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Thanks!