California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Applied Probability Theory **Spring 2021**
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Do **five** (5) of the following seven problems as indicated below. You do not need to simplify algebraic expressions in your final answers unless specifically asked to do so. All problems are worth the same number of points.

To receive full credit, make sure to **give reasons for your answers**, for example defining any relevant events and random variables, explaining how you set up an equation or what theorems you apply, and to clearly mark your answers. Note that correct answers without explanations do NOT receive full credit.

Exams are graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

SECTION 1 – Do two (2) of the three problems from this section. If you attempt all three, then the best two will be used for your grade.

Spring 2021 #1
Suppose that X and Y have a continuous joint distribution for which the joint probability density function is defined as follows:

$$f(x,y) = \begin{cases} 2, & \text{if } (x,y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

where A is a triangle region with vertices $(0,0),(0,1)$ and $(1,0)$.

(a) Determine the marginal probability density functions of X and Y.

(b) Are X and Y independent? Explain your answer.

(c) Determine the conditional density $f_{x|y}(x|y)$. Make sure to state the domain.

Spring 2021 #2

COVID-19 vaccinations have been offered to the public in California for several months now, and the number of vaccinated individuals is steadily going up. Three vaccines have been used, Moderna, Pfizer/BioNTec, and J&J Janssen. These vaccines have different levels of effectiveness, that is, prevention of getting COVID-19. According to the CDC, the effectiveness of Moderna is 94.5%, Pfizer/BioNTec is 95%, and J&J Janssen is 66.3%.

(a) Currently, 48% of fully vaccinated individuals in California received Pfizer/BioNTec, 41% received Moderna, and 11% received J&J Janssen. (Fully vaccinated means that an individual either received both shots for the Pfizer/BioNTec and Moderna vaccines, or the single shot for J&J Janssen.) What is the probability for a fully vaccinated California resident to get COVID-19? Carefully state all events and their respective probabilities. Make sure to distinguish between conditional and unconditional probabilities.

(b) For each of the three vaccines, compute the probability that a fully vaccinated California resident who got COVID-19 received the respective vaccine.

(c) Due to potential issues with blood clots in women under 50, the use of the J&J Janssen vaccine was temporarily halted, which reduced its percentage among the three vaccines. Without any computations, describe the effect of this change on the following probabilities:

- the probability that a fully vaccinated California resident gets the disease,
- the probability that a a fully vaccinated California resident who got COVID-19 received the J&J Janssen vaccine.

Give reasons for your answers.

Spring 2021 #3

Let X and Y be two independent random variables, where X has uniform distribution on $[0, 1]$ and Y has density $f(y) = 6y(1 - y) \quad \text{for } 0 < y < 1$.

(a) Find the density of $U = X + Y$. Make sure to carefully state the set of values for which $f_U(u)$ is non-zero.

(b) Set up the definite integral to compute the expected value of $(X + Y)^2$. Make sure to state the specifics for this problem, not just a generic formula. Do NOT compute the integral.

SECTION 2 – Do three (3) of the four problems from this section. If you attempt all four, then the three best will be used for your grade.

Spring 2021 #4

A game consists of rolling a single fair die continually until there are two sixes in a row.

(a) Find the expected number of rolls needed for this game.

(b) Find the expected number of sixes that you see in this game. (Hint: The answer is not 2, since for example, the sequence 2 6 3 4 **6 6** has three sixes.)

Spring 2021 #5

A Markov chain $\{X_n, n \geq 0\}$ with states $\{0, 1, 2\}$ has the transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 2/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Identify the classes of this Markov chain and determine which states are transient and which are recurrent. Give reasons for your answers.

(b) What is the probability that, starting from state 0, the process never enters state 1?

(c) If the initial distribution is $(4/5, 0, 1/5)$, what is the expected value of X_3? You may use that

$$\boldsymbol{P}^3 = \begin{pmatrix} 25/72 & 25/144 & 23/48 \\ 25/54 & 25/108 & 11/36 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Derive the the stationary distribution for this Markov chain.

Spring 2021 #6
Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.

(a) What is the expected time until the ninth immigrant arrives?

(b) What is the probability that the elapsed time between the 104th and the 105th arrival exceeds two days?

(c) What is the probability that two immigrants arrive between day 1 and day 4, and three immigrants arrive between day 3 and day 5? (Note: the time intervals $[1, 4]$ and $[3, 5]$ overlap.)

Spring 2021 #7

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion process.

(a) Let $Y(t) = \exp(B(t))$. Compute the expected value of Y at time t given the history of the process B up to time s. That is, for $s < t$, find $\mathbb{E}(Y(t)|B(u), 0 \leq u \leq s)$. You may need the fact that the moment generating function $\varphi_X(t)$ of a normal random variable X with mean μ and standard deviation σ is $\exp(\mu t + \sigma^2 t^2/2)$.

(b) Let T_a denote the first time the process B hits a, i.e., $T_a = \min\{t \geq 0 : B(t) = a\}$.

(b1) Prove that $\mathbb{P}(T_a \leq t) = 2\mathbb{P}(B(t) \geq a)$, for $a > 0$.

(b2) Compute the probability $\mathbb{P}(T_2 \leq 4)$, and express your answer in terms of Φ, which is the cumulative distribution function of the standard normal.