Radio Number for Square of Cycles *

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August 30, 2004

Abstract

Let G be a connected graph with diameter d. For any two vertices u and v, let $\mathrm{d}_G(u,v)$ denote the distance between u and v. A radio labelling for G is a function f that assigns to each vertex a non-negative integer (label) such that the in-equality $|f(u) - f(v)| \geq d - d_G(u,v) + 1$ holds for any two vertices u and v. The span of f is the difference of the maximum and the minimum labels assigned. The radio number of G is the minimum span over all radio labellings of G. The radio numbers for paths and cycles were studied by Chartrand et. al. and Zhang, and were completely determined by Liu and Zhu. We study the radio number for square cycles (cycles by adding edges between vertices of distance two apart). The radio numbers for the square of even cycles are completely determined. For the square of an odd cycle, we obtain a lower bound for the radio number, and show that the bound is achieved by many cases, while not achievable by some other cases.

1 Introduction

Radio labelling is motivated by the AM/FM radio channel assignment problem (cf. [1, 3]). Suppose we are given a set of stations (or transmitters), the task is to assign to each station (or transmitter) a channel (non-negative integer) such that the interference is avoided. The interference is closely related to the geographical location of the stations, the smaller distance between two stations, the stronger interference between them might occur. To

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avoid stronger interference, the larger separation of the channels assigned to the stations must be. To model this problem, we construct a graph, such that each station is represented by a vertex, and two vertices are adjacent when their corresponding stations are close. The ultimate goal is to find a *valid* labelling for the vertices such that the span (range) of the channels used is minimized.

For a given simple connected graph G, the *distance* between any two vertices u and v, denoted by $d_G(u,v)$ (or $d(u,v)$ when G is understood in the context), is the length (number of edges) of a shortest $(u,v)-$path. The *diameter* of G, denoted by $\mathrm{diam}(G)$ (or d if G is understood in the context), is the maximum distance between two vertices in G. A *radio labelling* (also known as *multi-level distance labeling* [5]) of G is a function, $f : V(G) \to \{0, 1, 2, \cdots\}$, such that the following holds for any u and v:

$$|f(u) - f(v)| \geq \mathrm{diam}(G) - d(u,v) + 1.$$

The *span* of f is the difference of the largest and the smallest channels used, $\max\limits_{u,v\in V(G)} \{f(v) - f(u)\}$. The *radio number* of G is defined as the minimum span of a radio labelling of G.

The radio numbers for different families of graphs have been studied by several authors. The radio number for paths and cycles was investigated by [3, 1, 12], and was completely solved by Liu and Zhu [5]. The radio number for *spiders* (trees with at most one vertex of degree more than 2) has been investigated by Liu [6].

The *square* of a graph G, denoted by G^2, has the same vertex set as G, and the edge set $E(G^2) = E(G) \cup \{uv : \mathrm{d}_G(u,v) = 2\}$. The radio number for the square of paths is completely settled in [7]. In this article, we investigate the radio number for the square of cycles (or *square cycles*), denoted by C_n^2. We completely determine the radio numbers for the squares of even cycles C_{2k}^2. For the radio number of the square of odd cycles, we obtain a lower bound, and show that the bound is achieved by many cases, while not achievable by some other cases.

2 A lower bound

Throughout the article, we denote $V(C_n) = V(C_n^2) = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, $E(C_n) = \{v_iv_{i+1} : i = 0, 1, \cdots, n-1\}$ and $E(C_n^2) = \{v_iv_{i+1}, v_iv_{i+2} : i = 0, 1, \cdots, n-1\}$. All the sub-indices are taken under modulo n. For instance, $v_n = v_0$, $v_{n+i} = v_i$, etc. For any $u, v \in V(C_n)$, we denote the distance between u and v, in C_n^2 and C_n, respectively, by $\mathrm{d}(u,v)$ and $\mathrm{d}_{C_n}(u,v)$. The diameter of C_n^2 is denoted by d.

For any radio labelling f of C_n^2, we re-name the vertices of C_n^2 by

$V(C_n^2) = \{x_0, x_1, \cdots, x_{n-1}\}$, where

$$0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_{n-1}).$$

Since we are searching for the minimum span of a radio labelling, without loss of generality, we assume $f(x_0) = 0$. Hence, the span of f is $f(x_{n-1})$. For any $i = 0, 1, 2, \cdots, n-2$, we set

$$d_i = d(x_i, x_{i+1}); \text{ and } f_i = f(x_{i+1}) - f(x_i).$$

A radio labelling f of a graph G is called *optimal* if the span of f is equal to the radio number of G.

Lemma 1 *Let f be a radio labelling for C_n^2. Then for any $i = 0, 1, \cdots, n-2$,*

$$f(x_{i+2}) - f(x_i) = f_{i+1} + f_i \geq \begin{cases} \lceil \frac{k+2}{2} \rceil, & \text{if } n = 4k; \\ \lceil \frac{k+1}{2} \rceil, & \text{if } n = 4k+1; \\ \lceil \frac{k+4}{2} \rceil, & \text{if } n = 4k+2; \\ \lceil \frac{k+3}{2} \rceil, & \text{if } n = 4k+3. \end{cases}$$

Proof Let $n = 4k + r$, $r = 0, 1, 2, 3$. For any $i = 0, 1, 2, \cdots, n-2$, let

$$l_i = d_{C_n}(x_i, x_{i+1}), \quad \text{and} \quad l_i^* = d_{C_n}(x_{i+2}, x_i).$$

Then, we have

$$d(x_i, x_{i+1}) = \left\lceil \frac{l_i}{2} \right\rceil, \text{ and } d(x_{i+2}, x_i) = \left\lceil \frac{l_i^*}{2} \right\rceil.$$

Assume $r = 0, 1$. Then $\text{diam}(C_n^2) = k$. If $l_i \leq k$, then $\text{d}(x_i, x_{i+1}) \leq \lceil \frac{k}{2} \rceil$. Hence, by definition, we have

$$f_i = f(x_{i+1}) - f(x_i) \geq k - d(x_i, x_{i+1}) + 1 \geq k - \left\lceil \frac{k}{2} \right\rceil + 1 \geq \left\lceil \frac{k}{2} \right\rceil.$$

This implies that $f(x_{i+2}) - f(x_i) = f_i + f_{i+1} > \lceil \frac{k}{2} \rceil + 1$. So the result follows.

Assume $l_i^* < k$. Then $\text{d}(x_i, x_{i+2}) \leq \lceil \frac{k}{2} \rceil$. Hence, by definition, we have

$$f(x_{i+2}) - f(x_i) \geq k - d(x_i, x_{i+2}) + 1 \geq \left\lceil \frac{k+2}{2} \right\rceil.$$

Thus, we assume $l_i, l_{i+1} \geq k+1$, and $l_i^* \geq k$. This implies that $l_i + l_{i+1} + l_i^* = n$. By definition, we have

$$\begin{array}{llll} f_i & = & f(x_{i+1}) - f(x_i) & \geq & k - \lceil \frac{l_i}{2} \rceil + 1, \\ f_{i+1} & = & f(x_{i+2}) - f(x_{i+1}) & \geq & k - \lceil \frac{l_{i+1}}{2} \rceil + 1, \\ f_i + f_{i+1} & = & f(x_{i+2}) - f(x_i) & \geq & k - \lceil \frac{l_i^*}{2} \rceil + 1. \end{array} \tag{2.1}$$

Summing the three inequalities in the above,

$$2(f(x_{i+2}) - f(x_i)) \geq 3k + 3 - \left(\left\lceil \frac{l_i}{2} \right\rceil + \left\lceil \frac{l_{i+1}}{2} \right\rceil + \left\lceil \frac{l_i^*}{2} \right\rceil \right). \tag{2.2}$$

If $n = 4k$, then at most two of the elements in $\{l_i, l_{i+1}, l_i^*\}$ are odd, so $\lceil \frac{l_i}{2} \rceil + \lceil \frac{l_{i+1}}{2} \rceil + \lceil \frac{l_i^*}{2} \rceil \leq \frac{4k+2}{2} = 2k + 1$.

If $n = 4k + 1$, then

$$\left\lceil \frac{l_i}{2} \right\rceil + \left\lceil \frac{l_{i+1}}{2} \right\rceil + \left\lceil \frac{l_i^*}{2} \right\rceil \leq \frac{4k+4}{2} = 2k + 2. \tag{2.3}$$

Therefore, the result follows by combining (2.2) and (2.3).

Similarly, one can show the cases for $r = 2, 3$ (in which $d = k + 1$, and one can first prove that the results follow if $l_i < k+1$ or $l_i^* < k$, then verify the rest by some calculation similar to the above). We leave the details to the reader. ∎

Theorem 2 *Let $n = 4k + r$, $r = 0, 1, 2, 3$. Then*

$$rn(C_n^2) \geq \begin{cases} \frac{2k^2+5k-1}{2}, & \text{if } r = 0 \text{ and } k \text{ is odd;} \\ \frac{2k^2+3k}{2}, & \text{if } r = 0 \text{ and } k \text{ is even;} \\ k^2 + k, & \text{if } r = 1 \text{ and } k \text{ is odd;} \\ k^2 + 2k, & \text{if } r = 1 \text{ and } k \text{ is even;} \\ k^2 + 5k + 1, & \text{if } r = 2 \text{ and } k \text{ is odd;} \\ k^2 + 4k + 1, & \text{if } r = 2 \text{ and } k \text{ is even;} \\ \frac{2k^2+7k+3}{2}, & \text{if } r = 3 \text{ and } k \text{ is odd;} \\ \frac{2k^2+9k+4}{2}, & \text{if } r = 3 \text{ and } k \text{ is even.} \end{cases}$$

Proof Let f be a radio labelling for C_n^2.

Assume $r = 0$ (so $n - 1$ is odd). By Lemma 1, the span of f has:

$$f(x_{n-1}) = f_0 + f_1 + f_2 + \cdots + f_{n-2} \geq \left(\frac{n-2}{2} \right) \times \left\lceil \frac{k+2}{2} \right\rceil + 1.$$

Considering the parity of k separately to the above, the results for $r = 0$ follow.

Assume $r = 1$ (so $n - 1$ is even). By Lemma 1, we get

$$f(x_{n-1}) = f_0 + f_1 + f_2 + \cdots + f_{n-2} \geq \left(\frac{n-1}{2} \right) \times \left\lceil \frac{k+1}{2} \right\rceil.$$

Considering the parity of k separately, the results for $r = 1$ follow.

The cases for $r = 2, 3$ can be proved by the same method. We leave the details to the reader. ∎

3 Square of Even Cycles

We prove that for any even cycle, the radio number is equal to the lower bound proved in Theorem 2. It suffices to give radio labellings with spans equal to the lower bound. Throughout this article, we express each radio labelling f, with $0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_{n-1})$, by the following two:

- A permutation τ on $\{0, 1, 2, \cdots, n-1\}$, with $\tau(0) = 0$ and $x_i = v_{\tau(i)}$.
- Unless indicated, we let $f(x_0) = f(v_0) = 0$, and $f(x_{i+1}) = f(x_i) + d + 1 - \mathrm{d}(x_{i+1}, x_i)$ for $i = 0, 1, \cdots, n-2$.

To show that each f, expressed in the above format, is a radio labelling, we shall (and it suffices to) verify that all the following hold, for any i:

(1) τ is a permutation.

(2) $f(x_{i+2}) - f(x_i) \geq d - d(x_i, x_{i+2}) + 1$.

(3) $f(x_{i+3}) - f(x_i) \geq d - d(x_i, x_{i+3}) + 1$.

(4) $f(x_{i+4}) - f(x_i) \geq d + 1$.

Theorem 3 *Let C_n^2 be an n-vertex square cycle where n is even. Then*

$$rn(C_n^2) = \begin{cases} \frac{2k^2+5k-1}{2}, & \text{if } n = 4k \text{ and } k \text{ is odd;} \\ \frac{2k^2+3k}{2}, & \text{if } n = 4k \text{ and } k \text{ is even;} \\ k^2 + 5k + 1, & \text{if } n = 4k+2 \text{ and } k \text{ is odd;} \\ k^2 + 4k + 1, & \text{if } n = 4k+2 \text{ and } k \text{ is even.} \end{cases}$$

Proof By Theorem 2, it suffices to find radio labellings with spans achieving the desired numbers. We consider cases.

$\boxed{\textbf{Case 1: } n = 4k \textbf{ and } k \textbf{ is odd}}$ Define $\tau(0) = 0$ and:

$$\tau(i+1) = \begin{cases} \tau(i) + 2k \pmod{n}, & \text{if } i \equiv 0, 2 \pmod{4}; \\ \tau(i) + k \pmod{n}, & \text{if } i \equiv 1 \pmod{4}; \\ \tau(i) + k + 1 \pmod{n}, & \text{if } i \equiv 3 \pmod{4}. \end{cases}$$

By the definition of τ, and the fact that k is odd, we have

$$d_i = \mathrm{d}(x_i, x_{i+1}) = \mathrm{d}(v_{\tau(i)}, v_{\tau(i+1)}) = \begin{cases} k, & \text{if } i \text{ is even;} \\ \frac{k+1}{2}, & \text{if } i \text{ is odd.} \end{cases}$$

Now we show that f is a radio labelling.

Claim (1): τ is a permutation. For $i = 0, 1, \cdots, k-1$, we re-write τ by:

$$\begin{array}{llll}
\tau(4i) & = (2k+k+2k+k+1)i & = (2i+0)k+i & \pmod{n}; \\
\tau(4i+1) & = 2ki+i+2k & = (2i+2)k+i & \pmod{n}; \\
\tau(4i+2) & = 2ki+i+2k+k & = (2i+3)k+i & \pmod{n}; \\
\tau(4i+3) & = 2ki+i+3k+2k & = (2i+1)k+i & \pmod{n}.
\end{array}$$

Assume $\tau(4i+j) = \tau(4i'+j')$. Then, $(2i-2i'+t-t')k \equiv i'-i \pmod{n}$ for some $t, t' \in \{0,1,2,3\}$. This implies, $i = i'$, as $0 \leq i'-i < k$. Therefore, it must be $t = t'$, because $t'-t \leq 3$. So, τ is a permutation.

Claim (2): $f(x_{i+2}) - f(x_i) \geq d - d(x_i, x_{i+2}) + 1$. By the definition of τ, we have $d_i + d_{i+1} = k + \frac{k+1}{2} = \frac{3k+1}{2}$, and $d(x_i, x_{i+2}) \in \{\frac{k-1}{2}, \frac{k+1}{2}\}$. So, $d(x_i, x_{i+2}) \geq \frac{k-1}{2}$. Therefore,

$$\begin{array}{rcl}
f(x_{i+2}) - f(x_i) & = & f_i + f_{i+1} \\
& = & (d - d_i + 1) + (d - d_{i+1} + 1) \\
& = & 2k + 2 - \frac{3k+1}{2} \\
& \geq & k - d(x_i, x_{i+2}) + 1.
\end{array}$$

Claim (3): $f(x_{i+3}) - f(x_i) \geq d - d(x_i, x_{i+3}) + 1$. By the definition of τ, we have

$$d(x_i, x_{i+3}) = \begin{cases} \frac{k+1}{2}, & \text{if } d_i + d_{i+1} + d_{i+2} = k + \frac{k+1}{2} + k; \\ 1, & \text{if } d_i + d_{i+1} + d_{i+2} = \frac{k+1}{2} + k + \frac{k+1}{2}. \end{cases}$$

By some calculation, we conclude that

$$\begin{array}{rcl}
f(x_{i+3}) - f(x_i) & = & (d - d_i + 1) + (d - d_{i+1} + 1) + (d - d_{i+2} + 1) \\
& \geq & d - d(x_i, x_{i+3}) + 1.
\end{array}$$

Claim (4): $f(x_{i+4}) - f(x_i) \geq k + 1$. By the definition of τ, we have $d_i + d_{i+1} + d_{i+2} + d_{i+3} = 3k + 1$. Therefore,

$$\begin{array}{rcl}
f(x_{i+4}) - f(x_i) & = & 4k - (d_i + d_{i+1} + d_{i+2} + d_{i+3}) + 4 \\
& = & 4k + 4 - (3k+1) \geq k + 1.
\end{array}$$

By Claims $(1-4)$, f is a radio labelling. The span of f is:

$$\begin{array}{rcl}
f(x_{n-1}) & = & d - d_0 + 1 + d - d_1 + 1 + \cdots + d - d_{n-2} + 1 \\
& = & (n-1)(d+1) - (d_0 + d_1 + d_2 + \cdots + d_{n-2}) \\
& = & (4k-1)(k+1) - \left((2k-1)\left(\dfrac{3k+1}{2}\right) + k\right) = \dfrac{2k^2 + 5k - 1}{2}.
\end{array}$$

Figure 1 shows an optimal radio labelling for C_{12}^2 with span 16.

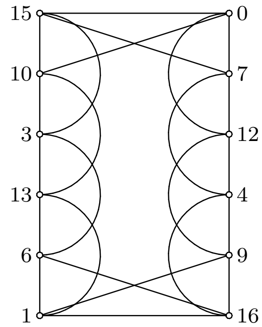

Figure 1: An optimal radio labelling for C_{12}^2.

Case 2: $n = 4k$ and k is even

If $k = 2$, then $(0, 3, 6, 1, 4, 7, 2, 5)$ gives an optimal radio labelling for $V(C_8^2) = (v_0, v_1, v_2, v_3, \cdots, v_7)$.

Assume $k \geq 4$. Define $\tau(0) = 0$ and:

$$\tau(i+1) = \begin{cases} \tau(i) + 2k & \pmod{n}, \quad \text{if } i \text{ is even;} \\ \tau(i) + k + 1 & \pmod{n}, \quad \text{if } i \text{ is odd.} \end{cases}$$

By the definition of τ, we obtain

$$d_i = d(x_i, x_{i+1}) = d(v_{\tau(i)}, v_{\tau(i+1)}) = \begin{cases} k, & \text{if } i \text{ is even;} \\ \frac{k+2}{2}, & \text{if } i \text{ is odd.} \end{cases}$$

Now we prove that f is a radio labelling.

Claim (1): τ is a permutation. For $i = 0, 1, \cdots, 2k-1$, we re-write τ by:

$$\begin{array}{lcll} \tau(2i) & = & (3k+1)i & \pmod{n}, \\ \tau(2i+1) & = & (3k+1)i + 2k & \pmod{n}. \end{array}$$

Assume $\tau(2i) = \tau(2j)$ or $\tau(2i+1) = \tau(2j+1)$ for some $0 \leq i, j \leq 2k-1$. Then we have $(3k+1)(i-j) \equiv 0 \pmod{4k}$. This implies that $i = j$, because $j - i \leq 2k - 1$, and $\gcd(4k, 3k+1) = 1$ (as k is even).

Assume $\tau(2i+1) = \tau(2j)$ for some $i, j \leq 2k-1$. Because $(3k+1) \equiv (1-k) \pmod{4k}$, we have $(i-j)(3k+1) \equiv (j-i)(k-1) \equiv -2k \pmod{4k}$. This implies that $(j-i+2)k \equiv (j-i) \pmod{4k}$. Because $j-i \leq 2k-1$, it must be either $j-i = 0$ or $j-i = k$. The former case implies that $(2k) \equiv 0 \pmod{4k}$, a contradiction; and the latter case implies that $(k+2) \equiv 1 \pmod{4}$, contradicting that k is even. Therefore, τ is a permutation.

Claim (2): $f(x_{i+2}) - f(x_i) \geq d - d(x_i, x_{i+2}) + 1$. By the definition of τ, we have, for any i:

$$\mathrm{d}(x_i, x_{i+2}) = \frac{k}{2}, \text{ and } d_i + d_{i+1} = k + \frac{k+2}{2} = \frac{3k+2}{2}.$$

Hence

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= (d - d_i + 1) + (d - d_{i+1} + 1) \\ &= 2k + 2 - \frac{3k+2}{2} \geq k - d(x_i, x_{i+2}) + 1. \end{aligned}$$

<u>Claim (3)</u>: $f(x_{i+3}) - f(x_i) \geq d - d(x_i, x_{i+3}) + 1$. By τ, we have

$$d(x_i, x_{i+3}) = \begin{cases} \frac{k+2}{2}, & \text{if } d_i + d_{i+1} + d_{i+2} = k + \frac{k+2}{2} + k; \\ 1, & \text{if } d_i + d_{i+1} + d_{i+2} = \frac{k+2}{2} + k + \frac{k+2}{2}. \end{cases}$$

By some calculation, we get

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= (d - d_i + 1) + (d - d_{i+1} + 1) + (d - d_{i+2} + 1) \\ &\geq d - d(x_i, x_{i+3}) + 1. \end{aligned}$$

<u>Claim (4)</u>: $f(x_{i+4}) - f(x_i) \geq k+1$. As $d_i + d_{i+1} + d_{i+2} + d_{i+3} = 3k+2$, so

$$\begin{aligned} f(x_{i+4}) - f(x_i) &= 4d - (d_i + d_{i+1} + d_{i+2} + d_{i+3}) + 4 \\ &= 4k + 4 - (3k+2) \ \geq k + 1. \end{aligned}$$

Therefore, f is a radio labelling with span $\frac{2k^2+3k}{2}$, as

$$\begin{aligned} f(x_{n-1}) &= (d - d_0 + 1) + (d - d_1 + 1) + \cdots + (d - d_{n-2} + 1) \\ &= (4k-1)k - (2k-1)(3k+2)/2 - k + (4k-1) = \frac{2k^2+3k}{2}. \end{aligned}$$

Figure 2 shows an optimal radio labelling for C_{16}^2 with span 22.

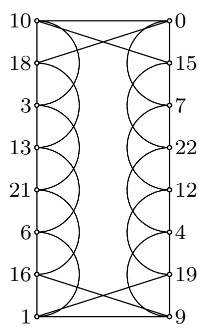

Figure 2: An optimal radio labelling for C_{16}^2.

$\boxed{\textbf{Case 3: } n = 4k+2}$ Define $\tau(0) = 0$ and:

$$\tau(i+1) = \begin{cases} \tau(i) + 2k + 1 & \pmod{n}, \quad \text{if } i \text{ is even}; \\ \tau(i) + k + 1 & \pmod{n}, \quad \text{if } i \text{ is odd}. \end{cases}$$

By τ, we obtain

$$d_i = d(x_i, x_{i+1}) = \begin{cases} k+1, & \text{if } i \text{ is even;} \\ \lceil \frac{k+1}{2} \rceil, & \text{if } i \text{ is odd.} \end{cases}$$

We now verify that f is a radio labelling.

<u>Claim (1)</u>: τ is a permutation. For $i = 0, 1, \cdots, 2k$, we re-write τ by the following:

$$\begin{array}{lcll} \tau(2i) & = & (3k+2)i & \pmod{n}, \\ \tau(2i+1) & = & (3k+2)i + 2k + 1 & \pmod{n}. \end{array}$$

Assume $\tau(2i) = \tau(2j)$ or $\tau(2i+1) = \tau(2j+1)$, for some $i, j \leq 2k$. Then, $(3k+2)(i-j) \equiv 0 \pmod{4k+2}$. Moreover, because $3k+2 = -k \pmod{n}$, we have $(j-i)k \equiv 0 \pmod{n}$. This implies that $i = j$, as $j - i \leq 2k < \frac{n}{2}$ and $\gcd(k, 4k+2) \leq 2$.

Assume $\tau(2i) = \tau(2j+1)$, for some $i, j \leq 2k$. Then, we have $(3k+2)(i-j) \equiv k(j-i) \equiv 2k+1 \pmod{4k+2}$. This implies that $(j-i-2)k \equiv 1 \pmod{4k+2}$. Hence, $\gcd(k, 4k+2) = 1$, and so k must be odd. Note that, $(2k+1)(k-1) = (4k+2)\left(\frac{k-1}{2}\right) \equiv 0 \pmod{4k+2}$, so $(2k-1)k \equiv 1 \pmod{4k+2}$. Therefore, $j-i-2 = 2k-1$, which is impossible, as $j-i \leq 2k$. Therefore, τ is a permutation.

<u>Claim (2)</u>: $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i, x_{i+2})+1$. By τ, we have $d_i + d_{i+1} = k+1+\lceil \frac{k+1}{2} \rceil$ and $d(x_i, x_{i+2}) = \lceil \frac{k}{2} \rceil$. Hence,

$$\begin{array}{rcl} f(x_{i+2}) - f(x_i) & = & 2(k+2) - (k+1+\lceil \dfrac{k+1}{2} \rceil) \\ & = & k+1-d(x_i, x_{i+2})+1. \end{array}$$

<u>Claim (3)</u>: $f(x_{i+3}) - f(x_i) \geq k+1-d(x_i, x_{i+3})+1$. By τ, we have

$$d(x_i, x_{i+3}) = \begin{cases} 1, & \text{if } d_i + d_{i+1} + d_{i+2} = k+1+2\lceil \frac{k+1}{2} \rceil; \\ \lceil \frac{k+1}{2} \rceil, & \text{if } d_i + d_{i+1} + d_{i+2} = 2k+2+\lceil \frac{k+1}{2} \rceil. \end{cases}$$

By some calculation, we conclude that

$$f(x_{i+3}) - f(x_i) = (d-d_i+1)+(d-d_{i+1}+1)+(d-d_{i+2}+1) \geq k+2-d(x_i, x_{i+3}).$$

<u>Claim (4)</u>: $f(x_{i+4}) - f(x_i) \geq k+2$. This follows by the fact that $d_i + d_{i+1} + d_{i+2} + d_{i+3} \leq 2(k+1+\frac{k+2}{2})$.

Hence, f is a radio labelling. The span of f has two possibilities:

If k is odd, then $d_{2i} + d_{2i+1} = k+1+\frac{k+1}{2} = \frac{3k+3}{2}$ for all i. Therefore,

$$\begin{array}{rcl} f(x_{n-1}) & = & (n-1)d + (n-1) - (d_0 + d_1 + d_2 + \cdots + d_{n-2}) \\ & = & (4k+1)(k+2) - (k(3k+3)+k+1) = k^2+5k+1. \end{array}$$

If k is even, then $d_{2i} + d_{2i+1} = k + 1 + \frac{k+2}{2} = \frac{3k+4}{2}$ for all i. So, we get $f(x_{n-1}) = k^2 + 4k + 1$. ∎

Figure 3 shows optimal radio labellings for C_{22}^2 and C_{10}^2 with spans 50 and 13, respectively.

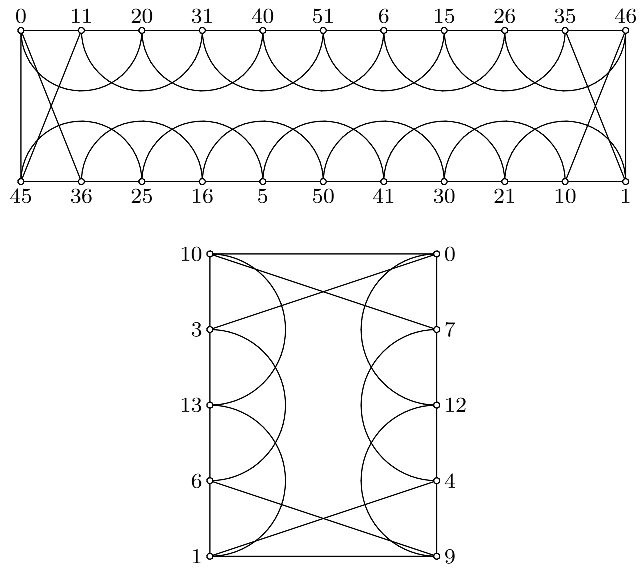

Figure 3: An optimal radio labelling for C_{22}^2 and C_{10}^2, respectively.

4 Square of Odd Cycles

The radio number for the square of odd cycles turned out to be more complicated than the even cycles. We show that the lower bound proved in Theorem 2 is sharp for many square of odd cycles, while this is not the case for some others.

Theorem 4 *Let C_n^2 be an n-vertex square cycle with $n = 4k + 1$. Then*

$$rn(C_n^2) = \begin{cases} k^2 + 2k, & \text{if } k \text{ is even;} \\ k^2 + k, & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

Moreover, if $k \equiv 1 \pmod 4$, then

$$k^2 + k + 1 \le rn(C_{4k+1}^2) \le k^2 + k + 2.$$

Proof If $k \equiv 0, 2, 3$, by Lemma 2, it suffices to find a radio labelling f with span equal to the desired number. In each of the following labelling f, we use the same notations as in the previous section.

$\boxed{\textbf{Case 1: } k \equiv 0 \pmod 4}$ Define τ by:

$$\begin{array}{lcll}
\tau(4i) & = & 2i(3k+1) & \pmod n, \quad 0 \le i \le k; \\
\tau(4i+1) & = & 2i(3k+1) + 2k - i & \pmod n, \quad 0 \le i \le k-1; \\
\tau(4i+2) & = & (2i+1)(3k+1) & \pmod n, \quad 0 \le i \le k-1; \\
\tau(4i+3) & = & (2i+1)(3k+1) + 2k - i & \pmod n, \quad 0 \le i \le k-1.
\end{array}$$

By the definition of τ, we obtain

$$d_{4i} = d_{4i+2} = \left\lceil \frac{2k-i}{2} \right\rceil, \text{ and } d_{4i+1} = d_{4i+3} = \left\lceil \frac{k+1+i}{2} \right\rceil.$$

We verify that f is a radio labelling with span equal to the desired number.

Claim (1): τ is a permutation. By some calculation, using the facts that $(3k+1) \equiv -k \pmod{4k+1}$ and $2(3k+1) \equiv 2k+1 \pmod{4k+1}$, we re-write τ as:

$$\begin{array}{lcll}
\tau(4i) & = & -2ik & \pmod n, \quad 0 \le i \le k; \\
\tau(4i+1) & = & 2(i+1)k & \pmod n, \quad 0 \le i \le k-1; \\
\tau(4i+2) & = & -(2i+1)k & \pmod n, \quad 0 \le i \le k-1; \\
\tau(4i+3) & = & (2i+1)k & \pmod n, \quad 0 \le i \le k-1.
\end{array}$$

Thus, for any $0 \le j \le n-1$, $\tau(j) = ak \pmod n$ for some $-2k \le a \le 2k$. Moreover, because there are $4k+1 = n$ integers in the interval $[-2k, 2k]$, we conclude that if $\tau(j) = ak$, $\tau(j') = a'k$ and $a = a'$, then $j = j'$.

Assume that $\tau(j) = \tau(j')$ for some j, j'. Let $\tau(j) = ak$ and $\tau(j') = a'k$, for some $-2k \le a, a' \le 2k$. Then $(a-a')k \equiv 0 \pmod{4k+1}$. This implies that $a = a'$, since $\gcd(k, 4k+1) = 1$ and $a - a' \le 4k$. By the discussion in the previous paragraph, it must be that $j = j'$. So, τ is a permutation.

Claim (2): $f(x_{i+2}) - f(x_i) \ge k - d(x_i, x_{i+2}) + 1$. It suffices to show that $d_i + d_{i+1} \le d(x_i, x_{i+2}) + k + 1$. Let $i = \tau(4j+t)$, and $i+2 = \tau(4j'+t')$ for some $0 \le t, t' \le 3$. By the definition of τ, it suffices to consider the following two cases:

Assume $j = j'$. Then $d(x_i, x_{i+2}) = \frac{k}{2}$, and $d_i + d_{i+1} = \lceil \frac{2k-j}{2} \rceil + \lceil \frac{k+1+j}{2} \rceil \le \frac{3k+2}{2}$. So, the claim follows.

Assume $j' = j+1$. Then $d(x_i, x_{i+2}) = \frac{k+2}{2}$, and $d_i + d_{i+1} = \lceil \frac{k+1+j}{2} \rceil + \lceil \frac{2k-(j+1)}{2} \rceil \le \frac{3k+2}{2}$. So, the claim follows.

Claim (3): $f(x_{i+3}) - f(x_i) \ge k+1-d(x_i, x_{i+3})+1$. It suffices to show that $d_i + d_{i+1} + d_{i+2} \le d(x_i, x_{i+3}) + 2k + 2$. Let $i = \tau(4j+t)$ and $i+3 = \tau(4j'+t')$, for some $0 \le t, t' \le 3$. By the definition of τ, we consider the following two cases:

Assume $j = j'$. Then, $d(x_i, x_{i+3}) \in \{\lceil \frac{k-j}{2} \rceil, \lceil \frac{j+1}{2} \rceil\}$, and

$$d_i + d_{i+1} + d_{i+2} = \begin{cases} \lceil \frac{2k-j}{2} \rceil + \lceil \frac{k+1+j}{2} \rceil + \lceil \frac{2k-j}{2} \rceil, & \text{if } d(x_i, x_{i+3}) = \lceil \frac{k-j}{2} \rceil \ ; \\ \lceil \frac{k+1+j}{2} \rceil + \lceil \frac{2k-j}{2} \rceil + \lceil \frac{k+1+j}{2} \rceil, & \text{if } d(x_i, x_{i+3}) = \lceil \frac{j+1}{2} \rceil. \end{cases}$$

By some calculation, we get $d_i + d_{i+1} + d_{i+2} \le d(x_i, x_{i+3}) + 2k + 2$. So, the claim follows.

Assume $j' = j + 1$. Then, $d(x_i, x_{i+3}) \in \{\lceil \frac{k-1-j}{2} \rceil, \lceil \frac{j+1}{2} \rceil\}$, and

$$d_i + d_{i+1} + d_{i+2} = \begin{cases} \lceil \frac{2k-j}{2} \rceil + \lceil \frac{k+1+j}{2} \rceil + \lceil \frac{2k-j-1}{2} \rceil, & d(x_i, x_{i+3}) = \lceil \frac{k-j-1}{2} \rceil; \\ \lceil \frac{k+1+j}{2} \rceil + \lceil \frac{2k-j-1}{2} \rceil + \lceil \frac{k+j+2}{2} \rceil, & d(x_i, x_{i+3}) = \lceil \frac{j+1}{2} \rceil. \end{cases}$$

By some calculation, we get $d_i + d_{i+1} + d_{i+2} \le d(x_i, x_{i+3}) + 2k + 2$.

Claim (4): $f(x_{i+4}) - f(x_i) \ge k + 1$. By the definition of τ, we have $d_i + d_{i+1} + d_{i+2} + d_{i+3} \le 3k$. So the result follows.

Therefore, f is a radio labelling.

Notice that $d_{2i} + d_{2i+1} = \frac{3k+2}{2}$ for all i. So the span of f is $k^2 + 2k$, because

$$\begin{aligned} f(x_{n-1}) &= (d - d_0 + 1) + (d - d_1 + 1) + \cdots + (d - d_{n-2} + 1) \\ &= (4k)(k+1) - (d_0 + d_1 + d_2 + \cdots + d_{n-2}) = k^2 + 2k. \end{aligned}$$

Figure 4 shows an optimal radio labelling for C_{17}^2 with span 24.

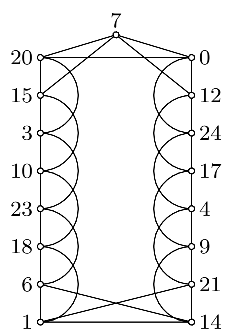

Figure 4: An optimal radio labelling for C_{17}^2.

$\boxed{\textbf{Case 2: } k \equiv 3 \pmod 4}$ For $0 \le i \le n-1$, let:

$$\tau(i) = \left(\frac{3k+1}{2}\right) i \pmod n.$$

Hence, for $0 \le i \le n-2$,

$$d_i = \left\lceil \frac{3k+1}{4} \right\rceil = \frac{3k+3}{4}.$$

We now show that the labelling f is a radio labelling.

Claim (1): τ is a permutation. Assume, to the contrary, that $\left(\frac{3k+1}{2}\right) i \equiv \left(\frac{3k+1}{2}\right) i' \pmod{4k+1}$ for some $0 \le i, i' \le 4k$ and $i \ne i'$. This is impossible, since by the Euclidean Algorithm (assuming $k = 4m+3$ for some m), one can easily show that $\gcd(4k+1, \frac{3k+1}{2}) = \gcd(16m+13, 6m+5) = 1$.

Claim (2): $f(x_{i+2}) - f(x_i) \ge k - d(x_i, x_{i+2}) + 1$. By the definition of τ, we have $d(x_i, x_{i+2}) = \frac{k+1}{2}$, and $d_i + d_{i+1} = 2 \cdot (\frac{3k+3}{4})$. So, the result follows.

Claim (3): $f(x_{i+3}) - f(x_i) \ge k - d(x_i, x_{i+3}) + 1$. By τ, we have $d(x_i, x_{i+3}) = \frac{k+1}{4}$, and $d_i + d_{i+1} + d_{i+2} = 3 \cdot (\frac{3k+3}{4})$. So the result follows.

Claim (4): $f(x_{i+4}) - f(x_i) \ge k+1$. This follows directly by $d_i + d_{i+1} + d_{i+2} + d_{i+3} = 3k+3$.

The span of f is $k^2 + k$, because $d_0 + d_1 + d_2 + \cdots + d_{n-2} = k(3k+3)$.

Figure 5 shows an optimal radio labelling for C_{29}^2 with span 56.

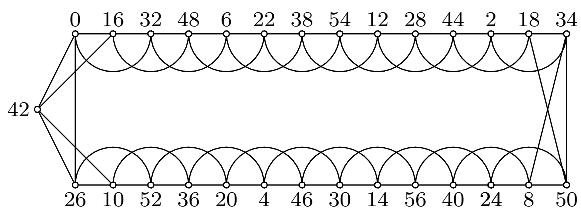

Figure 5: An optimal radio labelling for C_{29}^2.

Case 3: $k \equiv 2 \pmod 4$ To define the permutation τ, we first partition the vertices of C_n^2 into eight sets, in the following order:

$$A, D, G, B, E, H, C, F,$$

such that A consists of the first $(k/2)+1$ vertices, $A = \{v_0, v_1, v_2, \cdots, v_{k/2}\}$, and each other set consists of $k/2$ consecutive numbers. For instance, $D = \{v_{\frac{k}{2}+1}, \cdots, v_{k-1}, v_k\}$ and $F = \{v_{\frac{7k}{2}+1}, \cdots, v_{4k}\}$.

Secondly, we arrange the vertices in the above sets into Table 1: The permutation of τ is defined by the ordering given by the arrows Table 1. That is, $x_0 = v_0$, $x_1 = v_{\frac{3k}{2}+1}$, $\cdots$, $x_{4k} = v_{k/2}$, etc.

A	B	C	D	E	F	G	H
v_0	$\rightarrow v_{\frac{3k}{2}+1}$	$\rightarrow v_{3k+1}$	$\rightarrow v_{\frac{k}{2}+1}$	$\rightarrow v_{2k+1}$	$\rightarrow v_{\frac{7k}{2}+1}$	$\rightarrow v_{k+1}$	$\rightarrow v_{\frac{5k}{2}+1}$
$\rightarrow v_1$	$\rightarrow v_{\frac{3k}{2}+2}$	$\rightarrow v_{3k+2}$	$\rightarrow v_{\frac{k}{2}+2}$	$\rightarrow v_{2k+2}$	$\rightarrow v_{\frac{7k}{2}+2}$	$\rightarrow v_{k+2}$	$\rightarrow v_{\frac{5k}{2}+2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\rightarrow v_{\frac{k}{2}-1}$	$\rightarrow v_{\frac{3k}{2}+\frac{k}{2}}$	$\rightarrow v_{3k+\frac{k}{2}}$	$\rightarrow v_{\frac{k}{2}+\frac{k}{2}}$	$\rightarrow v_{2k+\frac{k}{2}}$	$\rightarrow v_{\frac{7k}{2}+\frac{k}{2}}$	$\rightarrow v_{k+\frac{k}{2}}$	$\rightarrow v_{\frac{5k}{2}+\frac{k}{2}}$
$\rightarrow v_{\frac{k}{2}}$							

Table 1:

By the arrangement of the vertices in Table 1, we observe that, for any i:

$$d_i = d(x_i, x_{i+1}) = \frac{3k+2}{4};\ d(x_i, x_{i+2}) = \frac{k}{2};\ \text{and } d(x_i, x_{i+3}) = \left\lceil \frac{k}{4} \right\rceil = \frac{k+2}{4}.$$

To verify that f is a radio labelling, it suffices to prove Claims $(2-4)$.
Claim (2): $f(x_{i+2}) - f(x_i) \geq k - d(x_i, x_{i+2}) + 1$. By the observation above, we have

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= 2k + 2 - (d_i + d_{i+1}) \\ &= 2k + 2 - (2 \cdot \frac{3k+2}{4}) \geq k - d(x_i, x_{i+2}) + 1. \end{aligned}$$

Claim (3): $f(x_{i+3}) - f(x_i) \geq k - d(x_i, x_{i+3}) + 1$. Similar to Claim (2), as $d_i + d_{i+1} + d_{i+2} = 3 \cdot (\frac{3k+2}{4})$, we get

$$f(x_{i+3}) - f(x_i) = 3k + 3 - (d_i + d_{i+1} + d_{i+2}) \geq k - d(x_i, x_{i+3}) + 1.$$

Claim (4): $f(x_{i+4}) - f(x_i) \geq k + 1$. This can be proved similar to Claim (3).

Therefore, f is a radio labelling. The span of f is $k^2 + 2k$, as $d_0 + d_1 + d_2 + \cdots + d_{n-2} = k(3k+2)$.

Figure 6 shows an optimal radio labelling for C_{25}^2 with span 48.

$\boxed{\textbf{Case 4: } k \equiv 1 \pmod 4}$ We first show that $rn(C_n^2) > k^2 + k$. Suppose, to the contrary, there exists a radio labelling f with span $k^2 + k$. Since k is odd, by Lemma 1 and Theorem 2, we have

$$f(x_{i+2}) - f(x_i) = \frac{k+1}{2},\ \text{for all } i. \tag{4.1}$$

Referring to the proof of Lemma 1, we get

$$l_i, l_{i+1}, l_i^* \text{ are all odd for any } i, \tag{4.2}$$

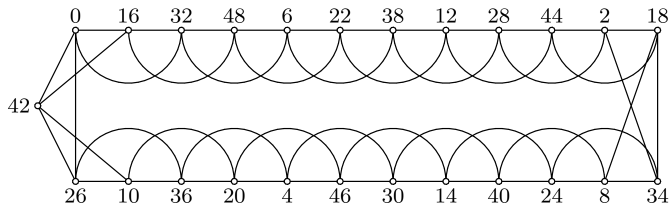

Figure 6: An optimal radio labelling for C_{25}^2.

and all the in-equalities in (2.1), (2.2) and (2.3) are equalities (note, because the span of f is k^2+k and because k is odd, it must be that $l_i, l_{i+1} \geq k+1$, and $l_i^* \geq k$). In particular, $f(x_{i+2}) - f(x_i) = k + 1 - \frac{l_i^*+1}{2}$ for any $0 \leq i \leq n-2$. Combining this with (4.1), we get

$$l_i^* = \mathrm{d}_{C_n}(x_{i+2}, x_i) = k, \text{ for all } i. \tag{4.3}$$

By definition, for any i, $1 \leq f(x_{i+1}) - f(x_i) \leq f(x_{i+2}) - f(x_i) = \frac{k-1}{2}$. So, we have $1 \leq k+1-\frac{l_i+1}{2} \leq \frac{k-1}{2}$, which implies

$$k+2 \leq l_i = \mathrm{d}_{C_n}(x_{i+1}, x_i) \leq 2k-1. \tag{4.4}$$

Because $f(x_0) = 0$, by (4.1) and (4.3), we have

$$f(x_{2i}) = f(v_{n-ki}) = \frac{(k+1)i}{2} \pmod{n}, \text{ for all } 0 \leq i \leq 2k.$$

Because $n = 4k+1$, the above implies that the following four blocks of vertices are occupied by $f(x_{2i})$, $i = 0, 1, 2, \cdots, 2k$:

$$\begin{aligned} A &= [v_0, \cdots, v_{\frac{k+1}{2}-1}], \\ B &= [v_{k+1}, \cdots, v_{k+\frac{k+1}{2}-1}], \\ C &= [v_{2k+1}, \cdots, v_{2k+\frac{k+1}{2}}], \\ D &= [v_{3k+1}, \cdots, v_{3k+\frac{k+1}{2}}]. \end{aligned}$$

Consider the remaining vertices (which are x_{2i+1}, $i = 0, 1, \cdots, 2k-1$):

$$\begin{aligned} W &= [v_{\frac{k+1}{2}}, \cdots, v_k], \\ X &= [v_{k+\frac{k+1}{2}}, \cdots, v_{2k}], \\ Y &= [v_{2k+\frac{k+1}{2}+1}, \cdots, v_{3k}], \\ Z &= [v_{3k+\frac{k+1}{2}+1}, \cdots, v_{4k}]. \end{aligned}$$

By (4.4), if $x_{2i} \in A$, then $x_{2i+1} \in X$, for some $i = 0, 4, 8, 12, \cdots, 2k-2$. If $x_{2i} = v_j \in A$ for some even j, by (4.3), then $x_{2i+1} = v_{j'} \in X$ for some odd j'. Notice that there are $(\frac{k-1}{4} + 1)$ vertices with even sub-indices t of v_t in block A, while there are $(\frac{k-1}{4})$ vertices with odd sub-indices t' of $v_{t'}$ in block X. By pigeonhole principle, this is impossible.

Now, it suffices to find a labelling with span $k^2 + k + 2$. Define τ by:

$$\begin{aligned}
\tau(2i) &= (3k+1)i, && i = 0, 1, \cdots 2k;\\
\tau(2i+1) &= (3k+1)i + 2k, && i = 0, 1;\\
\tau(2i+1) &= (3k+1)i + 2k - 1, && i = 2, 3;\\
\tau(2i+1) &= (3k+1)(i-4) + \tfrac{3k+1}{2}, && i = 4, 5, \cdots, 2k-1.
\end{aligned}$$

We verify that f is a radio labelling.

Claim (1): τ is a permutation. Assume $\tau(2i) = \tau(2i')$ for some $0 \le i, i' \le 2k$. Then $(3k+1)(i-i') \equiv 0 \pmod{n}$. Because $(3k+1) \equiv -k \pmod{n}$, we have $k(i'-i) \equiv 0 \pmod{n}$. This implies $i = i'$, since $i' - i \le 2k$.

Similarly, one can show that if $\tau(2i+1) = \tau(2i'+1)$ for some $i, i'=0, 1, 2, 3$, or $\tau(2i+1) = \tau(2i'+1)$ for some $i, i' \in \{4, 5, \cdots, 2k\}$, then $i = i'$.

Assume $\tau(2i) = \tau(2i'+1)$ for some $i = 0, 1, 2, \cdots, 2k$ and $i' = 4, 5, \cdots, 2k-1$. Because $4(3k+1) \equiv 1 \pmod{n}$, so we have $(3k+1)(i-i') \equiv \frac{3k-1}{2} \pmod{n}$. Multiplying by 4 to both sides, we get

$$4(3k+1)(i-i') \equiv (i-i') \equiv 6k-2 \equiv 2k-3 \pmod{n}.$$

Then, it must be $i = i'$, because $0 \le i - i' \le 2k-4$ (if $i > i'$), and $i' - i \le 2k-1$ (if $i' > i$). Note that, $-(2k-3) \equiv 2k+4 \pmod{n}$.

Assume $\tau(2i) = \tau(2i'+1)$ for some $i = 0, 1, 2, \cdots, 2k$ and $i' = 0, 1$. Then we have $(3k+1)(i-i') \equiv 2k \pmod{n}$, implying $(i'-i) \equiv 2 \pmod{n}$. This is impossible as $-1 \le i' - i \le 2k$.

Similarly, one can show that it is impossible that $\tau(2i) = \tau(2i'+1)$ for some $i = 0, 1, 2, \cdots, 2k$ and $i' = 2, 3$.

Assume $\tau(2i+1) = \tau(2i'+1)$ for some $i = 4, 5, \cdots, 2k-1$ and $i' = 0, 1$. Then, we obtain $(3k+1)(i-i') \equiv \frac{k+1}{2} \pmod{n}$. Multiplying both sides by 4, we get $i - i' \equiv 2k+2 \pmod{n}$, which is impossible, because $3 \le i - i' \le 2k-1$.

Similarly, one can show that it is impossible that $\tau(2i+1) = \tau(2i'+1)$ for some $i = 4, 5, \cdots, 2k-1$ and $i' = 2, 3$.

Claim (2): $f(x_{i+2}) - f(x_i) \ge k + 1 - d(x_i, x_{i+2})$ for all i. It suffices to show that $(d_i + d_{i+1}) - d(x_i, x_{i+2}) \le k+1$. By the definition of τ, we have $d(x_i, x_{i+2}) \ge \frac{k+1}{2}$ and $d_i + d_{i+1} \le \frac{6k+6}{4}$. Hence, the result follows.

Claim (3): $f(x_{i+3}) - f(x_i) \geq k + 1 - d(x_i, x_{i+3})$ for all i. It suffices to show that $(d_i + d_{i+1} + d_{i+2}) - d(x_i, x_{i+3}) \leq 2k + 2$. By τ, we obtain:

$$d(x_i, x_{i+3}) = \begin{cases} \frac{k+1}{2}, & \text{if } d_i + d_{i+1} + d_{i+2} \in \{\frac{5k+1}{2}, \frac{5k+3}{2}\}; \\ 1, & \text{if } d_i + d_{i+1} + d_{i+2} \in \{\frac{4k+2}{2}, \frac{4k+4}{2}, \frac{4k+6}{2}\}, \\ \frac{k-1}{4}, & \text{if } d_i + d_{i+1} + d_{i+2} = \frac{9k+7}{4}; \\ \frac{k+3}{4}, & \text{if } d_i + d_{i+1} + d_{i+2} = \frac{9k+11}{4}. \end{cases}$$

By some calculations, we conclude that the claim follows.
Claim (4): $f(x_{i+4}) - f(x_i) \geq k + 1$ for any i. By τ, we have $d_i + d_{i+1} + d_{i+2} + d_{i+3} \leq 3k + 3$ for any i. So, the result follows.

Therefore, f is a radio labelling. The span of f is $k^2 + k + 2$, because

$$\begin{aligned} f(x_{n-1}) =& (n-1)(k+1) - (d_0 + d_1 + d_2 + \cdots + d_{n-2}) \\ =& 4k^2 + 4k - \left(k + \frac{k+1}{2} + k + \frac{k+1}{2} + k + \frac{k+3}{2} + k + \frac{k+3}{2} + \right. \\ & \left. \left(\frac{3k+1}{4}\right)\left(\frac{4k-8}{2}\right) + \left(\frac{3k+5}{4}\right)\left(\frac{4k-8}{2}\right) \right) \\ =& k^2 + k + 2. \end{aligned}$$

∎

Figure 7 shows a radio labelling for C_{21}^2 with span 32.

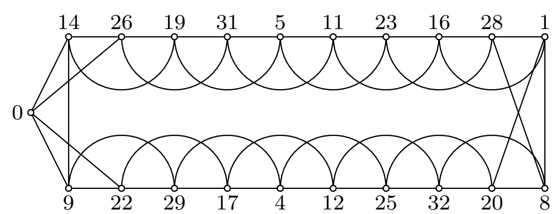

Figure 7: A radio labelling for C_{21}^2.

Conjecture 1 *If $k \equiv 3 \pmod 4$, then $rn(C_{4k+1}^2) = k^2 + k + 2$.*

For the radio numbers of C_{4k+3}^2, we first prove a lemma.

Lemma 5 *If k is odd and $\gcd(n, k) > 1$, then $rn(C_{4k+3}^2) > \frac{2k^2+7k+3}{2}$.*

Proof By Theorem 2, $rn(C_n^2) \geq \frac{2k^2+7k+3}{2}$. Assume to the contrary, there exists a radio labelling f with $f(x_0) = 0$ and span $f(x_{n-1}) = \frac{2k^2+7k+3}{2}$. Then by the proof of Lemma 1, we have $f_i + f_{i+1} = \frac{k+3}{2}$ for all i. Similar to the proof of Case 4 for $n = 4k+1$ case (see (4.1) $-$ (4.4)), by some calculation, we get for any i,

$$f(x_{i+2}) - f(x_i) = \frac{k+3}{2}, \text{ and } \ l_i^* = \mathrm{d}_{C_n}(x_{i+2}, x_i) = k.$$

This implies that,

$$f(x_{2i}) = f(v_{n-ki}) = \frac{(k+3)i}{2} \pmod{n}, \text{ for all } i.$$

This is impossible, since the function defined by $\tau(0) = 0$ and $\tau(2i) = n - ki \pmod{n}$ can not be a permutation on $\{0, 1, 2, \cdots n-1\}$, as $\gcd(n, k) \geq 3$ (since k is odd). ∎

Theorem 6 *Let C_n^2 be an n-vertex square cycle where $n = 4k+3$. If $k = 4m$ for some $m \in Z^+$, or $k = 4m+2$ for some $m \not\equiv 5 \pmod{7}$, then*

$$rn(C_n^2) = \frac{2k^2+9k+4}{2}.$$

If $k = 4m+2$ for some $m \equiv 5 \pmod{7}$, then

$$\frac{2k^2+9k+4}{2} \leq rn(C_n^2) \leq \frac{2k^2+9k+10}{2}.$$

If $k = 4m+1$ for some $m \in Z^+$, then

$$rn(C_n^2) = \begin{cases} \frac{2k^2+7k+3}{2}, & \text{if } m \equiv 0, 1 \pmod{3}; \\ \frac{2k^2+7k+5}{2}, & \text{if } m \equiv 2 \pmod{3}. \end{cases}$$

If $k = 4m+3$ for some $m \in Z^+$, then

$$rn(C_n^2) \geq \begin{cases} \frac{2k^2+7k+5}{2}, & \text{if } m \equiv 0 \pmod{3}; \\ \frac{2k^2+7k+3}{2}, & \text{if } m \equiv 1, 2 \pmod{3}. \end{cases}$$

Proof Let $n = 4k+3$. Then $d = k+1$.

By Lemma 5, the bound proved in Theorem 2 can not be achieved for the following two cases for $n = 4k+3$:

1) $k = 4m+1$ and $m \equiv 2 \pmod{3}$. Let $m = 3p+2$ for some $p \in Z^+$. Then, $\gcd(n, k) = \gcd(48p+39, 12p+9) \geq 3$.

2) $k = 4m+3$ and $m \equiv 0 \pmod 3$. Let $m = 3p$ for some $p \in Z^+$. Then $\gcd(n,k) = \gcd(48p+15, 12p+3) \geq 3$.

If 1) or 2) holds, then by Lemma 5, $rn(C^2_{4k+3}) \geq \frac{2k^2+7k+5}{2}$.

Hence, by Theorem 2 and Lemma 5, it suffices to give radio labellings for $k \equiv 0,1,2 \pmod 4$, with span achieving the desired numbers. We consider cases, and in each of the labelling f given, we use the same notations used in the previous section.

$\boxed{\textbf{Case 1: } k \equiv 0 \pmod 4}$ Let $k = 4m$. For $0 \leq i \leq n-1$, define τ by:

$$\tau(i) = \left(\frac{3k+2}{2}\right) i \pmod n.$$

Hence, for any $i = 0, 1, \cdots, n-2$,

$$d_i = \left\lceil \frac{3k+2}{4} \right\rceil = \frac{3k+4}{4}.$$

We now claim that f is a radio labelling.

Claim (1): τ is a permutation. By the Euclidean Algorithm, $\gcd(4k+3, \frac{3k+2}{2}) = \gcd(16m+3, 6m+1) = 1$. Therefore, τ is a permutation.

Claim (2): $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i, x_{i+2})+1$. By the definition of τ, we have $d(x_i, x_{i+2}) = \frac{k+2}{2}$ and $d_i + d_{i+1} = \frac{3k+4}{2}$. So, the result follows.

Claim (3): $f(x_{i+3}) - f(x_i) \geq k+1-d(x_i, x_{i+3})+1$. By τ, $d(x_i, x_{i+3}) = \frac{k}{4}$ and $d_i + d_{i+1} + d_{i+2} = \frac{9k+12}{4}$. Hence, the result follows.

Claim (4): $f(x_{i+4}) - f(x_i) \geq k+2$. This is true since $d_i + d_{i+1} + d_{i+2} + d_{i+3} = 3k+4$.

So, f is a radio labelling. The span of f is $\frac{2k^2+9k+4}{2}$, by some easy calculation.

Figure 8 shows an optimal radio labelling for C^2_{19} with span 36.

$\boxed{\textbf{Case 2: } k \equiv 2 \pmod 4}$ Let $k = 4m+2$ for $m \in Z^+$. So, $n = 16m+11$.

- **Subcase 2.1:** $m \equiv 0,1,2,3,4,6 \pmod 7$. For $0 \leq i \leq n-1$, define τ by:

$$\tau(i) = \left(\frac{3k+4}{2}\right) i \pmod n.$$

By the definition of τ, we have

$$d_i = \left\lceil \frac{3k+4}{4} \right\rceil = \frac{3k+6}{4}.$$

The labelling f is defined by $f(x_0) = f(v_0) = 0$, and for $0 \leq i \leq n-1$,

$$f(x_{i+1}) = \begin{cases} f(x_i) + k + 1 - d_i + 1, & \text{if } i \text{ is even;} \\ f(x_i) + k + 1 - d_i + 2, & \text{if } i \text{ is odd.} \end{cases}$$

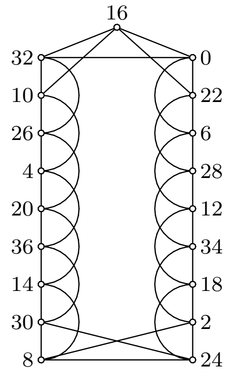

Figure 8: An optimal radio labelling for C_{19}^2.

We show that f is a radio labelling.

Claim (1): τ is a permutation. This is true, because $\gcd(n, \frac{3k+4}{2}) = \gcd(16m+11, 6m+5) = 1$.

Claim (2): $f(x_{i+2}) - f(x_i) \geq k+1-d(x_i,x_{i+2})+1$. By the definition of τ, we have $d(x_i,x_{i+2}) = \frac{k}{2}$ and $d_i + d_{i+1} = \frac{3k+6}{2}$. Hence,

$$f(x_{i+2}) - f(x_i) = 2k+5-(d_i+d_{i+1}) = k+1-d(x_i,x_{i+2})+1.$$

Claim (3): $f(x_{i+3}) - f(x_i) \geq k+1-d(x_i,x_{i+3})+1$. Because $d(x_i,x_{i+3}) = \frac{k+6}{4}$ and $d_i + d_{i+1} + d_{i+2} = \frac{9k+18}{4}$, we have

$$f(x_{i+2}) - f(x_i) \geq (d-d_i+1)+(d-d_{i+1}+2)+(d-d_{i+2}+1) \geq k+2-d(x_i,x_{i+3}).$$

Claim (4): $f(x_{i+4}) - f(x_i) \geq k+2$. Because $d_i+d_{i+1}+d_{i+2}+d_{i+3} = 3k+6$, so $f(x_{i+4}) - f(x_i) \geq 4(k+2)+2-(d_i+d_{i+1}+d_{i+2}+d_{i+3}) > k+2$.

The span of f is $\frac{2k^2+9k+4}{2}$, because

$$\begin{aligned} f(x_{n-1}) &= (d-d_i+1)+(d-d_{i+1}+2)+\cdots+(d-d_{n-2}+2) \\ &= (n-1)d + (1+2)\left(\frac{n-1}{2}\right) - \sum_{i=0}^{n-2} d_i = \frac{2k^2+9k+4}{2}. \end{aligned}$$

- **Subcase 2.2:** $m \equiv 5 \pmod 7$. It suffices to find a radio labelling with span $\frac{2k^2+9k+10}{2}$. Let $m = 7p+5$ for some $p \in Z^+$. Since $n = 4k+3$ and $k = 4m+2$, we have $gcd(n, \frac{3k+4}{2}) = 7$, and that $\frac{n}{7}$ is odd.

For $i = 0,1,2,\cdots,n-1$ and $j = 0,1,\cdots,6$, define τ by:

$$\tau(i) = \left(\frac{3k+4}{2}\right) i - j \pmod n, \quad \text{if} \quad \frac{jn}{7} \leq i < \frac{(j+1)n}{7}.$$

By the definition of τ, and since k is even, we obtain

$$d_i = \begin{cases} \frac{3k+2}{4}, & \text{for } i = \frac{jn}{7} - 1 \text{ and } j = 1, 2 \cdots, 6; \\ \frac{3k+6}{4}, & \text{otherwise.} \end{cases} \tag{4.5}$$

The labelling f is defined by $f(x_0) = f(v_0) = 0$, and

- For $j = 0, 2, 4, 6$ and $\frac{jn}{7} \le i < \frac{(j+1)n}{7}$:

$$f(x_{i+1}) = \begin{cases} f(x_i) + k + 1 - d_i + 1, & \text{if } i \text{ is even;} \\ f(x_i) + k + 1 - d_i + 2, & \text{if } i \text{ is odd.} \end{cases} \tag{4.6}$$

- For $j = 1, 3, 5$ and $\frac{jn}{7} \le i < \frac{(j+1)n}{7}$:

$$f(x_{i+1}) = \begin{cases} f(x_i) + k + 1 - d_i + 1, & \text{if } i \text{ is odd;} \\ f(x_i) + k + 1 - d_i + 2, & \text{if } i \text{ is even.} \end{cases} \tag{4.7}$$

We verify that f is a radio labelling.
Claim(1): τ is a permutation. Assume, to the contrary, that $\tau(i) = \tau(i')$ for some $0 \le i, i' \le n-1$. Then we have $\left(\frac{3k+4}{2}\right) i - j \equiv \left(\frac{3k+4}{2}\right) i' - j' \pmod{n}$ for some $0 \le j, j' \le 6$. This implies $(i - i')\left(\frac{3k+4}{2}\right) \equiv j - j' \pmod{n}$. Because $\frac{3k+4}{2} \equiv n \equiv 0 \pmod{7}$ and $0 \le |j - j'| \le 6$, it must be that $j = j'$. Hence, $|i - i'| < \frac{n}{7}$. (Recall that $gcd(n, \frac{3k+4}{2}) = 7$.) Let $n = 7t$ and $\frac{3k+4}{2} = 7s$ for some $s, t \in Z^+$ with $gcd(s,t) = 1$. Then we have $(i - i')7s \equiv 0 \pmod{7t}$, implying $(i - i')s \equiv 0 \pmod{t}$. This is impossible because $|i - i'| < \frac{n}{7} = t$ and $gcd(s,t) = 1$. Therefore, τ is a permutation.
Claim(2): $f(x_{i+2}) - f(x_i) \ge k + 1 - d(x_i, x_{i+2}) + 1$ for any i. By the definition of τ, we have $d(x_i, x_{i+2}) = \frac{k}{2}$. By (4.5), (4.6) and (4.7), we have

$$d_i + d_{i+1} = \begin{cases} \frac{3k+4}{2}, & \text{if } f_i = d - d_i + 1 \text{ and } f_{i+1} = d - d_{i+1} + 1; \\ \frac{3k+6}{2}, & \text{otherwise.} \end{cases}$$

Now we consider the two cases in the above separately.
If $d_i + d_{i+1} = \frac{3k+4}{2}$, then

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= (d - d_i + 1) + (d - d_{i+1} + 1) \\ &= k + 2 - d(x_i, x_{i+2}). \end{aligned}$$

If $d_i + d_{i+1} = \frac{3k+6}{2}$, then we obtain

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= (d - d_i + 1) + (d - d_{i+1} + 2) \\ &= k + 2 - d(x_i, x_{i+2}). \end{aligned}$$

Claim(3): $f(x_{i+3}) - f(x_i) \geq k+1-d(x_i, x_{i+3})+1$ for any i. By the definition of τ, we have $d(x_i, x_{i+3}) = \frac{k+6}{4}$. By (4.5), we have $d_i + d_{i+1} + d_{i+2} \in \{\frac{9k+18}{4}, \frac{9k+14}{4}\}$. Hence, by (4.6) and (4.7), we have

$$
\begin{aligned}
f(x_{i+3}) - f(x_i) &= f_i + f_{i+1} + f_{i+2} \\
&\geq (d-d_i+1)+(d-d_{i+1}+1)+(d-d_{i+2}+1) \\
&\geq 3k+6-\frac{9k+18}{4} \geq k+2-d(x_i, x_{i+3}).
\end{aligned}
$$

Claim(4): $f(x_{i+4}) - f(x_i) \geq k+2$ for any i. By (4.5), $d_i + d_{i+1} + d_{i+2} + d_{i+3} \in \{3k+6, 3k+5\}$. So, the result follows by a similar calculation to the above.

Therefore, f is a radio labelling, with span $\frac{2k^2+9k+10}{2}$, because

$$
\begin{aligned}
f(x_{n-1}) &= (d-d_0+1)+(d-d_1+2)+\cdots+(d-d_{\frac{n}{7}-1}+1) \\
&\quad +(d-d_{\frac{n}{7}}+1)+(d-d_{\frac{n}{7}+1}+2)+\cdots+(d-d_{\frac{2n}{7}-1}+1) \\
&\quad \vdots \\
&\quad +(d-d_{\frac{6n}{7}}+1)+(d-d_{\frac{6n}{7}+1}+2)+\cdots+(d-d_{n-2}+2) \\
&= (n-1)d+3(\frac{n-7}{2})+6-\left((n-7)(\frac{3k+6}{4})+6(\frac{3k+2}{4})\right) \\
&= \frac{2k^2+9k+10}{2}.
\end{aligned}
$$

Case 3: $k \equiv 1 \pmod 4$ Let $k = 4m+1$ for some $m \in Z^+$.

- **Subcase 3.1:** $m \equiv 0, 1 \pmod 3$. For $0 \leq i \leq n-1$, define τ by:

$$\tau(i) = \left(\frac{3k+3}{2}\right) i \pmod n.$$

Hence,

$$d_i = \left\lceil \frac{3k+3}{4} \right\rceil = \frac{3k+5}{4}.$$

The proof that f is a radio labelling with the desired span is similar to Case 2 of Theorem 4. We leave the details to the reader (cf. [11]).

Figure 9 shows an optimal radio labelling for C_{23}^2 with span 44.

- **Subcase 3.2:** $m \equiv 2 \pmod 3$. Define $\tau(0) = 0$ and

$$
\begin{array}{lcll}
\tau(2i+1) & = & 3(k+1)i+2k+1 \pmod n, & \text{if } 0 < 2i+1 < \frac{2n}{3}; \\
\tau(2i) & = & 3(k+1)i \pmod n, & \text{if } 1 \leq 2i < \frac{2n}{3}; \\
\tau(i) & = & 4k+2+(i-\frac{2n}{3})\left(\frac{3k+3}{2}\right) \pmod n, & \text{if } \frac{2n}{3} \leq i \leq n-1.
\end{array}
$$

Recall that $n = 4k+3$, $k = 4m+1$ and $m \equiv 2 \pmod 3$. So, $n = 16m+7$ and $\frac{2n}{3}$ is even. Therefore, the above definition of τ implies that when

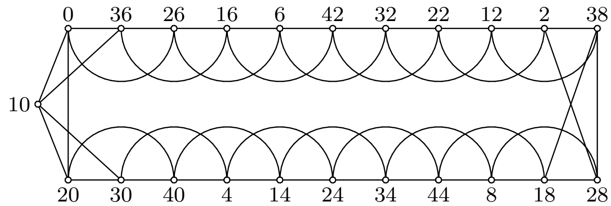

Figure 9: An optimal radio labelling for C_{23}^2.

$i = \frac{2n}{3}$, then $\tau(i-1) = 3k+1$ and $\mathrm{d}(x_{i-1}, x_i) = \frac{k+1}{2}$. By some calculation, we obtain

$$d_i = \begin{cases} k+1, & \text{if } i \text{ is even and } 0 \le i < \frac{2n}{3} - 1; \\ \frac{k+3}{2}, & \text{if } i \text{ is odd and } 0 \le i < \frac{2n}{3} - 1; \\ \frac{k+1}{2}, & \text{if } i = \frac{2n}{3} - 1; \\ \frac{3k+5}{4}, & \text{if } \frac{2n}{3} \le i < n-2; \end{cases}$$

We now verify that f is a radio labelling.

Claim (1): τ is a permutation. Because $m \equiv 2 \pmod 3$, so $n \equiv k \equiv 0 \pmod 3$. Therefore, we have

$$\begin{array}{llll} \tau(2i) & \equiv 0 & \pmod 3, & 0 \le 2i < \frac{2n}{3}, \\ \tau(2i+1) & \equiv 1 & \pmod 3, & 0 \le 2i+1 < \frac{2n}{3}, \\ \tau(i) & \equiv 2 & \pmod 3, & \frac{2n}{3} \le i \le n-1. \end{array}$$

Hence, it suffices to show that τ is one-to-one within each case.

Assume $\tau(2j) = \tau(2i)$ for some $0 \le 2i, 2j < \frac{2n}{3}$. Then $(k+1)(j-i) \equiv 0 \pmod{\frac{n}{3}}$. Letting $m = 3p+2$, we get $\gcd(k+1, \frac{n}{3}) = \gcd(12p+10, 16p+13) = 1$. Hence, $j \equiv i \pmod{\frac{n}{3}}$, implying that $i = j$, as $i, j < \frac{n}{3}$.

By the same argument, one can show that $\tau(2i+1) = (2k+1)+(3k+3)i \pmod n$ is one-to-one for $0 \le 2i+1 < \frac{2n}{3}$.

Assume $\tau(j) = \tau(i)$ for some $i, j \ge \frac{2n}{3}$. Then $\frac{k+1}{2}(j-i) \equiv 0 \pmod{n/3}$. This implies that $j = i$, since $j - i < n/3$. Therefore, τ is a permutation.

Claim (2): $f(x_{i+2}) - f(x_i) \ge k+1-d(x_i, x_{i+2})+1$. It suffices to show that $d_i + d_{i+1} - d(x_i, x_{i+2}) \le k+2$. By the definition of τ, $d(x_i, x_{i+2}) = \frac{k+1}{2}$ and $d_i + d_{i+1} \ge \frac{5k+7}{4}$. Thus, the result follows.

Claim (3): $f(x_{i+3}) - f(x_i) \ge k+1-d(x_i, x_{i+2})+1$. It suffices to show $d_i + d_{i+1} + d_{i+2} - d(x_i, x_{i+3}) \le 2k+4$. By the definition of τ, we obtain:

$$d(x_i, x_{i+3}) = \begin{cases} \frac{k+1}{2}, & \text{if } d_i + d_{i+1} + d_{i+2} = \frac{5k+7}{2}; \\ 1, & \text{if } d_i + d_{i+1} + d_{i+2} \in \{2k+4, 2k+3\}; \\ \frac{k+3}{4}, & \text{if } d_i + d_{i+1} + d_{i+2} \in \{\frac{9k+15}{4}, \frac{9k+11}{4}\}. \end{cases}$$

By some calculation, we conclude that $d_i + d_{i+1} + d_{i+2} - d(x_i, x_{i+3}) \leq 2k+4$.
Claim (4): $f(x_{i+4}) - f(x_i) \geq k+2$. The result follows by the fact that $d_i + d_{i+1} + d_{i+2} + d_{i+3} \leq 3k+5$.

Therefore, f is a radio labelling with span $\frac{2k^2+7k+5}{2}$, as

$$\begin{aligned}
f(x_{n-1}) &= (d - d_0 + 1) + (d - d_1 + 1) + \cdots + (d - d_{n-2} + 1) \\
&= (4k+2)(k+2) - (d_0 + d_2 + \cdots + d_{\frac{2n}{3}-2}) - \\
&\quad (d_1 + d_3 + \cdots + d_{\frac{2n}{3}-3}) - d_{\frac{2n}{3}-1} - (d_{\frac{2n}{3}} + \cdots + d_{n-2}) \\
&= 4k^2 + 10k + 4 - \\
&\quad (\frac{8k+6}{6})(k+1) - (\frac{4k}{3})(\frac{k+3}{2}) - \frac{k+1}{2} - (\frac{3k+5}{4})(\frac{4k}{3}) \\
&= \frac{2k^2+7k+5}{2}.
\end{aligned}$$

∎

Figure 10 shows an optimal radio labelling for C_{39}^2 with span 115.

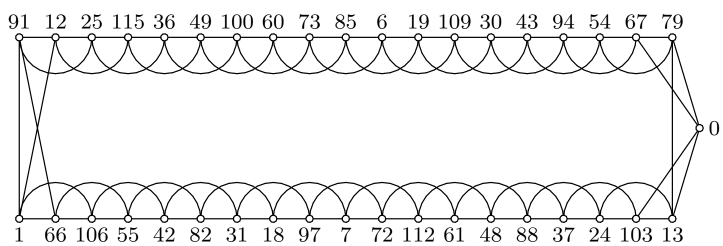

Figure 10: An optimal radio labelling for C_{39}^2.

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