REALISM AND ANTI-REALISM IN MATHEMATICS

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The purpose of this essay is (a) to survey and critically assess the various metaphysical views — i.e., the various versions of realism and anti-realism — that people have held (or that one might hold) about mathematics; and (b) to argue for a particular view of the metaphysics of mathematics. Section 1 will provide a survey of the various versions of realism and anti-realism. In section 2, I will critically assess the various views, coming to the conclusion that there is exactly one version of realism that survives all objections (namely, a view that I have elsewhere called *full-blooded platonism*, or for short, FBP) and that there is exactly one version of anti-realism that survives all objections (namely, *fictionalism*). The arguments of section 2 will also motivate the thesis that we do not have any good reason for favoring either of these views (i.e., fictionalism or FBP) over the other and, hence, that we do not have any good reason for believing or disbelieving in abstract (i.e., non-spatiotemporal) mathematical objects; I will call this the weak epistemic conclusion. Finally, in section 3, I will argue for two further claims, namely, (i) that we could *never* have any good reason for favoring either fictionalism or FBP over the other and, hence, could never have any good reason for believing or disbelieving in abstract mathematical objects; and (ii) that there is no fact of the matter as to whether fictionalism or FBP is correct and, more generally, no fact of the matter as to whether there exist any such things as abstract objects; I will call these two theses the strong epistemic conclusion and the metaphysical conclusion, respectively.

(I just said that in section 2, I will argue that FBP and fictionalism survive all objections; but if I'm right that there is no fact of the matter as to whether FBP or fictionalism is correct, then it can't be that these two views survive *all* objections, for surely my no-fact-of-the-matter argument constitutes an objection of some sort to both FBP and fictionalism. This, I think, is correct, but for the sake of simplicity, I will ignore this point until section 3. During sections 1 and 2, I will defend FBP and fictionalism against the various traditional objections to realism and anti-realism — e.g., the Benacerrafian objections to platonism and the Quine-Putnam objection to anti-realism — and in doing this, I will write as if I think FBP and fictionalism are completely defensible views; but my section-3 argument for the claim that there is no fact of the matter as to which of these two views is correct does undermine the two views.)

Large portions of this paper are reprinted, with a few editorial changes, from my book, *Platonism and Anti-Platonism in Mathematics* (Oxford University Press,

Handbook of the Philosophy of Science. Philosophy of Mathematics
Volume editor: Andrew D. Irvine. General editors: Dov M. Gabbay, Paul Thagard and John Woods.

1998)[1] — though I should say that there are also several new sections here. Now, of course, because of space restrictions, many of the points and arguments in the book have not been included here, but the overall plan of this essay mirrors that of the book. One important difference, however, is this: while the book is dedicated more to developing my own views and arguments than to surveying and critiquing the views of others, because this is a survey essay, the reverse is true here. Thus, in general, the sections of the book that develop my own views have been pared down far more than the sections that survey and critique the views of others. Indeed, in connection with my own views, all I really do in this essay is briefly sketch the main ideas and arguments and then refer the reader to the sections of the book that fill these arguments in. Indeed, I refer the reader to my book so many times here that, I fear, it might get annoying after a while; but given the space restrictions for the present essay, I couldn't see any other way to preserve the overall structure of the book — i.e., to preserve the defenses of FBP and fictionalism and the argument for the thesis that there is no fact of the matter as to which of these two views is correct — than to omit many of the points made in the book and simply refer the reader to the relevant passages.

1 A SURVEY OF POSITIONS

Mathematical realism (as I will use the term here) is the view that our mathematical theories are true descriptions of some real part of the world. *Mathematical anti-realism*, on the other hand, is just the view that mathematical realism is false; there are lots of different versions of anti-realism (e.g., formalism, if-thenism, and fictionalism) but what they all have in common is the view that mathematics does not have an ontology (i.e., that there are no objects that our mathematical theories are about) and, hence, that these theories do not provide true descriptions of some part of the world. In this section, I will provide a survey of the various versions of realism and anti-realism that have been endorsed, or that one might endorse, about mathematics. Section 1.1 will cover the various versions of realism and section 1.2 will cover the various versions of anti-realism.

1.1 Mathematical Realism

Within the realist camp, we can distinguish *mathematical platonism* (the view that there exist abstract mathematical objects, i.e., non-spatiotemporal mathematical objects, and that our mathematical theories provide true descriptions of such objects) from *anti-platonistic realism* (the view that our mathematical theories are true descriptions of concrete, i.e., spatiotemporal, objects). Furthermore, within anti-platonistic realism, we can distinguish between *psychologism* (the view that our mathematical theories are true descriptions of mental objects) and *mathematical physicalism* (the view that our mathematical theories are true descriptions of some non-mental part of physical reality). Thus, the three kinds of realism are platonism, psychologism, and physicalism. (One might think there is a fourth realistic view here, namely, Meinongianism. I will discuss this view below, but for now, let me just say that I do not think there is fourth version of realism here; I think that Meinongianism either isn't a realistic view or else is equivalent to platonism.)

[1] I would like to thank Oxford University Press for allowing the material to be reprinted.

I should note here that philosophers of mathematics sometimes use the term 'realism' interchangeably with 'platonism'. This, I think, is not because they deny that the logical space of possible views includes anti-platonistic realism, but rather, because it is widely thought that platonism is the only really tenable version of realism. I think that this is more or less correct, but since I am trying to provide a comprehensive survey, I will cover anti-platonistic realism as well as platonistic realism. Nontheless, since I think the latter is much more important, I will have far more to say about it. Before I go into platonism, however, I will say a few words about the two different kinds of anti-platonistic realism — i.e., physicalism and psychologism.

1.1.1 Anti-platonistic realism (physicalism and psychologism)

The main advocate of mathematical physicalism is John Stuart Mill [1843, book II, chapters 5 and 6]. The idea here is that mathematics is about ordinary physical objects and, hence, that it is an empirical science, or a natural science, albeit a very general one. Thus, just as botany gives us laws about plants, mathematics, according to Mill's view, gives us laws about all objects. For instance, the sentence '$2 + 1 = 3$' tells us that whenever we add one object to a pile of two objects, we will end up with three objects. It does not tell us anything about any abstract objects, like the numbers 1, 2, and 3, because, on this view, there are simply no such things as abstract objects. (There is something a bit arbitrary and potentially confusing about calling this view 'physicalism', because Penelope Maddy [1990b] has used the term 'physicalistic platonism' to denote her view that set theory is about sets that exist in spacetime — e.g., sets of biscuits and eggs. We will see below that her view is different from Mill's and, indeed, not entirely physicalistic — it is platonistic in at least some sense of the term. One might also call Mill's view 'empiricism', but that would be misleading too, because one can combine empiricism with non-physicalistic views (e.g., Resnik and Quine have endorsed empiricist platonist views[2]); moreover, the view I am calling 'physicalism' here is an *ontological* view, and in general, empiricism is an epistemological view. Finally, one might just call the view here 'Millianism'; I would have no objection to that, but it is not as descriptive as 'physicalism'.)

Recently, Philip Kitcher [1984] has advocated a view that is similar in certain ways to Millian physicalism. According to Kitcher, our mathematical theories are about the activities of an ideal agent; for instance, in the case of arithmetic, the activities involve the ideal agent pushing blocks around, i.e., making piles of

[2] The view is developed in detail by Resnik [1997], but see also Quine (1951, section 6).

blocks, adding blocks to piles, taking them away, and so on. I will argue in section 2.2.3, however, that Kitcher's view is actually better thought of as a version of anti-realism.

Let's move on now to the second version of anti-platonistic realism — that is, to psychologism. This is the view that mathematics is about mental objects, in particular, ideas in our heads; thus, for instance, on this view, '3 is prime' is about a certain mental object, namely, the idea of 3.

One might want to distinguish two different versions of psychologism; we can call these views *actualist psychologism* and *possibilist psychologism* and define them in the following way:

> *Actualist Psychologism* is the view that mathematical statements are about, and true of, actual mental objects (or mental constructions) in actual human heads.[3] Thus, for instance, the sentence '3 is prime' says that the mentally constructed object 3 has the property of primeness.

> *Possibilist Psychologism* is the view that mathematical statements are about what mental objects it's possible to construct. E.g., the sentence 'There is a prime number between 10,000,000 and (10,000,000! + 2)' says that it's possible to construct such a number, even if no one has ever constructed one.

But (according to the usage that I'm employing here) possibilist psychologism is not a genuinely psychologistic view at all, because it doesn't involve the adoption of a psychologistic *ontology* for mathematics. It seems to me that possibilist psychologism collapses into either a platonistic view (i.e., a view that takes mathematics to be about abstract objects) or an anti-realist view (i.e., a view that takes mathematics not to be about anything — i.e., a view like deductivism, formalism, or fictionalism that takes mathematics not to have an ontology). If one takes possible objects (in particular, possible mental constructions) to be real things, then presumably (unless one is a Lewisian about the metaphysical nature of possibilia) one is going to take them to be abstract objects of some sort, and hence, one's possibilist psychologism is going to be just a semantically weird version of platonism. (On this view, mathematics is about abstract objects, it is objective, and so on; the only difference between this view and standard platonism is that it involves an odd, non-face-value view of *which* abstract objects the sentences of mathematics are about.) If, on the other hand, one rejects the existence of possible objects, then one will wind up with a version of possibilist psychologism that is essentially anti-realistic: on this view, mathematics will not have an ontology. Thus, in this essay, I am going to use 'psychologism' to denote actualist psychologism.

By the way, one might claim that actualist psychologism is better thought of as a version of anti-realism than a version of realism; for one might think that

[3]Obviously, there's a question here about *whose* heads we're talking about. Any human head? Any decently trained human head? Advocates of psychologism need to address this issue, but I won't pursue this here.

mathematical realism is most naturally defined as the view that our mathematical theories provide true descriptions of some part of the world *that exists independently of us human beings*. I don't think anything important hangs on whether we take psychologism to be a version of realism or anti-realism, but for whatever it's worth, I find it more natural to think of psychologism as a version of realism, for the simple reason that (in agreement with other realist views and disagreement with anti-realist views) it provides an ontology for mathematics — i.e., it says that mathematics is *about objects*, albeit mental objects. Thus, I am going to stick with the definition of mathematical realism that makes actualist psychologism come out as a version of realism. However, we will see below (section 2.2.3) that it is indeed true that actualist psychologism bears certain important similarities to certain versions of anti-realism.

Psychologistic views seem to have been somewhat popular around the end of the nineteenth century, but very few people have advocated such views since then, largely, I think, because of the criticisms that Frege leveled against the psychologistic views that were around back then — e.g., the views of Erdmann and the early Husserl.[4] Probably the most famous psychologistic views are those of the intuitionists, most notably Brouwer and Heyting. Heyting for instance said, "We do not attribute an existence independent of our thought ... to ... mathematical objects," and Brouwer made several similar remarks.[5] However, I do not think we should interpret either of these philosophers as straightforward advocates of actualist psychologism. I think the best interpretation of their view takes it to be an odd sort of hybrid of an actualist psychologistic view of mathematical assertions and a possibilist psychologistic view of mathematical negations. I hope to argue this point in more detail in the future, but the basic idea is as follows. Brouwer-Heyting intuitionism is generated by endorsing the following two principles:

(A) A mathematical assertion of the form 'Fa' means 'We are actually in possession of a proof (or an effective procedure for producing a proof) that the mentally constructed mathematical object a is F'.

(B) A mathematical sentence of the form '$\sim P$' means "There is a derivation of a contradiction from 'P' ".

Principle (A) commits them pretty straightforwardly to an actualist psychologistic view of assertions. But (B) seems to commit them to a possibilist psychologistic view of negations, for on this view, in order to assert '$\sim Fa$', we need something that entails that we *couldn't* construct the object a such that it was F (not merely that we *haven't* performed such a construction) — namely, a derivation of a contradiction from 'Fa'. I think this view is hopelessly confused, but I also think

[4]See, for instance, Husserl [1891] and Frege [1894] and [1893–1903, 12–15]. Husserl's and Erdmann's works have not been translated into English, and so I am not entirely certain that either explicitly accepted what I am calling psychologism here. Resnik [1980, chapter 1] makes a similar remark; all he commits to is that Erdmann and Husserl — and also Locke [1689] — came close to endorsing psychologism.

[5]Heyting [1931, 53]; and see, e.g., Brouwer [1948, 90].

it is the most coherent view that is consistent with what Brouwer and Heyting actually say — though I cannot argue this point here. (By the way, none of this is relevant to Dummett's [1973] view; his version of intuitionism is not psychologistic at all.)[6,7]

1.1.2 Mathematical platonism

As I said above, platonism is the view that (a) there exist abstract mathematical objects — objects that are non-spatiotemporal and wholly non-physical and non-mental — and (b) our mathematical theories are true descriptions of such objects. This view has been endorsed by Plato, Frege, Gödel, and in some of his writings, Quine.[8] (One might think that it's not entirely clear what thesis (a) — that there exist abstract objects — really amounts to. I think this is correct, and in section 3.2, I will argue that because of this, there is no fact of the matter as to whether platonism or anti-platonism is true. For now, though, I would like to assume that the platonist thesis is entirely clear.)

There are a couple of distinctions that need to be drawn between different kinds of platonism. The most important distinction, in my view, is between the traditional platonist view endorsed by Plato, Frege, and Gödel (we might call this *sparse platonism*, or *non-plenitudinous platonism*) and a view that I have developed elsewhere [1992; 1995; 1998] and called *plenitudinous platonism*, or *full-blooded platonism*, or for short, FBP. FBP differs from traditional platonism in several ways, but all of the differences arise out of one bottom-level difference concerning the question of *how many* mathematical objects there are. FBP can be expressed very intuitively, but perhaps a bit sloppily, as the view that the mathematical realm is plenitudinous; in other words, the idea here is that all the mathematical objects that (logically possibly) *could* exist actually *do* exist, i.e., that there actually exist mathematical objects of all logically possible kinds. (More needs to be said about what exactly is meant by 'logically possible'; I address this in my [1998, chapter 3, section 5].) In my book, I said a bit more about how to define FBP, but Greg Restall [2003] has recently argued that still more work is

[6]Intuitionism itself (which can be defined in terms of principles (A) and (B) in the text) is not a psychologistic view. It is often assumed that it goes together naturally with psychologism, but in work currently in progress, I argue that intuitionism is independent of psychologism. More specifically, I argue that (i) intuitionists can just as plausibly endorse platonism or anti-realism as psychologism, and (ii) advocates of psychologism can (and indeed should) avoid intuitionism and hang onto classical logic. Intuitionism, then, isn't a view of the metaphysics of mathematics at all. It is a thesis about the semantics of mathematical discourse that is consistent with both realism and anti-realism. Now, my own view on this topic is that intuitionism is a wildly implausible view, but I will not pursue this here because it is not a version of realism or anti-realism. (And by the way, a similar point can be made about *logicism*: it is not a version of realism or anti-realism (it is consistent with both of these views) and so I will not discuss it here.)

[7]Recently, a couple of non-philosophers — namely, Hersh [1997] and Dehaene [1997] — have endorsed views that sound somewhat psychologistic. But I do not think these views should be interpreted as versions of the view that I'm calling psychologism (and I should note here that Hersh at least is careful to distance himself from this view).

[8]See, e.g., Plato's *Meno* and *Phaedo*; Frege [1893–1903]; Gödel [1964]; and Quine [1948; 1951].

required on this front; I will say more about this below, in section 2.1.3.

I should note here that the non-plenitudinousness of traditional platonism is, I think, more or less unreflective. That is, the question of whether the mathematical realm is plenitudinous was almost completely ignored in the literature until very recently; but despite this, the question is extremely important, for as I have argued — and I'll sketch the argument for this here (section 2.1) — platonists can defend their view if and only if they endorse FBP. That is, I have argued (and will argue here) that (a) FBP is a defensible view, and (b) non-plenitudinous versions of platonism are not defensible.

I don't mean to suggest, however, that I am the only philosopher who has ever defended a view like FBP. Zalta and Linsky [1995] have defended a similar view: they claim that "there are as many abstract objects of a certain sort as there possibly could be." But their conception of abstract objects is rather unorthodox, and for this reason, their view is quite different, in several respects, from FBP.[9] Moreover, they have not used FBP in the way that I have, arguing that platonists can solve the traditional problems with their view if and only if they endorse FBP. (I do not know of anyone else who has claimed that the mathematical realm is plenitudinous in the manner of FBP. In my book [1998, 7–8], I quote passages from Hilbert, Poincaré, and Resnik that bring the FBP-ist picture to mind, but I argue there that none of these philosophers really endorses FBP. Hilbert and Poincaré don't even endorse platonism, let alone FBP; Resnik does endorse (a structuralist version of) platonism, but it's unlikely that he would endorse an FBP-ist version of structuralistic platonism. It *may* be that Shapiro would endorse such a view, but he has never said this in print. In any event, whatever we end up saying about whether these philosophers endorse views like FBP, the main point is that they do not give FBP a prominent role, as I do. On my view, as we have seen, plenitudinousness is the key prong in the platonist view, and FBP is the only defensible version of platonism.)

A second divide in the platonist camp is between *object-platonism* and *structuralism*. I have presented platonism as the view that there exist abstract mathematical *objects* (and that our mathematical theories describe such objects). But this is not exactly correct. The real core of the view is the belief in the abstract, i.e., the belief that there is something real and objective that exists outside of spacetime and that our mathematical theories characterize. The claim that this abstract something is a collection of *objects* can be jettisoned without abandoning platonism. Thus, we can say that, strictly speaking, mathematical platonism is the view that our mathematical theories are descriptions of an abstract *mathematical realm*, i.e., a non-physical, non-mental, non-spatiotemporal aspect of reality.

Now, the most traditional version of platonism — the one defended by, e.g., Frege and Gödel — is a version of object-platonism. Object-platonism is the view that the mathematical realm is a system of abstract mathematical *objects*, such as numbers and sets, and that our mathematical theories, e.g., number theory and set theory, describe these objects. Thus, on this view, the sentence '3 is prime'

[9]See also Zalta [1983; 1988].

says that the abstract object that is the number 3 has the property of primeness. But there is a very popular alternative to object-platonism, *viz.*, structuralism. According to this view, our mathematical theories are not descriptions of particular systems of abstract objects; they are descriptions of abstract *structures*, where a structure is something like a *pattern*, or an "objectless template" — i.e., a system of *positions* that can be "filled" by any system of objects that exhibit the given structure. One of the central motivations for structuralism is that the "internal properties" of mathematical objects seem to be mathematically unimportant. What is mathematically important is structure — i.e., the relations that hold between mathematical objects. To take the example of arithmetic, the claim is that any sequence of objects with the right structure (i.e., any *ω-sequence*) would suit the needs of arithmetic as well as any other. What structuralists maintain is that arithmetic is concerned not with some particular one of these ω-sequences, but rather, with the structure or pattern that they all have in common. Thus, according to structuralists, there is no *object* that is the number 3; there is only the fourth position in the natural-number pattern.

Some people read Dedekind [1888] as having held a view of this general sort, though I think that this is a somewhat controversial interpretation. The first person to explicitly endorse the structuralist thesis as I have presented it here — i.e., the thesis that mathematics is about structure and that different systems of objects can "play the role" of, e.g., the natural numbers — was Benacerraf [1965]. But Benacerraf's version of the view was anti-platonistic; he sketched the view very quickly, but later, Hellman [1989] developed an anti-platonistic structuralism in detail. The main pioneers of platonistic structuralism — the view that holds that mathematics is about structures and positions in structures and that these structures and positions are real, objective, and abstract — are Resnik [1981; 1997] and Shapiro [1989; 1997], although Steiner [1975] was also an early advocate.

In my book, I argued that the dispute between object-platonists and structuralists is less important than structuralists think and, indeed, that platonists don't need to take a stand on the matter. Resnik and Shapiro think that by adopting structuralism, platonists improve their standing with respect to both of the great objections to platonism, i.e., the epistemological objection and the non-uniqueness objection, both of which will be discussed in section 2.1. But I have argued (and will sketch the argument here) that this is false. The first thing I have argued here is that structuralism doesn't do any work in connection with these problems after all (in connection with the epistemological problem, I argue this point in my [1998, chapter 2, section 6.5] and provide a brief sketch of the reasoning below, in section 2.1.1.4.3; and in connection with the non-uniqueness problem, I argue the point in my [1998, chapter 4, section 3] and provide a sketch of the reasoning below, in section 2.1.2.3). But the more important thing I've done is to provide FBP-ist solutions to these two problems that work for both structuralism and object-platonism [1998, chapters 3 and 4]; below (section 2.1), I will quickly sketch my account of how FBP-ists can solve the two problems; I will not take the space to argue that FBP is consistent with structuralism as well as with object-platonism, but the point is entirely obvious.[10]

The last paragraph suggests that there is no reason to favor structuralism over object-platonism. But the problem here is even deeper: it is not clear that structuralism is even *distinct* from object-platonism in an important way, for as I argue in my book (chapter 1, section 2.1), positions in structures — and, indeed, structures themselves — seem to be just special kinds of mathematical *objects*. Now, in light of this point, one might suggest that the structuralists' "objects-versus-positions" rhetoric is just a distraction and that structuralism should be defined in some other way. One suggestion along these lines, advanced by Charles Parsons,[11] is that structuralism should be defined as the view that mathematical objects have no internal properties, i.e., that there is no more to them than the relations that they bear to other mathematical objects. But (a) it seems that mathematical objects do have non-structural properties, e.g., being non-spatiotemporal and being non-red; and (b) the property of having only structural properties is *itself* a non-structural property (or so it would seem), and so the above definition of structuralism is simply incoherent. A second suggestion here is that structuralism should be defined as the view that the internal properties of mathematical objects are not mathematically *important*, i.e., that structure is what is important in mathematics. But whereas the last definition was too strong, this one is too weak. For as we'll see in section 2.1.2, traditional object-platonism is perfectly consistent with the idea that the internal properties of mathematical objects are not mathematically important; indeed, it seems to me that just about everyone who claims to be an object-platonist would *endorse* this idea. Therefore, this cannot be what separates structuralism from traditional object-platonism. Finally, structuralists might simply define their view as the thesis that mathematical objects are positions in structures that can be "filled" by other objects. But if I'm right that this thesis doesn't do any work in helping platonists solve the problems with their view, then it's not clear what the motivation for this thesis could be, or indeed, why it is philosophically important.[12]

I think it is often convenient for platonists to speak of mathematical theories as describing structures, and in what follows, I will sometimes speak this way. But as I see it, structures are mathematical objects, and what's more, they are made up of objects. We can think of the elements of mathematical structures as "positions" if we want to, but (a) they are still mathematical *objects*, and (b) as

[10] I have formulated FBP (and my solutions to the problems with platonism) in object-platonist terms, but it is obvious that this material could simply be reworded in structuralistic FBP-ist terms (or in a way that was neutral between structuralism and object-platonism).

[11] See the first sentence of Parsons [1990].

[12] Resnik has suggested to me that the difference between structuralists and object-platonists is that the latter often see facts of the matter where the former do not. One might put this in terms of property possession again; that is, one might say that according to structuralism, there are some cases where there is no fact of the matter as to whether some mathematical object a possesses some mathematical property P. But we will see below (sections 2.1.2–2.1.3) that object-platonists are not committed to all of the fact-of-the-matter claims (or property-possession claims) normally associated with their view. It will become clearer at that point, I think, that there is no important difference between structuralism and object-platonism.

we'll see below, there is no good *reason* for thinking of them as "positions".

1.2 Mathematical Anti-Realism

Anti-realism, recall, is the view that mathematics does not have an ontology, i.e., that our mathematical theories do not provide true descriptions of some part of the world. There are lots of different versions of anti-realism. One such view is *conventionalism*, which holds that mathematical sentences are analytically true. On this view, '2 + 1 = 3' is like 'All bachelors are unmarried': it is true solely in virtue of the meanings of the words appearing in it. Views of this sort have been endorsed by Ayer [1946, chapter IV], Hempel [1945], and Carnap [1934; 1952; 1956].

A second view here is *formalism*, which comes in a few different varieties. One version, known as *game formalism*, holds that mathematics is a game of symbol manipulation; on this view, '2 + 1 = 3' would be one of the "legal results" of the "game" specified by the axioms of PA (i.e., Peano Arithmetic). The only advocates of this view that I know of are those, e.g., Thomae, whom Frege criticized in his *Grundgesetze* (sections 88–131). A second version of formalism — *metamathematical formalism*, endorsed by Curry [1951] — holds that mathematics gives us truths about what holds in various formal systems; for instance, on this view, one truth of mathematics is that the sentence '2 + 1 = 3' is a theorem of the formal system PA. One might very well doubt, however, that metamathematical formalism is a genuinely anti-realistic view; for since this view says that mathematics is *about theorems and formal systems*, it seems to entail that mathematics has an ontology, in particular, one consisting of sentences. As a version of realism, however — that is, as the view that mathematics is about actually existing sentences — the view has nothing whatsoever to recommend it.[13] Finally, Hilbert sometimes seems to accept a version of formalism, but again, it's not clear that he really had an anti-realistic view of the metaphysics of mathematics (and if he did, it's not clear what the view was supposed to be). I think that Hilbert was by far the most brilliant of the formalists and that his views on the philosophy of mathematics were the most important, insightful, and original. But I also think that the metaphysical component of his view — i.e., where he stood on the question of realism — was probably the least interesting part of his view. His finitism and his earlier view that axiom systems provide definitions are far more important; I will touch on the axiom-systems-are-definitions thesis later on, but I will not discuss this view (or Hilbert's finitism) in the present section, because neither of these views is a version of anti-realism, and neither entails anti-realism. As for the question of Hilbert's metaphysics, in the latter portion of his career he seemed to endorse the view that finitistic arithmetical claims can be taken to be about sequences of strokes — e.g., '2 + 1 = 3' can be taken as saying something to the effect that if we concatenate '||' with '|', we get '|||' — and that mathematical claims that go beyond finitary arithmetic can be treated instrumentally, along the lines of game formalism. So the later Hilbert was an anti-realist about infinitary mathematics, but I think he is best interpreted as a platonist about finitary arithmetic, because it is most natural to take him as saying that finitary arithmetic is about stroke *types*, which are abstract objects.[14,15]

Another version of anti-realism — a view that, I think, can be characterized as a descendent of formalism — is *deductivism*, or *if-thenism*. This view holds that mathematics gives us truths of the form 'if A then T' (or 'it is necessary that if A then T') where A is an axiom, or a conjunction of several axioms, and T is a theorem that is provable from these axioms. Thus, for instance, deductivists claim that '2 + 1 = 3' can be taken as shorthand for the sentence '(it is necessary that) if the axioms of arithmetic are true, then 2 + 1 = 3'. Thus, on this view, mathematical sentences come out true, but they are not *about* anything. Putnam originally introduced this view, and Hellman later developed a structuralist version of it. But the early Hilbert also hinted at the view.[16]

Another anti-realistic view worth mentioning is Wittgenstein's (see, e.g., his [1956]). His view is related in certain ways to game formalism and conventionalism, but it is distinct from both. I do not want to try to give a quick formulation of this view, however, because I do not think it is possible to do this; to capture the central ideas behind Wittgenstein's philosophy of mathematics would take quite a bit more space. (I should point out here that Wittgenstein's view can be interpreted in a number of different ways, but I think it's safe to say that however we end up interpreting the view, it is going to be a version of anti-realism.)

Another version of anti-realism that I don't want to try to explain in full is due to Chihara [1990]. Chihara's project is to reinterpret all of mathematics, and it would take a bit of space to adequately describe how he does this, but the basic anti-realist idea is very simple: Chihara's goal is to replace sentences involving ontologically loaded existential quantification over mathematical objects (e.g., 'there is a set x such that...') with assertions about what open-sentence tokens it is possible to construct (e.g., 'it is possible to construct an open sentence

[13] One might endorse an anti-platonistic version of this view (maintaining that mathematics is about sentence *tokens*) or a platonistic version (maintaining that mathematics is about sentence *types*). But (a) the anti-platonistic version of this view is untenable, because there aren't *enough* tokens lying around the physical world to account for all of mathematical truth (indeed, to account even for finitistic mathematical truth). And (b) the platonistic version of this view has no advantage over traditional platonism, and it has a serious disadvantage, because it provides a non-standard, non-face-value semantics for mathematical discourse that flies in the face of actual mathematical practice (I will say more about this problem below, in section 2.2.2).

[14] See Hilbert [1925] for a formulation of the formalism/finitism that he endorsed later in his career. For his earlier view, including the idea that axioms are definitions, see his [1899] and his letters to Frege in [Frege, 1980].

[15] The idea that mathematics is about symbols — e.g., strokes — is a view that has been called *term formalism*. This view is deeply related to metamathematical formalism, and in particular, it runs into a problem that is exactly analogous to the problem with metamathematical formalism described above (note 13).

[16] See Putnam [1967a; 1967b], Hellman [1989], and Hilbert [1899] and his letters to Frege in [Frege, 1980].

x such that...'). Chihara thinks that (a) his reinterpreted version of mathematics does everything we need mathematics to do, and (b) his reinterpreted version of mathematics comes out true, even though it has no ontology (i.e., is not about some part of the world) because it merely makes claims about what is *possible*. In this respect, his view is similar to certain versions of deductivism; Hellman, for instance, holds that the axioms of our mathematical theories can be read as making claims about what is possible, while the theorems can be read as telling us what would follow if the axioms were true.

Another version of anti-realism — and I will argue in section 2.2 that this is the best version of anti-realism — is *fictionalism*. This view differs from other versions of anti-realistic anti-platonism in that it takes mathematical sentences and theories at *face value*, in the way that platonism does. Fictionalists agree with platonists that the sentence '3 is prime' is about the number 3[17] — in particular, they think it says that this number has the property of primeness — and they also agree that if there is any such thing as 3, then it is an abstract object. But they disagree with platonists in that they do not think that there is any such thing as the number 3 and, hence, do not think that sentences like '3 is prime' are true. According to fictionalists, mathematical sentences and theories are fictions; they are comparable to sentences like 'Santa Claus lives at the North Pole.' This sentence is not true, because 'Santa Claus' is a vacuous term, that is, it fails to refer. Likewise, '3 is prime' is not true, because '3' is a vacuous term — because just as there is no such person as Santa Claus, so there is no such thing as the number 3. Fictionalism was first introduced by Hartry Field [1980; 1989]; as we'll see, he saw the view as being wedded to the thesis that empirical science can be nominalized, i.e., restated so that it does not contain any reference to, or quantification over, mathematical objects. But in my [1996a] and [1998], I defend a version of fictionalism that is divorced from the nominalization program, and similar versions of fictionalism have been endorsed by Rosen [2001] and Yablo [2002].

One obvious question that arises for fictionalists is this: "Given that '$2+1=3$' is false, what is the difference between this sentence and, say, '$2+1=4$'?" The difference, according to fictionalism, is analogous to the difference between 'Santa Claus lives at the North Pole' and 'Santa Claus lives in Tel Aviv'. In other words, the difference is that '$2+1=3$' is part of a certain well-known mathematical story, whereas '$2+1=4$' is not. We might express this idea by saying that while neither '$2+1=3$' nor '$2+1=4$' is true *simpliciter*, there is another truth predicate (or pseudo-truth predicate, as the case may be) — *viz.*, 'is true in the story of mathematics' — that applies to '$2+1=3$' but not to '$2+1=4$'. This seems to be the view that Field endorses, but there is a bit more that needs to be said on

[17]I am using 'about' here in a *thin* sense. I say more about this in my book (see, e.g., chapter 2, section 6.2), but for present purposes, all that matters is that in this sense of 'about', 'S is about b' does not entail that there is any such thing as b. For instance, we can say that the novel *Oliver Twist* is about an orphan named 'Oliver' without committing to the existence of such an orphan. Of course, one might also use 'about' in a *thicker* way; in this sense of the term, a story (or a belief state, or a sentence, or whatever) can be about an object only if the object exists and the author (or believer or speaker or whatever) is "connected" to it in some appropriate way.

this topic. In particular, it is important to realize that the above remarks do not lend any metaphysical or ontological distinction to sentences like '$2+1=3$'. For according to fictionalism, there are *alternative* mathematical "stories" consisting of sentences that are not part of standard mathematics. Thus, the real difference between sentences like '$2+1=3$' and sentences like '$2+1=4$' is that the former are part of *our* story of mathematics, whereas the latter are not. Now, of course, fictionalists will need to explain why we use, or "accept", this particular mathematical story, as opposed to some alternative story, but this is not hard to do. The reasons are that this story is pragmatically useful, that it's aesthetically pleasing, and most important, that it dovetails with our conception of the natural numbers.

On the version of fictionalism that I defend, sentences like '3 is prime' are simply false. But it should be noted that this is not essential to the view. What is essential to mathematical fictionalism is that (a) there are no such things as mathematical objects, and hence, (b) mathematical singular terms are vacuous. Whether this means that sentences like '3 is prime' are false, or that they lack truth value, or something else, depends upon our theory of vacuity. I will adopt the view that such sentences are false, but nothing important will turn on this.[18]

It is also important to note here that the comparison between mathematical and fictional discourse is actually not central to the fictionalistic view of mathematics. The fictionalist view that we're discussing here is a view about mathematics only; it includes theses like (a) and (b) in the preceding paragraph, but it doesn't say anything at all about fictional discourse. In short, mathematical fictionalism — or at any rate, the version of fictionalism that I have defended, and I think that Field would agree with me on this — is entirely neutral regarding the analysis of fictional discourse. My own view (though in the present context this doesn't really matter) is that there are important differences between mathematical sentences and sentences involving fictional names. Consider, e.g., the following two sentence tokens:

(1) Dickens's original token of some sentence of the form 'Oliver was F' from *Oliver Twist*;

(2) A young child's utterance of 'Santa Claus lives at the North Pole'.

Both of these tokens, it seems, are untrue. But it seems to me that they are very different from one another and from ordinary mathematical utterances (fictionalistically understood). (1) is a bit of pretense: Dickens knew it wasn't true when he uttered it; he was engaged in a kind of pretending, or literary art, or some such thing. (2), on the other hand, is just a straightforward expression of a false belief. Mathematical fictionalists needn't claim that mathematical utterances are analogous to either of these utterances: they needn't claim that when

[18]It should be noted here that fictionalists allow that *some* mathematical sentences are true, albeit vacuously so. For instance, they think that sentences like 'All natural numbers are integers' — or, for that matter, 'All natural numbers are zebras' — are vacuously true for the simple reason that there are no such things as numbers. But we needn't worry about this complication here.

we use mathematical singular terms, we're engaged in a bit of make-believe (along the lines of (1)) or that we're straightforwardly mistaken (along the lines of (2)). There are a number of different things fictionalists can say here; for instance, one line they could take is that there is a bit of imprecision in what might be called our communal intentions regarding sentences like '3 is prime', so that these sentences are somewhere between (1) and (2). More specifically, one might say that while sentences like '3 is prime' are best read as being "about" abstract objects — i.e., thinly about abstract objects (see note 17) — there is nothing built into our usage or intentions about whether there really do exist abstract objects, and so it's not true that we're explicitly involved in make-believe, and it's not true that we clearly intend to be talking about an actually existing platonic realm. But again, this is just one line that fictionalists could take. (See my [2009] for more on this and, in particular, how fictionalists can respond to the objection raised by Burgess [2004].)

One might think that '3 is prime' is less analogous to (1) or (2) than it is to, say, a sentence about Oliver uttered by an informed adult who intends to be saying something true about Dickens's novel, e.g.,

(3) Oliver Twist lived in London, not Paris.

But we have to be careful here, because (a) one might think (indeed, I do think) that (3) is best thought of as being about Dickens's *novel*, and not Oliver, and (b) fictionalists do *not* claim that sentences like '3 is prime' are about the story of mathematics (they think this sentence is about 3 and is true-in-the-story-of-mathematics, but not true *simpliciter*). But some people — e.g., van Inwagen [1977], Zalta [1983; 1988], Salmon [1998], and Thomasson [1999] — think that sentences like (3) are best interpreted as being about Oliver Twist, the actual literary character, which on this view is an abstract object; a fictionalist who accepted this platonistic semantics of (3) could maintain that '3 is prime' is analogous to (3).

Finally, I end by discussing Meinongianism. There are two different versions of this view; the first, I think, is just a terminological variant of platonism; the second is a version of anti-realism. The first version of Meinongianism is more well known, and it is the view that is commonly ascribed to Meinong, though I think this interpretation of Meinong is controversial. In any event, the view is that our mathematical theories provide true descriptions of objects that have some sort of being (that *subsist*, or that *are*, in some sense) but do not have full-blown existence. This sort of Meinongianism has been almost universally rejected. The standard argument against it (see, e.g., [Quine 1948]) is that it is not genuinely distinct from platonism; Meinongians have merely created the illusion of a different view by altering the meaning of the term 'exist'. On the standard meaning of 'exist', any object that *is* — that has any being at all — exists. Therefore, according to standard usage, Meinongianism entails that mathematical objects exist (of course, Meinongians wouldn't assent to the sentence 'Mathematical objects exist', but this, it seems, is simply because they don't know what 'exist' means); but Meinongianism clearly doesn't take mathematical objects to exist in space-time, and so on this view, mathematical objects are abstract objects. Therefore, Meinongianism is not distinct from platonism.[19]

The second version of Meinongianism, defended by Routley [1980] and later by Priest [2003], holds that (a) things like numbers and universals don't exist at all (i.e., they have no sort of being whatsoever), but (b) we can still say true things about them — e.g., we can say (truly) that 3 is prime, even though there is no such thing as 3. Moreover, while Azzouni [1994] would not use the term 'Meinongianism', he has a view that is very similar to the Routley-Priest view. For he seems to want to say that (a) as platonists and fictionalists assert, mathematical sentences — e.g., '3 is prime' and 'There are infinitely many transfinite cardinals' — should be read at face value, i.e., as being about mathematical objects (in at least some thin sense); (b) as platonists assert, such sentences are true; and (c) as fictionalists assert, there are really no such things as mathematical objects that exist independently of us and our mathematical theorizing. I think that this view is flawed in a way that is similar to the way in which the first version of Meinongianism is flawed, except that here, the problem is with the word 'true', rather than 'exists'. The second version of Meinongianism entails that a mathematical sentence of the form 'Fa' can be true, even if there is no such thing as the object a (Azzouni calls this a sort of truth by convention, for on his view, it applies by stipulation; but the view here is different from the Ayer-Hempel-Carnap conventionalist view described above). But the problem is that it seems to be built into the standard meaning of 'true' that if there is no such thing as the object a, then sentences of the form 'Fa' cannot be literally true. Or equivalently, it is a widely accepted criterion of ontological commitment that if you think that the sentence 'a is F' is literally true, then you are committed to the existence of the object a. One might also put the point here as follows: just as the first version of Meinongianism isn't genuinely distinct from platonism and only creates the illusion of a difference by misusing 'exists', so too the second version of Meinongianism isn't genuinely distinct from fictionalism and only creates the illusion of a difference by misusing

[19] Priest [2003] argues that (a) Meinongianism is different from traditional platonism, because the latter is non-plenitudinous; and (b) Meinongianism is different from FBP, because the former admits as legitimate the objects of *inconsistent* mathematical theories as well as consistent ones; and (c) if platonists go for a plenitudinous view that also embraces the inconsistent (i.e., if they endorse what Beall [1999] has called *really full-blooded platonism*), then the view looks more like Meinongianism than platonism. But I think this last claim is just false; unless Meinongians can give some appropriate content to the claim that, e.g., 3 is but doesn't exist, it seems that the view should be thought of as a version of platonism. (I should note here that in making the above argument, Priest was very likely thinking of the *second* version of Meinongianism, which I will discuss presently, and so my argument here should not be thought of as a refutation of Priest's argument; it is rather a refutation of the idea that Priest's argument can be used to save first-version Meinongianism from the traditional argument against it. Moreover, as we'll see, I do not think the second version of Meinongianism is equivalent to platonism, and so Priest's argument will be irrelevant there.) Finally, I might also add here that just as there are different versions of platonism that correspond to points (a)–(c) above, so too we can define analogous versions of Meinongianism. So I don't think there's any difference between the two views on this front either.

'true'; in short, what they call truth isn't *real* truth, because on the standard meaning of 'true' — that is, the meaning of 'true' in *English* — if a sentence has the form '*Fa*', and if there is no such thing as the object *a*, then '*Fa*' isn't true. To simply stipulate that such a sentence *is* true is just to alter the meaning of 'true'.

2 CRITIQUE OF THE VARIOUS VIEWS

I will take a somewhat roundabout critical path through the views surveyed above. In section 2.1, I will discuss the main criticisms that have been leveled against platonism; in section 2.2, I will critically assess the various versions of anti-platonism, including the various anti-platonistic versions of realism (i.e., physicalism and psychologism); finally, in section 2.3, I will discuss a lingering worry about platonism. I follow this seemingly circuitous path for the simple reason that it seems to me to generate a logically pleasing progression through the issues to be discussed — even if it doesn't provide a clean path through realism first and anti-realism second.

2.1 Critique of Platonism

In this section, I will consider the two main objections to platonism. In section 2.1.1, I will consider the epistemological objection, and in section 2.1.2, I will consider the non-uniqueness (or multiple-reductions) objection. (There are a few other problems with platonism as well, e.g., problems having to do with mathematical reference, the applications of mathematics, and Ockham's razor. I will address these below.) As we will see, I do not think that any of these objections succeeds in refuting platonism, because I think there are good FBP-ist responses to all of them, though we will also see that these objections (especially the epistemological one) do succeed in refuting non-full-blooded versions of platonism.

2.1.1 The Epistemological Argument Against Platonism

In section 2.1.1.1, I will formulate the epistemological argument; in sections 2.1.1.2–2.1.1.4, I will attack a number of platonist strategies for responding to the argument; and in section 2.1.1.5, I will explain what I think is the correct way for platonists to respond.

2.1.1.1 Formulating the Argument While this argument goes all the way back to Plato, the *locus classicus* in contemporary philosophy is Benacerraf's [1973]. But Benacerraf's version of the argument rests on a causal theory of knowledge that has proved vulnerable. A better formulation of the argument is as follows:

(1) Human beings exist entirely within spacetime.

(2) If there exist any abstract mathematical objects, then they exist outside of spacetime.

Therefore, it seems very plausible that

(3) If there exist any abstract mathematical objects, then human beings could not attain knowledge of them.

Therefore,

(4) If mathematical platonism is correct, then human beings could not attain mathematical knowledge.

(5) Human beings have mathematical knowledge.

Therefore,

(6) Mathematical platonism is not correct.

The argument for (3) is everything here. If it can be established, then so can (6), because (3) trivially entails (4), (5) is beyond doubt, and (4) and (5) trivially entail (6). Now, (1) and (2) do not deductively entail (3), and so even if we accept (1) and (2), there is room here for platonists to maneuver — and as we'll see, this is precisely how most platonists have responded. However, it is important to notice that (1) and (2) provide a strong *prima facie* motivation for (3), because they suggest that mathematical objects (if there are such things) are totally inaccessible to us, i.e., that information cannot pass from mathematical objects to human beings. But this gives rise to a *prima facie* worry (which may or may not be answerable) about whether human beings could acquire knowledge of abstract mathematical objects (i.e., it gives rise to a *prima facie* reason to think that (3) is true). Thus, we should think of the epistemological argument not as *refuting* platonism, but rather as issuing a challenge to platonists. In particular, since this argument generates a *prima facie* reason to doubt that human beings could acquire knowledge of abstract mathematical objects, and since platonists are committed to the thesis that human beings can acquire such knowledge, the challenge to platonists is simply to explain *how* human beings could acquire such knowledge.

There are three ways that platonists can respond to this argument. First, they can argue that (1) is false and that the human mind is capable of, somehow, forging contact with the mathematical realm and thereby acquiring information about that realm; this is Gödel's strategy, at least on some interpretations of his work. Second, we can argue that (2) is false and that human beings can acquire information about mathematical objects via normal perceptual means; this strategy was pursued by the early Maddy. And third, we can accept (1) and (2) and try to explain how (3) could be false anyway. This third strategy is very different from the first two, because it involves the construction of what might be called a *no-contact* epistemology; for the idea here is to accept the thesis that human beings cannot come into any sort of information-transferring contact with mathematical objects — this is the result of accepting (1) and (2) — and to try to explain how humans could nonetheless acquire knowledge of abstract objects. This third strategy has been the most popular among contemporary philosophers. Its advocates include Quine, Steiner, Parsons, Hale, Wright, Resnik, Shapiro, Lewis, Katz, and myself.

In sections 2.1.1.2–2.1.1.4, I will describe (and criticize) the strategy of rejecting (1), the strategy of rejecting (2), and all of the various no-contact strategies in the literature, except for my own. Then in section 2.1.1.5, I will describe and defend my own no-contact strategy, i.e., the FBP-based epistemology defended in my [1995] and [1998].

2.1.1.2 Contact with the Mathematical Realm: The Gödelian Strategy of Rejecting (1) On Gödel's [1964] view, we acquire knowledge of abstract mathematical objects in much the same way that we acquire knowledge of concrete physical objects: just as we acquire information about physical objects via the faculty of sense perception, so we acquire information about mathematical objects by means of a faculty of *mathematical intuition*. Now, other philosophers have endorsed the idea that we possess a faculty of mathematical intuition, but Gödel's version of this view involves the idea that the mind is non-physical in some sense and that we are capable of forging contact with, and acquiring information from, non-physical mathematical objects. (Others who endorse the idea that we possess a faculty of mathematical intuition have a *no-contact* theory of intuition that is consistent with a materialist philosophy of mind. Now, some people might argue that Gödel had such a view as well. I have argued elsewhere [1998, chapter 2, section 4.2] that Gödel is better interpreted as endorsing an immaterialist, contact-based theory of mathematical intuition. But the question of what view Gödel actually held is irrelevant here.)

This reject-(1) strategy of responding to the epistemological argument can be quickly dispensed with. One problem is that rejecting (1) doesn't seem to help solve the lack-of-access problem. For even if minds are immaterial, it is not as if that puts them into informational contact with mathematical objects. Indeed, the idea that an immaterial mind could have some sort of information-transferring contact with abstract objects seems just as incoherent as the idea that a physical brain could. Abstract objects, after all, are causally inert; they cannot generate information-carrying signals at all; in short, information can't pass from an abstract object to *anything*, material or immaterial. A second problem with the reject-(1) strategy is that (1) is, in fact, true. Now, of course, I cannot argue for this here, because it would be entirely inappropriate to break out into an argument against Cartesian dualism in the middle of an essay on the philosophy of mathematics, but it is worth noting that what is required here is a very strong and implausible version of dualism. One cannot motivate a rejection of (1) by merely arguing that there are real mental states, like beliefs and pains, or by arguing that our mentalistic idioms cannot be reduced to physicalistic idioms. One has to argue for the thesis that there actually exists immaterial human mind-stuff.

2.1.1.3 Contact in the Physical World: The Maddian Strategy of Rejecting (2) I now move on to the idea that platonists can respond to the epistemological argument by rejecting (2). The view here is still that human beings are capable of acquiring knowledge of mathematical objects by coming into contact with them, i.e., receiving information from them, but the strategy now is not to bring human beings up to platonic heaven, but rather, to bring the inhabitants of platonic heaven down to earth. Less metaphorically, the idea is to adopt a naturalistic conception of mathematical objects and argue that human beings can acquire knowledge of these objects via *sense perception*. The most important advocate of this view is Penelope Maddy (or rather, the *early* Maddy, for she has since abandoned the view).[20] Maddy is concerned mainly with set theory. Her two central claims are (a) that sets are spatiotemporally located — a set of eggs, for instance, is located right where the eggs are — and (b) that we can acquire knowledge of sets by perceiving them, i.e., by seeing, hearing, smelling, feeling, and tasting them in the usual ways. Let's call this view *naturalized platonism*.

I have argued against naturalized platonism elsewhere [1994; 1998, chapter 2, section 5]. I will just briefly sketch one of my arguments here.

The first point that needs to be made in this connection is that despite the fact that Maddy takes sets to exist in spacetime, her view still counts as a version of *platonism* (albeit a non-standard version). Indeed, the view *has* to be a version of platonism if it is going to be (a) relevant to the present discussion and (b) tenable. Point (a) should be entirely obvious, for since we are right now looking for a solution to the epistemological problem with *platonism*, we are concerned only with platonistic views that reject (2), and not anti-platonistic views. As for point (b), if Maddy were to endorse a thoroughgoing anti-platonism, then her view would presumably be a version of physicalism, since she claims that there do exist sets and that they exist in spacetime, right where their members do; in other words, her view would presumably be that sets are purely physical objects. But this sort of physicalism is untenable. One problem here (there are actually many problems with this view; see section 2.2.3 below) is that corresponding to every physical object there are infinitely many sets. Corresponding to an egg, for instance, there is the set containing the egg, the set containing that set, the set containing *that* set, and so on; and there is the set containing the egg and the set containing the egg, and so on and on and on. But all of these sets have the same physical base; that is, they are made of the exact same matter and have the exact same spatiotemporal location. Thus, in order to maintain that these sets are different things, Maddy has to claim that they differ from one another in *non-physical* ways and, hence, that sets are at least partially non-physical objects. Now, I suppose one might adopt a psychologistic view here according to which sets are *mental* objects (e.g., one might claim that only physical objects exist "out there

[20] See Maddy [1980; 1990]. She abandons the view in her (1997) for reasons completely different from the ones I present here. Of course, Maddy isn't the first philosopher to bring abstract mathematical objects into spacetime. Aside from Aristotle, Armstrong [1978, chapter 18, section V] attempts this as well, though he doesn't develop the idea as thoroughly as Maddy does.

in the world" and that we then come along and somehow construct all the various different sets in our minds); but as Maddy is well aware, such views are untenable (see section 2.2.3 below). Thus, the only initially plausible option for Maddy (or indeed for anyone who rejects (2)) is to maintain that there is something non-physical and non-mental about sets. Thus, she has to claim that sets are abstract, in some appropriate sense of the term, although, of course, she rejects the idea that they are abstract in the traditional sense of being non-spatiotemporal.

Maddy, I think, would admit to all of this, and in my book (chapter 2, section 5.1) I say what I think the relevant sense of abstractness is. I will not pursue this here, however, because it is not relevant to the argument that I will mount against Maddy's view. All that matters to my argument is that according to Maddy's view, sets are abstract, or non-physical, in at least some non-trivial sense.

What I want to argue here is that human beings cannot receive any relevant perceptual data from naturalized-platonist sets (i.e., sets that exist in spacetime but are nonetheless non-physical, or abstract, in some non-traditional sense) — and hence that platonists cannot solve the epistemological problem with their view by rejecting (2). Now, it's pretty obvious that I can acquire perceptual knowledge of physical objects and aggregates of physical matter; but again, there is more to a naturalized-platonist set than the physical stuff with which it shares its location — there is something *abstract* about the set, over and above the physical aggregate, that distinguishes it from the aggregate (and from the infinitely many other sets that share the same matter and location). Can I perceive this abstract component of the set? It seems that I cannot. For since the set and the aggregate are made of the same matter, both lead to the same retinal stimulation. Maddy herself admits this [1990, 65]. But if I receive only one retinal stimulation, then the perceptual data that I receive about the set are identical to the perceptual data that I receive about the aggregate. More generally, when I perceive an aggregate, I do not receive *any* data about *any* of the infinitely many corresponding naturalized-platonist sets that go beyond the data that I receive about the aggregate. This means that naturalized platonists are no better off here than traditional platonists, because we receive no more perceptual information about naturalized-platonist sets than we do about traditional non-spatiotemporal sets. Thus, the Benacerrafian worry still remains: there is still an unexplained epistemic gap between the information we receive in sense perception and the relevant facts about sets. (It should be noted that there are a couple of ways that Maddy could respond to this argument. However, I argued in my book (chapter 2, section 5.2) that these responses do not succeed.)

2.1.1.4 Knowledge Without Contact We have seen that mathematical platonists cannot solve the epistemological problem by claiming that human beings are capable of coming into some sort of contact with (i.e., receiving information from) mathematical objects. Thus, if platonists are to solve the problem, they must explain how human beings could acquire knowledge of mathematical objects without the aid of any contact with them. Now, a few different no-contact platonists (most notably, Parsons [1980; 1994], Steiner [1975], and Katz [1981; 1998]) have started out their arguments here by claiming that human beings possess a (no-contact) faculty of mathematical intuition. But as almost all of these philosophers would admit, the epistemological problem cannot be solved with a mere appeal to a no-contact faculty of intuition; one must also explain how this faculty of intuition could be reliable — and in particular, how it could lead to *knowledge* — given that it's a *no-contact* faculty. But to explain how the faculty that generates our mathematical intuitions and beliefs could lead to knowledge, despite the fact that it's a no-contact faculty, is not significantly different from explaining how we could acquire knowledge of mathematical objects, despite the fact that we do not have any contact with such objects. Thus, no progress has been made here toward solving the epistemological problem with platonism.[21] (For a longer discussion of this, see my [1998, chapter 2, section 6.2].)

In sections 2.1.1.4.1–2.1.1.4.3, I will discuss and criticize three different attempts to explain how human beings could acquire knowledge of abstract objects without the aid of any information-transferring contact with such objects. Aside from my own explanation, which I will defend in section 2.1.1.5, these three explanations are (as far as I know) the only ones that have been suggested. (It should be noted, however, that two no-contact platonists — namely, Wright [1983, section xi] and Hale [1987, chapters 4 and 6] — have tried to solve the epistemological problem *without* providing an explanation of how we could acquire knowledge of non-spatiotemporal objects. I do not have the space to pursue this here, but in my book (chapter 2, section 6.1) I argue that this cannot be done.)

2.1.1.4.1 Holism and Empirical Confirmation: Quine, Steiner, and Resnik One explanation of how we can acquire knowledge of mathematical objects despite our lack of contact with them is hinted at by Quine [1951, section 6] and developed by Steiner [1975, chapter 4] and Resnik [1997, chapter 7]. The claim here is that we have good reason to believe that our mathematical theories are true, because (a) these theories are central to our overall worldview, and (b) this worldview has been repeatedly confirmed by empirical evidence. In other words, we don't need contact with mathematical objects in order to know that our theories of these objects are true, because *confirmation is holistic*, and so these theories are confirmed every day, along with the rest of our overall worldview.

One problem with this view is that confirmation holism is, in fact, false. Confirmation may be holistic with respect to the *nominalistic* parts of our empirical theories (actually, I doubt even this), but the mathematical parts of our empir-

[21] Again, most platonists who appeal to a no-contact faculty of intuition would acknowledge my point here, and indeed, most of them go on to offer explanations of how no-contact intuitions could be reliable (or what comes to the same thing, how we could acquire knowledge of abstract mathematical objects without the aid of any contact with such objects). The exception to this is Parsons; he never addresses the worry about how a no-contact faculty of intuition could generate knowledge of non-spatiotemporal objects. This is extremely puzzling, for it's totally unclear how an appeal to a no-contact faculty of intuition can help solve the epistemological problem with platonism if it's not conjoined with an explanation of reliability.

ical theories are *not* confirmed by empirical findings. Indeed, empirical findings provide no reason whatsoever for supposing that the mathematical parts of our empirical theories are true. I will sketch the argument for this claim below, in section 2.2.4, by arguing that the nominalistic contents of our empirical theories could be true even if their platonistic contents are fictional (the full argument can be found in my [1998, chapter 7]).

A second problem with the Quine-Steiner-Resnik view is that it leaves unexplained the fact that mathematicians are capable of acquiring mathematical knowledge without waiting to see if their theories get applied and confirmed in empirical science. The fact of the matter is that mathematicians acquire mathematical knowledge *by doing mathematics*, and then empirical scientists come along and use our mathematical theories, which we already know are true. Platonists need to explain how human beings could acquire this pre-applications mathematical knowledge. And, of course, what's needed here is precisely what we needed to begin with, namely, an explanation of how human beings could acquire knowledge of abstract mathematical objects despite their lack of contact with such objects. Thus, the Quinean appeal to applications hasn't helped at all — platonists are right back where they started.

2.1.1.4.2 Necessity: Katz and Lewis A second version of the no-contact strategy, developed by Katz [1981; 1998] and Lewis [1986, section 2.4], is to argue that we can know that our mathematical theories are true, without any sort of information-transferring contact with mathematical objects, because these theories are *necessarily* true. The reason we need information-transferring contact with ordinary physical objects in order to know what they're like is that these objects could have been different. For instance, we have to look at fire engines in order to know that they're red, because they could have been blue. But on the Katz–Lewis view, we don't need any contact with the number 4 in order to know that it's the sum of two primes, because it is necessarily the sum of two primes.

This view has been criticized by Field [1989, 233–38] and myself [1998, chapter 2, section 6.4]. In what follows, I will briefly sketch what I think is the main problem.

The first point to note here is that even if mathematical truths are necessarily true, Katz and Lewis still need to explain how we know that they're true. The mathematical realm might have the particular nature that it has of necessity, but that doesn't mean that we could know what its nature is. How could human beings know that the mathematical realm is composed of structures of the sort we study in mathematics — i.e., the natural number series, the set-theoretic hierarchy, and so on — rather than structures of some radically different kind? It is true that *if* the mathematical realm is composed of structures of the familiar sort, then it follows of necessity that 4 is the sum of two primes. But again, how could we know that the mathematical realm is composed of structures of the familiar kind?

It is important that this response not be misunderstood. I am not demanding here an account of how human beings could know that there exist any mathematical objects at all. That, I think, would be an illegitimate skeptical demand; as is argued in Katz's [1981, chapter VI] and my [1998, chapter 3], all we can legitimately demand from platonists is an account of how human beings could know the *nature* of mathematical objects, *given* that such objects exist. But in demanding that Katz and Lewis provide an account of how humans could know that there are objects answering to our mathematical theories, I mean to be making a demand of this latter sort. An anti-platonist might put the point here as follows: "Even if we assume that there exist mathematical objects — indeed, even if we assume that the mathematical objects that exist do so of necessity — we cannot assume that *any* theory we come up with will pick out a system of actually existing objects. Platonists have to explain how we could know *which* mathematical theories are true and which aren't. That is, they have to explain how we could know which kinds of mathematical objects exist."

The anti-platonist who makes this last remark has overlooked a move that platonists can make: they can say that, in fact, we *can* assume that any purely mathematical theory we come up with will pick out a system of actually existing objects (or, more precisely, that any such theory that's *internally consistent* will pick out a system of objects). Platonists can motivate this claim by adopting FBP. For if all the mathematical objects that possibly *could* exist actually *do* exist, as FBP dictates, then every (consistent) purely mathematical theory picks out a system of actually existing mathematical objects. It is important to note, however, that we should not think of this appeal to FBP as showing that the Katz-Lewis necessity-based epistemology can be made to work. It would be more accurate to say that what's going on here is that we are *replacing* the necessity-based epistemology with an FBP-based epistemology. More precisely, the point is that once platonists appeal to FBP, there is no more reason to appeal to necessity at all. (This point is already implicit in the above remarks, but it is made very clear by my own epistemology (see section 2.1.1.5 below, and my 1998, chapter 3), for I have shown how to develop an FBP-based epistemology that doesn't depend upon any claims about the necessity of mathematical truths.) The upshot of this is that the appeal to necessity isn't doing any epistemological work at all; FBP is doing all the work. Moreover, for the reasons already given, the necessity-based epistemology cannot be made to work without falling back on the appeal to FBP. Thus, the appeal to necessity seems to be utterly unhelpful in connection with the epistemological problem with platonism.

But this is not all. The appeal to necessity is not just epistemologically unhelpful; it is also *harmful*. The reason is that the thesis that our mathematical sentences and theories are necessary is dubious at best. Consider, for instance, the null set axiom, which says that there exists a set with no members. Why should we think that this sentence is necessarily true? It seems pretty obvious that it isn't logically or conceptually necessary, for it is an existence claim, and such claims aren't logically or conceptually true.[22] Now, one might claim that

[22] I should note, however, that in opposition to this, Hale and Wright [1992] have argued that the existence of mathematical objects is conceptually necessary. But Field [1993] has argued

our mathematical theories are *metaphysically* necessary, but it's hard to see what this could really amount to. One might claim that sentences like '$2 + 2 = 4$' and '$7 > 5$' are metaphysically necessary for the same reason that, e.g., 'Cicero is Tully' is metaphysically necessary — because they are true in all worlds in which their singular terms denote, or something along these lines — but this doesn't help at all in connection with existence claims like the null set axiom. We can't claim that the null set axiom is metaphysically necessary for anything like the reason that 'Cicero is Tully' is metaphysically necessary. If we tried to do this, we would end up saying that 'There exists an empty set' is metaphysically necessary because it is true in all worlds in which there exists an empty set. But of course, this is completely unacceptable, because it suggests that *all* existence claims — e.g., 'There exists a purple hula hoop' — are metaphysically necessary. In the end, it doesn't seem to me that there is any interesting sense in which 'There exists an empty set' is necessary but 'There exists a purple hula hoop' is not.

2.1.1.4.3 Structuralism: Resnik and Shapiro Resnik [1997, chapter 11, section 3] and Shapiro [1997, chapter 4, section 7] both claim that human beings can acquire knowledge of abstract mathematical structures, without coming into any sort of information-transferring contact with such structures, by simply constructing mathematical axiom systems; for they argue that axiom systems provide *implicit definitions* of structures. I want to respond to this in the same way that I responded to the Katz–Lewis appeal to necessity. The problem is that the Resnik-Shapiro view does not explain how we could know *which* of the various axiom systems that we might formulate actually pick out structures that exist in the mathematical realm. Now, as was the case with Katz and Lewis, if Resnik and Shapiro adopt FBP, or rather, a structuralist version of FBP, then this problem can be solved; for it follows from (structuralist versions of) FBP that any consistent purely mathematical axiom system that we formulate will pick out a structure in the mathematical realm. But as was the case with the Katz-Lewis epistemology, what's going on here is not that the Resnik-Shapiro epistemology is being salvaged, but rather that it's being replaced by an FBP-based epistemology.

It is important to note in this connection that FBP is not built into structuralism; one could endorse a non-plenitudinous or non-full-blooded version of structuralism, and so it is FBP and not structuralism that delivers the result that Resnik and Shapiro need. In fact, structuralism is entirely irrelevant to the implicit-definition strategy of responding to the epistemological problem, because one can claim that axiom systems provide implicit definitions of collections of mathematical objects as easily as one can claim that they provide implicit definitions of structures. What one needs, in order to make this strategy work, is FBP, not structuralism. (Indeed, I argue in my book (chapter 2, section 6.5) that similar remarks apply to everything Resnik and Shapiro say about the epistemology of mathematics: despite their rhetoric, structuralism doesn't play an essential role in their arguments, and so it is epistemologically irrelevant.)

convincingly that their argument is flawed.

Finally, I should note here, in defense of not just Resnik and Shapiro, but Katz and Lewis as well, that it may be that the views of these four philosophers are best interpreted as involving (in some sense) FBP. But the problem is that these philosophers don't *acknowledge* that they need to rely upon FBP, and so obviously — and more importantly — they don't *defend* the reliance upon FBP. In short, all four of these philosophers could have given FBP-based epistemologies without radically altering their metaphysical views, but none of them actually did.

(This is just a sketch of one problem with the Resnik-Shapiro view; for a more thorough critique, see my [1998, chapter 2, section 6.5].)

2.1.1.5 An FBP-Based Epistemology Elsewhere [1992; 1995; 1998], I argue that if platonists endorse FBP, then they can solve the epistemological problem with their view without positing any sort of information-transferring contact between human beings and abstract objects. The strategy can be summarized as follows. Since FBP says that all the mathematical objects that possibly could exist actually do exist, it follows that if FBP is correct, then all consistent purely mathematical theories truly describe some collection of abstract mathematical objects. Thus, to acquire knowledge of mathematical objects, all we need to do is acquire knowledge that some purely mathematical theory is *consistent*. (It doesn't matter how we come up with the theory; some creative mathematician might simply "dream it up".) But knowledge of the consistency of a mathematical theory — or any other kind of theory, for that matter — does not require any sort of contact with, or access to, the objects that the theory is about. Thus, the Benacerrafian lack-of-access problem has been solved: we can acquire knowledge of abstract mathematical objects without the aid of any sort of information-transferring contact with such objects.

Now, there are a number of objections that might occur to the reader at this point. Here, for instance, are four different objections that one might raise:

1. Your account of how we could acquire knowledge of mathematical objects seems to assume that we are capable of *thinking about* mathematical objects, or *dreaming up stories about* such objects, or *formulating theories about* them. But it is simply not clear how we could do these things. After all, platonists need to explain not just how we could acquire *knowledge* of mathematical objects, but also how we could do things like have *beliefs* about mathematical objects and *refer* to mathematical objects.

2. The above sketch of your epistemology seems to assume that it will be easy for FBP-ists to account for how human beings could acquire knowledge of the consistency of purely mathematical theories without the aid of any contact with mathematical objects; but it's not entirely clear how FBP-ists could do this.

3. You may be right that if FBP is true, then all consistent purely mathematical theories truly describe *some* collection of mathematical objects, or some part

of the mathematical realm. But *which* part? How do we know that it will be true of the part of the mathematical realm that its authors intended to characterize? Indeed, it seems mistaken to think that such theories will characterize *unique* parts of the mathematical realm at all.

4. All your theory can explain is how it is that human beings could *stumble onto* theories that truly describe the mathematical realm. On the picture you've given us, the mathematical community accepts a mathematical theory T for a list of reasons, one of which being that T is consistent (or, more precisely, that mathematicians believe that T is consistent). Then, since FBP is true, it turns out that T truly describes part of the mathematical realm. But since mathematicians have no conception of FBP, they do not know *why* T truly describes part of the mathematical realm, and so the fact that it does is, in some sense, *lucky*. Thus, let's suppose that T is a purely mathematical theory that we know (or reliably believe) is consistent. Then the objection to your epistemology is that you have only an FBP-ist account of

   (M1) our ability to know that *if* FBP is true, *then* T truly describes part of the mathematical realm.[23]

   You do not have an FBP-ist account of

   (M2) our ability to know that T truly describes part of the mathematical realm,

   because you have said nothing to account for

   (M3) our ability to know that FBP is true.

In my book (chapters 3 and 4), I respond to all four of the above worries, and I argue that FBP-ists can adequately respond to the epistemological objection to platonism by using the strategy sketched above. I do not have the space to develop these arguments here, although I should note that some of what I say below (section 2.1.2) will be relevant to one of the above objections, namely, objection number 3.

In addition to the above objections concerning my FBP-ist epistemology, there are also a number of objections that one might raise against FBP itself. For instance, one might think that FBP is inconsistent with the *objectivity* of mathematics, because one might think that FBP entails that, e.g., the continuum hypothesis (CH) has no determinate truth value, because FBP entails that both CH and ~CH truly describe parts of the mathematical realm. Or, indeed, one might think that because of this, FBP leads to *contradiction.* In my book (chapters 3 and 4), and my [2001] and [2009], I respond to both of these worries — i.e., the worries about objectivity and contradiction — as well as several other worries about FBP. Indeed, I argue not just that FBP is the best version of platonism there is, but that

[23]The FBP-ist account of (M1) is simple: we can learn what FBP says and recognize that if FBP is true, then *any* theory like T (i.e., any consistent purely mathematical theory) truly describes part of the mathematical realm.

it is entirely defensible — i.e., that it can be defended against all objections (or at any rate, all the objections that I could think of at the time, except for the objection inherent in my argument for the claim that there is no fact of the matter as to whether FBP or fictionalism is true (see section 3 below)). I do not have anywhere near the space to develop all of these arguments here, though, and instead of trying to summarize all of this material, I simply refer the reader to my earlier writings. However, I should say that responses (or at least partial responses) to the two worries mentioned at the start of this paragraph — i.e., the worries about objectivity and contradiction — will emerge below, in sections 2.1.2–2.1.3, and I will also address there some objections that have been raised to FBP since my book appeared. (I don't want to respond to these objections just yet, because my responses will make more sense in the wake of my discussion of the non-uniqueness problem, which I turn to now.)

2.1.2 The Non-Uniqueness Objection to Platonism

2.1.2.1 Formulating the Argument Aside from the epistemological argument, the most important argument against platonism is the non-uniqueness argument, or as it's also called, the multiple-reductions argument. Like the epistemological argument, this argument also traces to a paper of Benacerraf's [1965], but again, my formulation will diverge from Benacerraf's. In a nutshell, the non-uniqueness problem is this: platonism suggests that our mathematical theories describe *unique* collections of abstract objects, but in point of fact, this does not seem to be the case. Spelling the reasoning out in a bit more detail, and couching the point in terms of arithmetic, as is usually done, the argument proceeds as follows.

(1) If there are any sequences of abstract objects that satisfy the axioms of Peano Arithmetic (PA), then there are infinitely many such sequences.

(2) There is nothing "metaphysically special" about any of these sequences that makes it stand out from the others as *the* sequence of natural numbers.

Therefore,

(3) There is no unique sequence of abstract objects that is the natural numbers.

But

(4) Platonism entails that there *is* a unique sequence of abstract objects that is the natural numbers.

Therefore,

(5) Platonism is false.

The only vulnerable parts of the non-uniqueness argument are (2) and (4). The two inferences — from (1) and (2) to (3) and from (3) and (4) to (5) — are both fairly trivial. Moreover, as we will see, (1) is virtually undeniable. (And note that we cannot make (1) any less trivial by taking PA to be a second-order theory and, hence, categorical. This will only guarantee that all the models of PA are isomorphic to one another. It will not deliver the desired result of there being only one model of PA.) So it seems that platonists have to attack either (2) or (4). That is, they have to choose between trying to *salvage* the idea that our mathematical theories are about unique collections of objects (rejecting (2)) and *abandoning* uniqueness and endorsing a version of platonism that embraces the idea that our mathematical theories are not (or at least, might not be) about unique collections of objects (rejecting (4)). In section 2.1.2.4, I will argue that platonists can successfully solve the problem by using the latter strategy, but before going into this, I want to say a few words about why I think they can't solve the problem using the former strategy, i.e., the strategy of rejecting (2).

2.1.2.2 Trying to Salvage the Numbers I begin by sketching Benacerraf's argument in *favor* of (2). He proceeds here in two stages: first, he argues that no sequence of *sets* stands out as *the* sequence of natural numbers, and second, he extends the argument so that it covers sequences of other sorts of objects as well. The first claim, i.e., the claim about sequences of sets, is motivated by reflecting on the numerous set-theoretic reductions of the natural numbers. Benacerraf concentrates, in particular, on the reductions given by Zermelo and von Neumann. Both of these reductions begin by identifying 0 with the null set, but Zermelo identifies $n+1$ with the singleton $\{n\}$, whereas von Neumann identifies $n+1$ with the union $n \cup \{n\}$. Thus, the two progressions proceed like so:

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$$

and

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$$

Benacerraf argues very convincingly that there is no non-arbitrary reason for identifying the natural numbers with one of these sequences rather than the other or, indeed, with any of the many other set-theoretic sequences that would seem just as good here, e.g., the sequence that Frege suggests in his reduction.

Having thus argued that no sequence of sets stands out as *the* sequence of natural numbers, Benacerraf extends the point to sequences of other sorts of objects. His argument here proceeds as follows. From an arithmetical point of view, the only properties of a given sequence that *matter* to the question of whether it is the sequence of natural numbers are *structural* properties. In other words, nothing about the individual objects in the sequence matters — all that matters is the structure that the objects jointly possess. Therefore, any sequence with the right structure will be as good a candidate for being the natural numbers as any other sequence with the right structure. In other words, any *ω-sequence* will be as good a candidate as any other. Thus, we can conclude that no one sequence of objects stands out as *the* sequence of natural numbers.

It seems to me that if Benacerraf's argument for (2) can be blocked at all, it will have to be at this second stage, for I think it is more or less beyond doubt that no sequence of *sets* stands out as *the* sequence of natural numbers. So how can we attack the second stage of the argument? Well, one strategy that some have followed is to argue that all Benacerraf has shown is that numbers cannot be *reduced* to objects of any other kind; e.g., Resnik argues [1980, 231] that while Benacerraf has shown that numbers aren't sets or functions or chairs, he hasn't shown that numbers aren't objects, because he hasn't shown that numbers aren't *numbers*. But this response misses an important point, namely, that while the first stage of Benacerraf's argument is couched in terms of reductions, the second stage is not — it is based on a premise about the arithmetical irrelevance of non-structural properties. But one might think that we can preserve the spirit of Resnik's idea while responding more directly to the argument that Benacerraf actually used. In particular, one might try to do this in something like the following way.

"There is some initial plausibility to Benacerraf's claim that only structural facts are relevant to the question of whether a given sequence of objects is the sequence of natural numbers. For (a) only structural facts are relevant to the question of whether a given sequence is *arithmetically adequate*, i.e., whether it satisfies PA; and (b) since PA is our best theory of the natural numbers, it would seem that it captures *everything we know* about those numbers. But a moment's reflection reveals that this is confused, that PA does *not* capture everything we know about the natural numbers. There is nothing in PA that tells us that the number 17 is not the inventor of Cocoa Puffs, but nonetheless, we know (pre-theoretically) that it isn't. And there is nothing in PA that tells us that numbers aren't sets, but again, we know that they aren't. Likewise, we know that numbers aren't functions or properties or chairs. Now, it's true that these facts about the natural numbers aren't *mathematically important* — that's why none of them is included in PA — but in the present context, that is irrelevant. What matters is this: while Benacerraf is right that if there are any sequences of abstract objects that satisfy PA, then there are many, the same cannot be said about our *full conception of the natural numbers* (FCNN). We know, for instance, that no sequence of sets or functions or chairs satisfies FCNN, because it is built into our conception of the natural numbers that they do not have members, that they cannot be sat on, and so forth. Indeed, we seem to know that no sequence of things that aren't natural numbers satisfies FCNN, because part of our conception of the natural numbers is that they are natural numbers. Thus, it seems that we know of only *one* sequence that satisfies FCNN, *viz.*, the sequence of natural numbers. But, of course, this means that (2) is false, that one of the sequences that satisfies PA stands out as *the* sequence of natural numbers."

Before saying what I think is wrong with this response to the non-uniqueness argument, I want to say a few words about FCNN, for I think this is an important notion, independently of the present response to the non-uniqueness argument. I say more about this in my [1998] and my [2009], but in a nutshell, FCNN is just the collection of everything that we, as a community, believe about the natural numbers. It is not a formal theory, and so it is not first-order or second-order, and it does not have any axioms in anything like the normal sense. Moreover, it is likely that there is no clear fact of the matter as to precisely which sentences are contained in FCNN (although for *most* sentences, there *is* a clear fact of the matter — e.g., '3 is prime' and '3 is not red' are clearly contained in FCNN, whereas '3 is not prime' and '3 is red' are clearly not). Now, I suppose that one might think it is somehow illegitimate for platonists to appeal to FCNN, or alternatively, one might doubt the claim that it is built into FCNN that numbers aren't, e.g., sets or properties. I cannot go into this here, but in my book [1998, chapter 4], I argue that there is, in fact, nothing illegitimate about the appeal to FCNN, and I point out that in the end, my own response to Benacerraf doesn't depend on the claim that it is built into FCNN that numbers aren't sets or properties.

What, then, is wrong with the above response to the non-uniqueness argument? In a nutshell, the problem is that this response begs the question against Benacerraf, because it simply helps itself to "the natural numbers". We can take the point of Benacerraf's argument to be that if all the ω-sequences were, so to speak, "laid out before us", we could have no good reason for singling one of them out as *the* sequence of natural numbers. Now, the above response does show that the situation here is not as grim as Benacerraf made it seem, because it shows that *some* ω-sequences can be ruled out as definitely *not* the natural numbers. In particular, any ω-sequence that contains an object that we recognize as a non-number — e.g., a function or a chair or (it seems to me, though again, I don't need this claim here) a set — can be ruled out in this way. In short, any ω-sequence that doesn't satisfy FCNN can be so ruled out. But platonists are not in any position to claim that all ω-sequences but one can be ruled out in this way; for since they think that abstract objects exist *independently of us*, they must admit that there are very likely numerous kinds of abstract objects that we've never thought about and, hence, that there are very likely numerous ω-sequences that satisfy FCNN and differ from one another only in ways that no human being has ever imagined. I don't see any way for platonists to escape this possibility, and so it seems to me very likely that (2) is true and, hence, that (3) is also true.

(I say a bit more on this topic, responding to objections and so on, in my book (chapter 4, section 2); but the above remarks are good enough for our purposes here.)

2.1.2.3 Structuralism Probably the most well-known platonist response to the non-uniqueness argument — developed by Resnik [1981; 1997] and Shapiro [1989; 1997] — is that platonists can solve the non-uniqueness problem by merely adopting a platonistic version of Benacerraf's own view, i.e., a platonistic version of *structuralism*. Now, given the way I formulated the non-uniqueness argument above, structuralists would reject (4), because on their view, arithmetic is not about some particular sequence of *objects*. Thus, it might seem that the non-uniqueness problem just doesn't arise at all for structuralists.

This, however, is confused. The non-uniqueness problem *does* arise for structuralists. To appreciate this, all we have to do is reformulate the argument in (1)–(5) so that it is about *parts of the mathematical realm* instead of objects. I did this in my book (chapter 4, section 3). On this alternate formulation, the two crucial premises — i.e., (2) and (4) — are rewritten as follows:

(2′) There is nothing "metaphysically special" about any part of the mathematical realm that makes it stand out from all the other parts as *the* sequence of natural numbers (or natural-number positions or whatever).

(4′) Platonism entails that there *is* a unique part of the mathematical realm that is the sequence of natural numbers (or natural-number positions or whatever).

Seen in this light, the move to structuralism hasn't helped the platonist cause at all. Whether they endorse structuralism or not, they have to choose between trying to salvage uniqueness (attacking (2′)) and abandoning uniqueness, i.e., constructing a platonistic view that embraces non-uniqueness (attacking (4′)). Moreover, just as standard versions of object-platonism seem to involve uniqueness (i.e., they seem to accept (4) and reject (2)), so too the standard structuralist view seems to involve uniqueness (i.e., it seems to accept (4′) and reject (2′)). For the standard structuralist view seems to involve the claim that arithmetic is about *the* structure that all ω-sequences have in common — that is, *the* natural-number structure, or pattern.[24] Finally, to finish driving home the point that structuralists have the same problem here that object-platonists have, we need merely note that the argument I used above (section 2.1.2.2) to show that platonists cannot plausibly reject (2) also shows that they cannot plausibly reject (2′). In short, the point here is that since structures exist independently of us in an abstract mathematical realm, it seems very likely that there are numerous things in the mathematical realm that count as structures, that satisfy FCNN, and that differ from one another only in ways that no human being has ever imagined.

In my book (chapter 4) I discuss a few responses that structuralists might make here, but I argue that none of these responses works and, hence, that (2′) is every bit as plausible as (2). A corollary of these arguments is that contrary to what is commonly believed, structuralism is wholly irrelevant to the non-uniqueness objection to platonism, and so we can (for the sake of rhetorical simplicity) forget about the version of the non-uniqueness argument couched in terms of parts of the mathematical realm, and go back to the original version couched in terms of mathematical objects — i.e., the version in (1)–(5). In the next section, I will

[24] Actually, I should say that this is how *I interpret* the standard structuralist view, for to the best of my knowledge, no structuralist has ever explicitly discussed this point. This is a bit puzzling, since one of the standard arguments for structuralism is supposed to be that it provides a way of avoiding the non-uniqueness problem. I suppose that structuralists just haven't noticed that there are general versions of the non-uniqueness argument that apply to their view as well as to object-platonism. They seem to think that the non-uniqueness problem just disappears as soon as we adopt structuralism.

sketch an argument for thinking that platonists can successfully respond to the non-uniqueness argument by rejecting (4), i.e., by embracing non-uniqueness; and as I pointed out in my book, structuralists can mount an exactly parallel argument for rejecting (4′). So again, the issue of structuralism is simply irrelevant here.

(Before leaving the topic of (2) entirely, I should note that I do not think platonists should *commit* to the truth of (2). My claim is that platonists should say that (2) is very likely true, and that we humans could never know that it was false, but that it simply doesn't matter to the platonist view whether (2) is true or not (or more generally, whether any of our mathematical theories picks out a unique collection of objects). This is what I mean when I say that platonists should reject (4): they should reject the claim that their view is committed to uniqueness.)

2.1.2.4 The Solution: Embracing Non-Uniqueness The only remaining platonist strategy for responding to the non-uniqueness argument is to reject (4). Platonists have to give up on uniqueness, and they have to do this in connection not just with arithmetical theories like PA and FCNN, but with all of our mathematical theories. They have to claim that while such theories truly describe collections of abstract mathematical objects, they do not pick out *unique* collections of such objects (or more precisely, that if any of our mathematical theories does describe a unique collection of abstract objects, it is only by blind luck that it does).

Now, this stance certainly represents a departure from traditional versions of platonism, but it cannot be seriously maintained that in making this move, we *abandon* platonism. For since the core of platonism is the belief in abstract objects — and since the core of mathematical platonism is the belief that our mathematical theories truly describe such objects — it follows that the above view is a version of platonism. Thus, the only question is whether there is some reason for thinking that platonists cannot make this move, i.e., for thinking that platonists are *committed* to the thesis that our mathematical theories describe unique collections of mathematical objects. In other words, the question is whether there is any *argument* for (4) — or for a generalized version of (4) that holds not just for arithmetic but for all of our mathematical theories.

It seems to me — and this is the central claim of my response to the non-uniqueness objection — that there *isn't* such an argument. First of all, Benacerraf didn't give any argument at all for (4).[25] Moreover, to the best of my knowledge, no one else has ever argued for it either. But the really important point here is that, *prima facie*, it seems that there couldn't *be* a cogent argument for (4) — or for a generalized version of (4) — because, on the face of it, (4) and its generalization are both highly implausible. The generalized version of (4) says that

[25] Actually, Benacerraf's [1965] paper doesn't even assert that (4) is true. It is arguable that (4) is implicit in that paper, but this is controversial. One might also maintain that there is an argument for (4) implicit in Benacerraf's 1973 argument for the claim that we ought to use the same semantics for mathematese that we use for ordinary English. I will respond to this below.

(P) Our mathematical theories truly describe collections of abstract mathematical objects

entails

(U) Our mathematical theories truly describe *unique* collections of abstract mathematical objects.

This is a *really* strong claim. And as far as I can tell, there is absolutely no reason to believe it. Thus, it seems to me that platonists can simply accept (P) and reject (U). Indeed, they can endorse (P) together with the *contrary* of (U); that is, they can claim that while our mathematical theories do describe collections of abstract objects, none of them describes a unique collection of such objects. In short, platonists can avoid the so-called non-uniqueness "problem" by simply *embracing* non-uniqueness, i.e., by adopting *non-uniqueness platonism* (NUP).

In my book (chapter 4, section 4) — and see also my [2001] and [2009] in this connection — I discuss NUP at length. I will say just a few words about it here. According to NUP, when we do mathematics, we have objects of a certain kind in mind, namely, the objects that correspond to our *full conception* for the given branch of mathematics. For instance, in arithmetic, we have in mind objects of the kind picked out by FCNN; and in set theory, we have in mind objects of the kind picked out by our full conception of the universe of sets (FCUS); and so on. These are the objects that our mathematical theories are about; in other words, they are the *intended* objects of our mathematical theories. This much, I think, is consistent with traditional platonism: NUP-ists claim that while our mathematical theories might be satisfied by all sorts of different collections of mathematical objects, or parts of the mathematical realm, they are only really about the *intended* parts of the mathematical realm, or the *standard* parts, where what is intended or standard is determined, very naturally, by our intentions, i.e., by our full conception of the objects under discussion. (Sometimes, we don't have any substantive pretheoretic conception of the relevant objects, and so the intended structures are just the structures that satisfy the relevant axioms.) But NUP-ists differ from traditional platonists in maintaining that in any given branch of mathematics, it may very well be that there are multiple intended parts of the mathematical realm — i.e., multiple parts that dovetail with all of our intentions for the given branch of mathematics, i.e., with the FC for the given branch of mathematics.

Now, according to NUP, when we do mathematics, we often don't worry about the fact that there might be multiple parts of the mathematical realm that count as intended for the given branch of mathematics. Indeed, we often ignore this possibility altogether and proceed as if there is just one intended part of the mathematical realm. For instance, in arithmetic, we proceed as if there is a unique sequence of objects that is the natural numbers. According to NUP-ists, proceeding in this way is very convenient and completely harmless. The reason it's convenient is that it's just intuitively pleasing (for us, anyway) to do arithmetic in this way, assuming that we're talking about a unique structure and thinking about that structure in the normal way. And the reason it's harmless is that we simply aren't interested in the differences between the various ω-sequences that satisfy FCNN. In other words, because all of these sequences are structurally equivalent, they are indistinguishable with respect to the sorts of facts and properties that we are trying to characterize in doing arithmetic, and so no harm can come from proceeding as if there is only one sequence here.

One might wonder what NUP-ists take the truth conditions of mathematical sentences to be. Their view is that a purely mathematical sentence is *true simpliciter* (as opposed to true in some specific model or part of the mathematical realm) iff it is true in all of the intended parts of the mathematical realm for the given branch of mathematics (and there is at least one such part of the mathematical realm). (This is similar to what traditional (U)-platonists say; the only difference is that NUP-ists allow that for any given branch of mathematics, there may be *numerous* intended parts of the mathematical realm.) Now, NUP-ists go on to say that a mathematical sentence is *false simpliciter* iff it's false in all intended parts of the mathematical realm. Thus, NUP allows for failures of bivalence (and I argue in my [2009] that this does not lead to any problems; in particular, it doesn't require us to stop using classical logic in mathematical proofs). Now, some failures of bivalence will be mathematically uninteresting — e.g., if we have two intended structures that are isomorphic to one another, then any sentence that's true in one of these structures and false in the other will be mathematically uninteresting (and note that within the language of mathematics, there won't even be such a sentence). But suppose that we develop a theory of Fs, for some mathematical kind F, and suppose that our concept of an F is not perfectly precise, so that there are multiple structures that all fit perfectly with our concept of an F, and our intentions regarding the word 'F', but that aren't structurally equivalent to one another. Then, presumably, there will be some mathematically interesting sentences that are true in some intended structures but false in others, and so we will have some mathematically interesting failures of bivalence. We will have to say that there is no fact of the matter as to whether such sentences are true or false, or that they lack truth value, or some such thing. This *may* be the case right now with respect to the continuum hypothesis (CH). It may be that our full conception of set is compatible with both ZF+CH hierarchies and ZF+$\sim$CH hierarchies. If so, then hierarchies of both sorts count as intended structures, and hence, CH is true in some intended structures and false in others, and so we will have to say that CH has no determinate truth value, or that there is no fact of the matter as to whether it is true or false, or some such thing. On the other hand, it may be that there is a fact of the matter here. Whether there is a fact of the matter depends upon whether CH or $\sim$CH follows from axioms that are true in all intended hierarchies, i.e., axioms that are built into our conception of set. Thus, on this view, the question of whether there is a fact of the matter about CH is a *mathematical* question, not a philosophical question. Elsewhere [2001; 2009], I have argued at length that (a) this is the best view to adopt in connection with CH, and (b) NUP (or rather, FBP-NUP) is the only version of realism that yields this view of CH.[26]

This last sentence suggests that platonists have independent reasons for favoring NUP over traditional (U)-platonism — i.e., that it is not the case that the only reason for favoring NUP is that it provides a solution to the non-uniqueness objection. There is also a second independent reason here, which can be put in the following way: (a) as I point out in my book (chapter 4, section 4), FBP leads very naturally into NUP — i.e., it fits much better with NUP than with (U)-platonism — and (b) as we have seen here (and again, this point is argued in much more detail in my book (chapters 2 and 3)), FBP is the best version of platonism there is; indeed, we've seen that FBP is the only tenable version of platonism, because non-full-blooded (i.e., non-plenitudinous) versions of platonism are refuted by the epistemological argument.

But the obvious question that needs to be answered here is whether there are any good arguments for the opposite conclusion, i.e., for thinking that traditional (U)-platonism is superior to NUP, or to FBP-NUP. Well, there are many arguments that one might attempt here. That is, there are many objections that one might raise to FBP-NUP. In my book, I responded to all the objections that I could think of (see chapter 3 for a defense of the FBP part of the view and chapter 4 for a defense of the NUP part of the view). Some of these objections were discussed above; I cannot go through all of them here, but in section 2.1.3, I will respond to a few objections that have been raised against FBP-NUP since my book appeared, and in so doing, I will also touch on some of the objections mentioned above.

In brief, then, my response to the non-uniqueness objection to platonism is this: the fact that our mathematical theories fail to pick out unique collections of mathematical objects (or *probably* fail to do this) is simply not a problem for platonists, because they can endorse NUP, or FBP-NUP.

I have now argued that platonists can adequately respond to both of the Benacerrafian objections to platonism. These two objections are widely considered to be the only objections that really challenge mathematical platonism, but there are some other objections that platonists need to address — objections not to FBP-NUP in particular, but to platonism in general. For instance, there is a worry about how platonists can account for the applicability of mathematics; there are worries about whether platonism is consistent with our abilities to refer to, and have beliefs about, mathematical objects; and there is a worry based on Ockham's razor. I responded to these objections in my book (chapters 3, 4, and 7); I cannot discuss all of ~~them~~ here, but below (section 2.3) I will say a few words about the

[26] These remarks are relevant to the problem of accounting for the objectivity of mathematics, which I mentioned in section 2.1.3.5. It is important to note that FBP-ists can account for lots of objectivity in mathematics. On this view, sentences like '3 is prime' are objectively true, and indeed, sentences that are undecidable in currently accepted mathematical theories can be objectively true. E.g., I think it's pretty clear that the Gödel sentence for Peano Arithmetic and the axiom of choice are both true in all intended parts of the mathematical realm. But unlike traditional platonism, FBP also allows us to account for how it *could* be that *some* undecidable sentences do not have objective truth values, and as I argue in my [2001] and [2009], this is a strength of the view.

Ockham's-razor-based objection.

2.1.3 Responses to Some Recent Objections to FBP-NUP

2.1.3.1 Background to Restall's Objections Greg Restall [2003] has raised some objections to FBP-NUP. Most of his criticisms concern the question of how FBP is to be *formulated*. In my book [1998, section 2.1], I offered a few different formulations of FBP, although I wasn't entirely happy with any of them. I wrote:

> The idea behind FBP is that the ordinary, actually existing mathematical objects exhaust all of the logical possibilities for such objects; that is, that there actually exist mathematical objects of all logically possible kinds; that is, that all the mathematical objects that logically possibly *could* exist actually *do* exist; that is, that the mathematical realm is plenitudinous. Now, I do not think that any of the four formulations of FBP given in the previous sentence avoids all ... difficulties ..., but it seems to me that, between them, they make tolerably clear what FBP says.

I'm now no longer sure that these definitions are unacceptable — this depends on what we say about *logical possibilities*, and *kinds*, and how clear we take 'plenitudinous' to be. Moreover, to these four formulations, I might add a fifth, suggested to me by a remark of Zalta and Linsky: There are as many mathematical objects as there logically possibly could be.[27] In any event, I want to stand by what I said in my book: together, these formulations of FBP make it clear enough what the view is.

Restall doesn't object to any of these definitions of FBP; rather, he objects to two other definitions — definitions that, in my book, I explicitly distanced myself from. One of these definitions is a statement of second-order modal logic. After making the above informal remarks about FBP, I said that I do not think "that there is any really adequate way to formalize FBP", that "it is a mistake to think of FBP as a formal theory", and that "FBP is, first and foremost, an informal philosophy of mathematics" (p. 6). But having said this, I added that one might try to come close to formalizing FBP with this:

(1) $(\forall Y)(\Diamond(\exists x)(Mx \& Yx) \supset (\exists x)(Mx \& Yx))$ — where 'Y' is a second-order variable and 'Mx' means 'x is a mathematical object'.

The second definition of FBP that Restall attacks can be put like this:

(0) Every logically consistent purely mathematical theory truly describes a part of the mathematical realm. (Note that to say that T truly describes a part P of the mathematical realm is not just to say that P is a *model* of T, for theories can have very unnatural models;[28] rather, the idea here is that if T truly describes P, then T is intuitively and straightforwardly *about* P — that is, P is a part of the mathematical realm that is, so to speak, *lifted straight off* of the theory, and not some convoluted, unnatural model.)

[27]This isn't an exact quote, but see their [1995, 533] for a similar remark.

Now, as we saw above, it is true that thesis (0) *follows from* FBP and, indeed, that (0) is an important feature of my FBP-ist epistemology; but I never intended to use (0) as a *definition* of FBP (I make this point in my book (chapter 1, endnote 13)). One reason for this is as follows: if (0) is true, then it requires explanation, and as far as I can see, the explanation could only be that the mathematical realm is plenitudinous.[29] Thus, by defining FBP as the view that the mathematical realm is plenitudinous, I am simply zeroing in on something that is, in some sense, prior to (0); so again, on this approach, (0) doesn't define FBP — it *follows* from FBP. Moreover, this way of proceeding dovetails with the fact that FBP is, at bottom, an *ontological* thesis, i.e., a thesis about which mathematical objects exist. The thesis that the mathematical realm is plenitudinous (which is what I take FBP to be) is an ontological thesis of this sort, but intuitively, (0) is not; intuitively, (0) is a thesis about mathematical theories, not mathematical objects.

Nonetheless, Restall's objections are directed toward (1) and (0), taken as definitions. Now, since I don't endorse (1) or (0) as definitions, these objections are irrelevant. Nonetheless, I want to discuss Restall's objections to show that they don't raise any problems for the definitions I do use (or any other part of my view). So let us turn to his objections now.

2.1.3.2 Restall's Objection Regarding Formalization Restall begins by pointing out that if FBP-ists are going to use a definition along the lines of (1), they need to insist that the second-order predicate Y be a mathematical predicate. I agree with this; as I made clear in the book, FBP is supposed to be restricted to *purely* mathematical theories, and so, obviously, I should have insisted that Y be purely mathematical. Thus, letting 'Math (Y)' mean 'Y is a purely mathematical property', we can replace (1) with

(3) $(\forall \mathrm{Y})[(\mathrm{Math}(\mathrm{Y}) \ \& \ \Diamond(\exists \mathrm{x})(\mathrm{Mx} \ \& \ \mathrm{Yx})) \supset (\exists \mathrm{x})(\mathrm{Mx} \ \& \ \mathrm{Yx})]$.

Restall then goes on to argue that (3) is unacceptable because it is contradictory; for, Restall argues, since CH and ~CH are both logically possible, it follows from (3) that CH and ~CH are both true.

As I pointed out above (section 2.1.1.5), this worry arises not just for definitions like (3), but for FBP in general. In particular, one might worry that because FBP

[28]Moreover, T could truly describe a part of the mathematical realm that isn't a *model* at all; e.g., one might say of a given set theory that it truly describes the part of the mathematical realm that consists of all pure sets. But there is no model that corresponds to this part of the mathematical realm, because the domain of such a model would be the set of all sets, and there is no such thing.

[29]Alternatively, one might try to explain (0) by appealing to Henkin's theorem that all syntactically consistent first-order theories have models, but this won't work; see my book (chapter 3, note 10) for more on this.

entails that all consistent purely mathematical theories truly describe collections of abstract objects, and because ZF+CH and ZF+~CH are both consistent purely mathematical theories, FBP entails that CH and ~CH are both true. I responded to this objection in my book (chapter 3, section 4); I won't repeat here everything I said there, but I would like to briefly explain how I think FBP-ists can respond to this worry. (And after doing this, I will also say a few words about the status of (3) in this connection.)

The main point that needs to be made here is that FBP does not lead to contradiction, because it does not entail that either CH or ~CH is true. It entails that they both truly describe parts of the mathematical realm, but it does not entail that they are *true*, for as we saw above, on the FBP-NUP-ist view, a mathematical statement is *true simpliciter* iff it is true in all intended parts of the mathematical realm (and there is at least one such part); so truly describing a part of the mathematical realm is not sufficient for truth. A second point to be made here is that while FBP entails that both ZF+CH and ZF+~CH truly describe parts of the mathematical realm, there is nothing wrong with this, because on this view, they describe *different* parts of that realm. That is, they describe different hierarchies. (Again, this is just a sketch of my response to the worry about contradiction; for my full response, see my book (chapter 3, section 4).)

What do these considerations tell us about formalizations like (3)? Well, it reveals another problem with them (which we can add to the problems I mentioned in my book), namely, that such formalizations fail to capture the true spirit of FBP because they don't distinguish between *truly describing a part of the mathematical realm* and *being true*. To solve this problem, we would have to replace the occurrences of 'Yx' in (3) with "'Yx' truly describes x", or something to this effect. But of course, if we did this, we would no longer have a formalization of the sort I was considering.

2.1.3.3 Restall's Objection Regarding FCNN Next, Restall argues against the following potential definitions of FBP:

(5) Every consistent mathematical theory has a model; and

(7) Every consistent mathematical theory truly describes some part of the mathematical realm.

I wouldn't use either of these definitions, however; if I were going to use a definition of this general sort, I would use (0) rather than (5) or (7). Again, I don't think of (0) as definitional, but if I were going to fall back to a definition of this general kind, it would be to (0) and not to (5) or (7). I disapprove of (5) because it uses 'has a model' instead of 'truly describes part of the mathematical realm', and as I pointed out above, these are not equivalent; and I disapprove of (7) because it isn't restricted to *purely* mathematical theories. Because of this, Restall's objections to (5) and (7) are irrelevant.

At this point, however, Restall claims that even if we restrict our attention to purely mathematical theories — and hence, presumably, move to a definition like (0) — two problems still remain. I will address one of these problems here and the other in the next section. The first alleged problem can be put like this: (a) if FBP applies only to purely mathematical theories, then it won't apply to FCNN; but (b) if FBP doesn't apply to FCNN, "then we need some *other* reason to conclude that FCNN truly describes some mathematical structure" (Restall, 2003, p. 908).

My response to this is simple: I never claimed (and don't need the claim) that FCNN truly describes part of the mathematical realm. The purpose of the FBP-NUP-ist's appeal to FCNN is to limit the set of structures that count as *intended* structures of arithmetic; the claim, put somewhat roughly, is that a structure counts as an intended structure of arithmetic just in case FCNN truly describes it.[30] But it is not part of FBP-NUP that FCNN *does* truly describe part of the mathematical realm. If it doesn't truly describe any part of the mathematical realm (even on the assumption that FBP is true), then that's a problem for *arithmetic*, not for the FBP-NUP-ist philosophy of arithmetic — it means that there is something wrong with our conception of the natural numbers, because it means that (even if FBP is true) there are no structures that correspond to our number-theoretic intentions and, hence, that our arithmetical theories aren't true. Now, for whatever it's worth, I think it's pretty obvious that there *isn't* anything wrong with our conception of the natural numbers, and so I think that if FBP is true, then FCNN does truly describe part of the mathematical realm. For (a) it seems pretty obvious that FCNN is consistent, and given this, FBP entails that the purely mathematical part of FCNN (i.e., the part consisting of sentences like the axioms and theorems of PA, and sentences like 'Numbers aren't sets') truly describes part of the mathematical realm; and (b) I think it's pretty obvious that the mixed part of FCNN (i.e., the part containing sentences like 'Numbers aren't chairs') is more or less trivial and, in particular, that it doesn't rule out all of the parts of the mathematical realm that are truly described by the purely mathematical part of FCNN; it is just very implausible to suppose that there are mixed sentences built into the way that we *conceive* of the natural numbers that rule out *all* of the "candidate structures" (from the vast, plenitudinous mathematical realm) that are truly described by the purely mathematical part of FCNN. Of course, this is *conceivable* — it *could* be (in some sense) that it's built into FCNN that 2 is such that snow is purple. But this just seems very unlikely. (Of course, it is also very unlikely that it's built into FCNN that 2 is such that snow is white; our conception of 2 is pretty obviously neutral regarding the color of snow, although I think it does follow from our conception of 2 that it isn't *made* of snow.) In any event, if the above remarks are correct, and if FBP is true, then it is *very likely* that FCNN truly describes part of the mathematical realm. But again, the FBP-NUP-ist doesn't need this result.

[30] I say this is "somewhat rough" because it is a bit simplified; in particular, it assumes that FCNN is consistent. I say a few words about how to avoid this assumption in my [2001], especially in endnotes 5, 18, and 20 (and the corresponding text).

2.1.3.4 Restall's Objection Regarding Non-Uniqueness The second alleged problem that still remains after we restrict FBP to purely mathematical theories (and the last problem that Restall raises) is that definitions of FBP along the lines of (0) are inconsistent with NUP. Restall claims that if NUP is true, and if we have a standard semantics, so that only one thing can be identical to the number 3, then mathematical theories don't truly describe their objects in the manner of (0).

First of all, it strikes me as an utter contortion of issues to take this as an objection to (0)-type definitions of FBP. Restall's objection can be put in the following way: "If you embrace (0)-type FBP and NUP, then you'll have to endorse the thesis that

(M) The numeral '3' doesn't have a unique reference; i.e., there are multiple things that are referents of '3'.

But (M) is absurd, for if '3' refers to two different objects x and y, then we'll have $x = 3$ and $y = 3$ and $x \neq y$, which is a contradiction. Therefore, we have to give up on (0)-type FBP or on NUP." It seems to me, however, that it is clearly NUP, and not FBP, that is the culprit in giving rise to (M); for (a) any version of NUP, whether it is FBP-ist or not, will run into (M)-type problems, but (b) this is not true of FBP — if it is not combined with NUP, it will not run into any such problem. Conclusion: this argument isn't an argument against FBP, or (0)-type definitions of FBP, at all; rather, it is an argument against NUP.

Nonetheless, as an argument against NUP, it is worth considering. Now, the first point I want to make in this connection is that the overall problem here is one that I addressed in my book. I pointed out myself that FBP-NUP entails (M), and I spent several pages (84–90) arguing that it is *acceptable* for platonists to endorse (M) and responding to several arguments for the contrary claim that it is *not* acceptable for platonists to endorse (M). Restall has a different argument for thinking (M) unacceptable, however, and so I want to address his argument.

Restall's argument against (M) is that it leads to contradiction, because if '3' refers to two different objects x and y, then we'll have $x = 3$ and $y = 3$ and $x \neq y$. But in fact, my FBP-NUP-ist view doesn't lead to this contradiction. Of course, there are some theories that endorse (M) that *do* lead to this contradiction. Consider, for instance, a theory that (a) talks about two different structures — e.g., $0^*, 1^*, 2^*, 3^*, \ldots$; and $0', 1', 2', 3' \ldots$ — that both satisfy FCNN and, hence, are both candidates for being the natural numbers, and (b) says that '$3 = 3^*$', '$3 = 3'$', and '$3^* \neq 3'$' are all true. This theory is obviously contradictory. But this isn't my FBP-NUP-ist view; in particular, FBP-NUP doesn't lead to the result that sentences like '$3 = 3^*$' and '$3 = 3'$' are true. Why? Because neither of these sentences is true in all intended parts of the mathematical realm — which, recall, is what is required, according to FBP-NUP, for a mathematical sentence to be true, or true *simpliciter*. Sentences like '$3 = 3^*$' and '$3 = 3'$' are true in some intended structures, but they are not true in *all* intended structures.

(Of course, according to FBP-NUP, sentences like this aren't *false simpliciter* either, and so we have here a failure of bivalence, though of course, not a mathematically interesting or important failure of bivalence. See section 2.1.2.4 above.)

2.1.3.5 Colyvan and Zalta: Non-Uniqueness vs. Incompleteness It is worth noting that if they wanted to, FBP-ists could avoid committing to NUP and (M). To see how, notice first that FBP-ists can say that among all the abstract mathematical objects that exist in the plenitudinous mathematical realm, some are *incomplete objects*. (Some thought would need to be put into defining 'incomplete', but here's a quick definition off the top of my head that might need to be altered: an object o is *incomplete with respect to the property P* iff there is no fact of the matter as to whether o possesses P.) Given this, and on the assumption that FCNN does truly describe part of the mathematical realm, FBP-ists could claim that FCNN picks out a *unique* part of the mathematical realm, namely, the part that (a) satisfies FCNN and (b) has no features that FCNN doesn't entail that it has. Call this view *incompleteness-FBP*. Zalta [1983] endorses a version of platonism that's similar to this in a couple of ways (but also different in a few important ways — e.g., on his view, FCNN doesn't play any role at all), and in a review of my book, he and co-author Mark Colyvan [1999] point out that no argument is given in my book for thinking that NUP-FBP is superior to incompleteness-FBP.

Colyvan and Zalta are right that I didn't address this in my book, so let me say a few words about why I think FBP-ists should favor NUP-FBP over incompleteness-FBP. It seems to me that incompleteness-FBP would be acceptable only if it were built into our intentions, in ordinary mathematical discourse, that we are speaking of objects that don't have any properties that aren't built into our intentions. Now, of course, it is an empirical question whether this *is* built into our intentions, but it seems to me implausible to claim that it is. If I am right about this, then in fact, our arithmetical intentions just don't zero in on unique objects. Now, I suppose one might object that regardless of whether the above kind of incompleteness is built into our intentions, *uniqueness* is built into our intentions, so that if FCNN doesn't pick out a unique part of the mathematical realm, then it doesn't count as being true. But I think this is just false. If God informed us that there are two different structures that satisfy FCNN and that differ from one another only in ways that no human being has ever imagined (and presumably these differences would be non-structural and, hence, mathematically uninteresting), I do not think the mathematical community (or common sense opinion) would treat this information as falsifying our arithmetical theories. Indeed, I think we wouldn't care that there were two such structures and wouldn't feel that we needed to choose between them in order to make sure that our future arithmetical claims were true. And this is evidence that a demand for uniqueness is not built into FCNN. In other words, it suggests that NUP doesn't fly in the face of our mathematical intentions and that it is perfectly acceptable to say, as NUP-FBP-ists do, that in mathematics, truth *simpliciter* can be defined in terms of truth in all intended parts of the mathematical realm.

2.2 *Critique of Anti-Platonism*

2.2.1 *Introduction: The Fregean Argument Against Anti-Platonism*

There are, I suppose, numerous arguments against mathematical anti-platonism (or, what comes to the same thing, in favor of mathematical platonism), but it seems to me that there is only one such argument with a serious claim to cogency. The argument I have in mind is due to Frege [1884; 1893–1903], though I will present it somewhat differently than he did. The argument is best understood as a pair of embedded inferences to the best explanation. In particular, it can be put in the following way:

(i) The only way to account for the truth of our mathematical theories is to adopt platonism.

(ii) The only way to account for the fact that our mathematical theories are applicable and/or indispensable to empirical science is to admit that these theories are true.

Therefore,

(iii) Platonism is true and anti-platonism is false.

Now, *prima facie*, it might seem that (i) is sufficient to establish platonism by itself. But (ii) is needed to block a certain response to (i). Anti-platonists might claim that the alleged fact to be explained in (i) — that our mathematical theories are true — is really no fact at all. More specifically, they might respond to (i) by denying that our mathematical theories are true and endorsing *fictionalism* — which, recall, is the view that (a) mathematical sentences like '$2 + 1 = 3$' do purport to be about abstract objects, but (b) there are no such things as abstract objects, and so (c) these sentences are not true. The purpose of (ii) is to argue that this sort of fictionalist response to (i) is unacceptable; the idea here is that our mathematical theories have to be true, because if they were fictions, then they would be no more useful to empirical scientists than, say, the novel *Oliver Twist* is. (This argument — i.e., the one contained in (ii) — is known as the *Quine-Putnam indispensability argument*, but it does trace to Frege.[31])

I think that the best — and, in the end, the only tenable — anti-platonist response to the Fregean argument in (i)–(iii) is the fictionalist response. Thus, what I want to do here is (a) defend fictionalism (I will do this in section 2.2.4, as well as the present section), and (b) attack the various non-fictionalistic versions of anti-platonism (I will argue against non-fictionalistic versions of anti-realistic anti-platonism in section 2.2.2, and I will argue against the two realistic versions of anti-platonism, i.e., physicalism and psychologism, in section 2.2.3). Now, in connection

[31] Frege appealed only to *applicability* here; see his [1893-1903, section 91]. The appeal to *indispensability* came with Quine (see, e.g., his [1948] and [1951]) and Putnam [1971; 1975].

with task (a) — i.e., the defense of fictionalism — the most important objection that needs to be addressed is just the Quine-Putnam objection mentioned in the last paragraph. I will explain how fictionalists can respond to this objection in section 2.2.4. It is worth noting, however, that there are a few other "minor" objections that fictionalists need to address. Here, for instance, are a few worries that one might have about fictionalism, aside from the Quine-Putnam worry:

1. One might worry that fictionalism is not genuinely anti-platonistic, i.e., that any plausible formulation of the view will involve a commitment to abstract objects. E.g., one might think that (a) fictionalists need to appeal to modal notions like *necessity* and *possibility* (or perhaps, *consistency*) and (b) the only plausible ways of interpreting these notions involve appeals to abstract objects, e.g., possible worlds. Or alternatively, one might claim that when fictionalists endorse sentences like "'3 is prime' is true-in-the-story-of-mathematics," they commit to abstract objects, e.g., sentence types and stories. (One might also worry that Field's nominalization program commits fictionalists to spacetime points and the use of second-order logic, and so one might think that, for these reasons, the view is not genuinely anti-platonistic; but we needn't worry here about objections to Field's nominalization program, because I am going to argue below that fictionalists don't need to — and, indeed, *shouldn't* — rely upon that program.)

2. One might worry that fictionalists cannot account for the objectivity of mathematics; e.g., one might think that fictionalists can't account for how there could be a correct answer to the question of whether the continuum hypothesis (CH) is true or false.

3. One might worry that fictionalism flies in the face of mathematical and scientific practice, i.e., that the thesis that mathematics consists of a body of truths is inherent in mathematical and scientific practice.

In my book (chapter 1, section 2.2, chapter 5, section 3, and the various passages cited in those two sections), I respond to these "minor" objections to fictionalism — i.e., objections other than the Quine-Putnam objection. I will not take the space to respond to all of these worries here, but I want to say just a few words about worry 2, i.e., about the problem of objectivity.

The reader might recall from section 2.1.1.5 that an almost identical problem of objectivity arises for FBP. (The same problem arises for both FBP and fictionalism because both views entail that from a purely *metaphysical* point of view, ZF+CH and ZF+$\sim$CH are equally "good" theories; FBP says that both of these theories truly describe parts of the mathematical realm, and fictionalism says that both of these theories are fictional.) Now, in section 2.1.2.4, I hinted at how FBP-ists can respond to this worry, and it is worth pointing out here that fictionalists can say essentially the same thing. FBP-ists should say that whether ZF+CH or ZF+$\sim$CH is correct comes down to the question of which of these theories (if either) is true in all of the intended parts of the mathematical realm, and that this in turn comes down to whether CH or ~CH is inherent in *our notion of set.* Likewise, fictionalists should say that the question of whether CH is "correct" is determined by whether it's part of the story of set theory, and that this is determined by whether CH would have been true (in all intended parts of the mathematical realm) if there had existed sets, and that this in turn is determined by whether CH is inherent in our notion of set. So even though CH is undecidable in current set theories like ZF, the question of the correctness of CH could still have an objectively correct answer, according to fictionalism, in the same way that the question of whether 3 is prime has an objectively correct answer on the fictionalist view. But fictionalists should also allow, in agreement with FBP-ists, that it *may* be that neither CH nor ~CH is inherent in our notion of set and, hence, may be that there is no objectively correct answer to the CH question. (I say a bit more about this below, but for a full defense of the FBP-ist/fictionalist view of CH, see my [2001] and [2009], as well as the relevant discussions in my book (chapter 3, section 4, and chapter 5, section 3).)

Assuming, then, that the various "minor" objections to fictionalism can be answered, the only objection to that view that remains is the Quine-Putnam indispensability objection. In section 2.2.4, I will defend fictionalism against this objection. (Field tried to respond to the Quine–Putnam objection by arguing that mathematics is not indispensable to empirical science. In contrast, I have argued, and will argue here, that fictionalists can (a) admit (for the sake of argument) that there *are* indispensable applications of mathematics to empirical science and (b) account for these indispensable applications from a fictionalist point of view, i.e., without admitting that our mathematical theories are true.) Before I discuss this, however, I will argue against the various non-fictionalistic versions of anti-platonism (sections 2.2.2–2.2.3).

2.2.2 Critique of Non-Fictionalistic Versions of Anti-Realistic Anti-Platonism

In the next two sections, I will critique the various non-fictionalistic versions of anti-platonism. I will discuss non-fictionalistic versions of anti-realistic anti-platonism in the present section, and I will discuss realistic anti-platonism (i.e., physicalism and psychologism) in the next section, i.e., section 2.2.3.

Given the result that the Quine-Putnam worry is the only important worry about fictionalism, it is easy to show that no version of anti-realistic anti-platonism possesses any advantage over fictionalism. For it seems to me that all versions of anti-realism encounter the same worry about applicability and indispensability that fictionalism encounters. Consider, for example, deductivism. Unlike fictionalists, deductivists try to salvage mathematical truth. But the truths they salvage cannot be lifted straight off of our mathematical theories. That is, if we take the theorems of our various mathematical theories at *face value*, then according to deductivists, they are not true. What deductivists claim is that the theorems of our mathematical theories "suggest" or "represent" certain closely related mathematical assertions that *are* true. For instance, if T is a theorem of Peano

Arithmetic (PA), then according to deductivists, it represents, or stands for, the truth '$AX \supset T$', or '$\Box(AX \supset T)$', where AX is the conjunction of all of the axioms of PA used in the proof of T. Now, it should be clear that deductivists encounter the same problem of applicability and indispensability that fictionalists encounter. For while sentences like '$AX \supset T$' are true, according to deductivists, AX and T and PA are *not* true, and so it is still mysterious how mathematics could be applicable (or, indeed, indispensable) to empirical science.

Now, one might object here that the problem of applicability and indispensability that deductivists face is *not the same* as the problem that fictionalists face, because deductivists have their "surrogate mathematical truths", i.e., their conditionals, and they might be able to solve the problem of applicability by appealing to these truths. But this objection is confused. If these "surrogate mathematical truths" are really *anti-platonistic* truths — and they have to be if they are going to be available to deductivists — then fictionalists can endorse them as easily as deductivists can, and moreover, they can appeal to them in trying to solve the problem of applicability. The only difference between fictionalists and deductivists in this connection is that the former do not try to use any "surrogate mathematical truths" to *interpret mathematical theory.* But they can still *endorse* these truths and appeal to them in accounting for applicability and/or indispensability. More generally, the point is that deductivism doesn't provide anti-platonists with *any* truths that aren't available to fictionalists. Thus, deductivists do not have any advantage over fictionalists in connection with the problem of applicability and indispensability.[32]

In my book (chapter 5, section 4), I argue that analogous points can be made about *all* non-fictionalist versions of anti-realistic anti-platonism — e.g., conventionalism, formalism, and so on. In particular, I argue that (a) all of these views give rise to *prima facie* worries about applicability and indispensability, because they all make the sentences and theories of mathematics factually empty in the sense that they're not "about the world", because they all maintain that our mathematical singular terms are *vacuous*, i.e., fail to refer; and (b) none of these views has any advantage over fictionalism in connection with the attempt to solve the problem of applications, because insofar as these views deny the existence of mathematical objects, their proponents do not have available to them any means of solving the problem that aren't also available to fictionalists.

These remarks suggest that, for our purposes, we could lump all versions of anti-realistic anti-platonism together and treat them as a single view. Indeed, I argued in my book (chapter 5, section 4) that if I replaced the word 'fictionalism' with the expression 'anti-realistic anti-platonism' throughout the book, all the same points could have been made; I would have had to make a few stylistic changes, but

[32]Thus, for instance, fictionalists are free to endorse Hellman's [1989, chapter 3] account of applicability. For whatever it's worth, I do not think that Hellman's account of applicability is a good one, because I think that the various problems with the conditional interpretation of mathematics carry over to the conditional interpretation of empirical theory. I will say a few words about these problems below.

nothing substantive would have needed to be changed, because all the important features of fictionalism that are relevant to the arguments I mounted in my book are shared by all versions of anti-realistic anti-platonism.

But I did not proceed in that way in the book; instead, I took fictionalism as a representative of anti-realistic anti-platonism and concentrated on it. The reason, very simply, is that I think there are good reasons for thinking that fictionalism is the *best* version of anti-realistic anti-platonism. One argument (not the only one) can be put in the following way.

The various versions of anti-realistic anti-platonism do not differ from fictionalism (or from one another) in any metaphysical or ontological way, because they all deny the existence of mathematical objects. (This, by the way, is precisely why they don't differ in any way that is relevant to the arguments concerning fictionalism that I develop in my book.) With a couple of exceptions, which I'll discuss in a moment, the various versions of anti-realism differ from fictionalism (and from one another) only in the interpretations that they provide for mathematical theory. But as soon as we appreciate this point, the beauty of fictionalism and its superiority over other versions of anti-realism begin to emerge. For whereas fictionalism interprets our mathematical theories in a very standard, straightforward, face-value way, other versions of anti-realism — e.g., deductivism, formalism, and Chihara's view — advocate controversial, non-standard, non-face-value interpretations of mathematics that seem to fly in the face of actual mathematical practice. Now, in my book (chapter 5, section 4), I say a bit about *why* these non-standard interpretations of mathematical theory are implausible; but since I don't really need this result — since I could lump all the versions of anti-realism together — I will not pursue this here. (It is worth noting, however, that in each case, the point is rather obvious — or so it seems to me. If we see the various non-standard interpretations of mathematics as claims about the semantics of actual mathematical discourse, they just don't seem plausible. E.g., it doesn't seem plausible to suppose, with deductivists, that ordinary utterances of '3 is prime' really mean '(Necessarily) if there are natural numbers, then 3 is prime'. If we're just doing empirical semantics (that is, if we're just trying to discover the actual semantic facts about actual mathematical discourse), then it seems very plausible to suppose that '3 is prime' means that 3 is prime — which, of course, is just what fictionalists say.[33])

There are two versions of non-fictionalistic anti-realism, however, that *don't*

[33] At least one advocate of reinterpretation anti-realism — namely, Chihara — would admit my point here; he does not claim that his theory provides a good interpretation of actual mathematical discourse. But given this, what possible reason could there be to adopt Chihara's view? If (a) the fictionalistic/platonistic semantics of mathematical discourse is the correct one, and (b) there's no reason to favor Chihara's anti-realism over fictionalism — after all, it encounters the indispensability problem, provides no advantage in solving that problem, and so on — then isn't fictionalism the superior view? It seems to me that if point (a) above is correct, and if (as fictionalists and Chihara agree) there are no such things as abstract objects, then fictionalism is the correct view of actual mathematics. Chihara's view might show that we could have done mathematics differently, in a way that would have made our mathematical assertions come out true, but I don't see why this provides any motivation for Chihara's view.

offer non-standard interpretations of mathematical discourse. But the problems with these views are just as obvious. One view here is the second version of Meinongianism discussed in section 1.2 above; advocates of this view agree with the platonist/fictionalist semantics of mathematese; the only point on which they differ from fictionalists is in their claim that the sentences of mathematics are true; but as we saw in section 1.2, second-version Meinongians obtain this result only by using 'true' in a non-standard way, maintaining that a sentence of the form '*Fa*' can be true even if its singular term (i.e., '*a*') doesn't refer to anything. The second view here is conventionalism, which holds that mathematical sentences like '3 is prime' are analytically true. Now, advocates of this view *might* fall back on a non-standard-interpretation strategy, maintaining that the *reason* '3 is prime' is analytic is that it really means, say, 'If there are numbers, then 3 is prime' — or whatever. But if conventionalists don't fall back on a reinterpretation strategy, then their thesis is just implausible, and for much the same reason that second-version Meinongianism is implausible: if we read '3 is prime' (or better, 'There is a prime number between 2 and 4') at face value, then it's clearly not analytic, because (a) in order for this sentence to be true, there has to exist such a thing as 3, and (b) sentences with existential commitments are not analytic, because they cannot be conceptually true, or true in virtue of meaning, or anything else along these lines.

One might object to the argument that I have given here — i.e., the argument for the supremacy of fictionalism over other versions of anti-realism — on the grounds that fictionalism *also* runs counter to mathematical practice. In other words, one might think that it is built into mathematical and/or scientific practice that mathematical sentences like '3 is prime' are *true*. But in my book (chapter 5, section 3), I argue that this is not the case.

(This is just a sketch of my argument for taking fictionalism to be the best version of anti-realism; for more detail, see my book (chapter 5, section 4) and for a different argument for th supremacy of fictionalism over other versions of anti-realism, see my [2008].)

2.2.3 Critique of Realistic Anti-Platonism (i.e., Physicalism and Psychologism)

In this section, I will argue against the two realistic versions of anti-platonism, thus completing my argument for the claim that fictionalism is the only tenable version of anti-platonism. I will discuss psychologism first and then move on to physicalism.

I pointed out in section 1.1.1 that psychologism is a sort of watered-down version of realism; for while it provides an ontology for mathematics, the objects that it takes mathematical theories to be about do not exist independently of us and our theorizing (for this reason, one might even deny that it is a version of realism, but this doesn't matter here). Because of this, psychologism is similar in certain ways to fictionalism. For one thing, psychologism and fictionalism both involve the idea that mathematics comes entirely from us, as opposed to something independent of us. Now, of course, fictionalists and psychologists put the idea here in different ways: fictionalists hold that our mathematical theories are fictional stories and, hence, not true, whereas advocates of psychologism allow that these theories are true, because the "characters" of the fictionalist's stories exist in the mind; but this is a rather *empty* sort of truth, and so psychologism does not take mathematics to be *factual* in a very deep way. More importantly, psychologism encounters the same worry about applicability and indispensability that fictionalism encounters; for it is no less mysterious how a story about ideas in our heads could be applicable to physical science than how a fictional story could be so applicable.

What, then, does the distinction between psychologism and fictionalism really come to? Well, the difference certainly *doesn't* lie in the assertion of the *existence* of the mental entities in question. Fictionalists admit that human beings do have ideas in their heads that correspond to mathematical singular terms. They admit, for instance, that I have an idea of the number 3. Moreover, they admit that we can make claims about these mental entities that correspond to our mathematical claims; corresponding to the sentence '3 is prime', for instance, is the sentence 'My idea of 3 is an idea of a prime number'. The only difference between fictionalism and psychologism is that the latter, unlike the former, involves the claim that our mathematical theories are *about* these ideas in our heads. In other words, advocates of psychologism maintain that the sentences '3 is prime' and 'My idea of 3 is an idea of a prime number' say essentially the *same thing*, whereas fictionalists deny this. Therefore, it seems to me that the relationship between fictionalism and psychologism is essentially equivalent to the relationship between fictionalism and the versions of anti-realistic anti-platonism that I discussed in section 2.2.2. In short, psychologism interprets mathematical theory in an empty, non-standard way in an effort to salvage mathematical truth, but it still leads to the Quine-Putnam indispensability problem in the same way that fictionalism does, and moreover, it doesn't provide anti-platonists with any means of solving this problem that aren't also available to fictionalists, because it doesn't provide anti-platonists with any entities or truths that aren't available to fictionalists.

It follows from all of this that psychologism can be handled in the same way that I handled the various versions of non-fictionalistic anti-realism and, hence, that I do not really need to refute the view. But as is the case with the various versions of non-fictionalistic anti-realism, it is easy to see that fictionalism is superior to psychologism, because the psychologistic interpretation of mathematical theory and practice is implausible. The arguments here have been well-known since Frege destroyed this view of mathematics in 1884. First of all, psychologism seems incapable of accounting for any talk about the class of *all* real numbers, since human beings could never construct them all. Second, psychologism seems to entail that assertions about very large numbers (in particular, numbers that no one has ever thought about) are all untrue; for if none of us has ever constructed some very large number, then any proposition about that number will, according to psychologism, be vacuous. Third, psychologism seems incapable of accounting for mathematical *error*: if George claims that 4 is prime, we cannot argue with him, because he is presumably saying that *his* 4 is prime, and for all we know, this could very well be *true*.[34] And finally, psychologism turns mathematics into a branch of psychology, and it makes mathematical truths contingent upon psychological truths, so that, for instance, if we all died, '$2+2=4$' would suddenly become untrue. As Frege says, "Weird and wonderful ... are the results of taking seriously the suggestion that number is an idea."[35]

Let me turn now to Millian physicalism. The idea here, recall, is that mathematics is simply a very general natural science and, hence, that it is about ordinary physical objects. Thus, just as astronomy gives us laws concerning all astronomical bodies, so arithmetic and set theory give us laws concerning all objects and piles of objects. The sentence '$2+1=3$', for instance, says that whenever we add one object to a pile of two objects, we end up with a pile of three objects.

Let me begin my critique of physicalism by reminding the reader that in section 2.1.1.3, I argued that because (a) there are infinitely many numerically distinct sets corresponding to every physical object and (b) all of these sets share the same physical base (i.e., are made of the same matter and have the same spatiotemporal location), it follows that (c) there must be something non-physical about these sets, over and above the physical base, and so it could not be true that sets are purely physical objects. A second problem with physicalism is that there simply isn't enough physical stuff in the universe to satisfy our mathematical theories. ZF, for instance, tells us that there are infinitely many transfinite cardinals. It is not plausible to suppose that this is a true claim about the physical world. A third problem with physicalism is that (a) it seems to entail that mathematics is an empirical science, contingent on physical facts and susceptible to empirical falsification, but (b) it seems that mathematics is not empirical and that its truths cannot be empirically falsified. (These arguments are all very quick; for a more thorough argument against the Millian view, see my book (chapter 5, section 5).)

Some of the problems with Millian physicalism are avoided by Kitcher's view [1984, chapter 6]. But as I argue in my book (chapter 5, section 5), Kitcher avoids these problems only by collapsing back into an *anti-realistic* version of anti-platonism, i.e., a view that takes mathematical theory to be *vacuous*. In particular, on Kitcher's view — and he readily admits this [1984, 117] — mathematical theories make claims about non-existent objects, namely, ideal agents. Thus, since Kitcher's view is a version of anti-realism, it can be handled in the same way that I handled all of the other versions of non-fictionalistic anti-realism: (a) I do not have to provide a refutation of Kitcher's view, because it would be acceptable to lump it together with fictionalism; and (b) while Kitcher's view has no advantage

[34] One might reply that the notion of error can be analyzed in terms of non-standardness, but I suspect that this could be cashed out only in terms of *types*. That is, the claim would have to be that a person's theory of arithmetic could be erroneous, or bad, if her concepts of 1, 2, 3, etc. were not of the culturally accepted types. But to talk of types of 1's, 2's, 3's, etc. is to collapse back into platonism.

[35] See Frege [1884, section 27]. Just about all of the arguments mentioned in this paragraph trace to Frege. His arguments against psychologism can be found in his [1884, introduction and section 27; 1893-1903, introduction; 1894 and 1919].

over fictionalism (it still encounters the indispensability problem, delivers no way of solving that problem that's not also available to fictionalists, and so on), we do have reason to favor fictionalism over Kitcher's view, because the latter involves a non-standard, non-face-value interpretation of mathematical discourse that flies in the face of actual mathematical practice. (Once again, this is just a sketch of my argument for the claim that fictionalism is superior to Kitcher's view; for more detail, see my book (chapter 5, section 5).)

2.2.4 Indispensability

I have now criticized all of the non-fictionalistic versions of anti-platonism, but I still need to show that fictionalists can respond to the Quine–Putnam indispensability argument (other objections to fictionalism were discussed in section 2.2.1). The Quine-Putnam argument is based on the premises that (a) there are indispensable applications of mathematics to empirical science and (b) fictionalists cannot account for these applications. There are two strategies that fictionalists can pursue in trying to respond to this argument. The first strategy, developed by Field [1980], is to argue that

(NI) Mathematics is *not indispensable* to empirical science; and

(AA) The mere fact that mathematics is applicable to empirical science — i.e., applicable in a dispensable way — can be accounted for without abandoning fictionalism.

Most critics have been willing to grant thesis (AA) to Field,[36] but (NI) is extremely controversial. To motivate this premise, one has to argue that all of our empirical theories can be *nominalized*, i.e., reformulated in a way that avoids reference to, and quantification over, abstract objects. Field tries to do this by simply showing how to carry out the nominalization for one empirical theory, namely, Newtonian Gravitation Theory. Field's argument for (NI) has been subjected to a number of objections,[37] and the consensus opinion among philosophers of mathematics seems to be that his nominalization program cannot be made to work. I am not convinced that Field's program cannot be carried out — the most important objection, in my opinion, is Malament's [1982] objection that it is not clear how Field's program can be extended to cover quantum mechanics, but in my [1996b], and in my book (chapter 6), I explain how Field's program can be so extended — but I will not pursue this here, because in the end, I do not think fictionalists should respond to the Quine-Putnam objection via Field's nominalization strategy. I think they should pursue another strategy.

The strategy I have in mind here is (a) to grant (for the sake of argument) that there *are* indispensable applications of mathematics to empirical science — i.e.,

[36] But see Shapiro [1983] for one objection to Field's argument for (AA), and see Field [1989, essay 4] for a response.

[37] Malament [1982] discusses almost all of these objections, but see also Resnik [1985] and Chihara [1990, chapter 8, section 5].

that mathematics is hopelessly and inextricably woven into some of our empirical theories — and (b) to simply *account* for these indispensable applications from a fictionalist point of view. I developed this strategy in my book (chapter 7), as well as my [1996a] and [1998b]; the idea has also been pursued by Rosen [2001] and Yablo [2002], and a rather different version of the view was developed by Azzouni [1994] in conjunction with his non-fictionalistic version of nominalism. I cannot even come close here to giving the entire argument for the claim that fictionalists can successfully block the Quine–Putnam argument using this strategy, but I would like to rehearse the most salient points.

The central idea behind this view is that because abstract objects are causally inert, and because our empirical theories don't assign any causal role to them, it follows that the truth of empirical science depends upon two sets of facts that are entirely independent of one another, i.e., that hold or don't hold independently of one another. One of these sets of facts is purely platonistic and mathematical, and the other is purely physical (or more precisely, purely nominalistic). Consider, for instance, the sentence

(A) The physical system S is forty degrees Celsius.

This is a *mixed* sentence, because it makes reference to physical and abstract objects (in particular, it says that the physical system S stands in the Celsius relation to the number 40). But, trivially, (A) does not assign any causal role to the number 40; it is not saying that the number 40 is *responsible* in some way for the fact that S has the temperature it has. Thus, if (A) is true, it is true in virtue of facts about S and 40 that are entirely independent of one another, i.e., that hold or don't hold independently of one another. And again, the same point seems to hold for all of empirical science: since no abstract objects are causally relevant to the physical world, it follows that if empirical science is true, then its truth depends upon two entirely independent sets of facts, *viz.*, a set of purely nominalistic facts and a set of purely platonistic facts.

But since these two sets of facts are *independent* of one another — that is, hold or don't hold independently of one another — it could very easily be that (a) there does obtain a set of purely physical facts of the sort required here, i.e., the sort needed to make empirical science true, but (b) there are no such things as abstract objects, and so there *doesn't* obtain a set of purely platonistic facts of the sort required for the truth of empirical science. In other words, it could be that the *nominalistic content* of empirical science is correct, even if its platonistic content is fictional. But it follows from this that mathematical fictionalism is perfectly consistent with the claim that empirical science paints an essentially accurate picture of the physical world. In other words, fictionalists can endorse what I have called *nominalistic scientific realism* [1996a; 1998, chapter 7; 1998b]. The view here, in a nutshell, is that there do obtain purely physical facts of the sort needed to make empirical science true (regardless of whether there obtain mathematical facts of the sort needed to make empirical science true); in other words, the view is that the physical world holds up *its end* of the "empirical-science bargain".

Nominalistic scientific realism is different from standard scientific realism. The latter entails that our empirical theories are strictly true, and fictionalists cannot make this claim, because that would commit them to the existence of mathematical objects. Nonetheless, nominalistic scientific realism is a genuinely *realistic* view; for if it is correct — i.e., if there does obtain a set of purely physical facts of the sort needed to make empirical science true — then even if there are no such things as mathematical objects and, hence, our empirical theories are (strictly speaking) not true, the physical world is nevertheless *just the way empirical science makes it out to be*. So this is, indeed, a kind of scientific realism.

What all of this shows is that fictionalism is consistent with the actual role that mathematics plays in empirical science, whether that role is indispensable or not. It simply doesn't matter (in the present context) whether mathematics is indispensable to empirical science, because even if it is, the picture that empirical science paints of the physical world could still be essentially accurate, even if there are no such things as mathematical objects.

Now, one might wonder what mathematics is doing in empirical science, if it doesn't need to be true in order for empirical science to be essentially accurate. The answer, I argue, is that mathematics appears in empirical science as a *descriptive aid*; that is, it provides us with an easy way of saying what we want to say about the physical world. In my book, I argue that (a) this is indeed the role that mathematics plays in empirical science, and (b) it follows from this that mathematics doesn't need to be true in order to do what it's supposed to do in empirical science.

(Again, this is just a quick summary; for the full argument that fictionalism can be defended against the Quine–Putnam argument along these lines, see my book (chapter 7), as well as my [1996a] and [1998b].)

(Given that I think that Field's response to the Quine–Putnam argument may be defensible, why do I favor my own response, i.e., the response just described in the last few paragraphs? Well, one reason is that my response is simply less controversial — i.e., it's not open to all the objections that Field's response is open to. A second reason is that my response fits better with mathematical and scientific practice (I argue this point in my book (chapter 7, section 3)). A third reason is that whereas Field's strategy can yield only a piecemeal response to the problem of the applications of mathematics, I account for all applications of mathematics at the same time and in the same way (again, I argue for this in my book (chapter 7, section 3)). And a fourth reason is that unlike Field's view, my view can be generalized so that it accounts not just for the use made of mathematics in empirical science, but also for the use made there of *non-mathematical*-abstract-object talk — e.g., the use made in belief psychology of 'that'-clauses that purportedly refer to propositions (the argument for this fourth reason is given in my [1998b]).)

2.3 Critique of Platonism Revisited: Ockham's Razor

I responded above to the two Benacerrafian objections to platonism, i.e., the epistemological objection and the non-uniqueness objection. These are widely regarded as the two most important objections to platonism, but there are other objections that platonists need to address. For one thing, as I pointed out above, there are a number of objections that one might raise against FBP-NUP in particular; I discussed these above (section 2.1) and in more detail in my book (chapters 3 and 4). But there are also some remaining objections to platonism in general; e.g., there is a worry about how platonists can account for the applicability of mathematics, and there are worries about whether platonism is consistent with our abilities to refer to, and have beliefs about, mathematical objects. In my book, I responded to these remaining objections (e.g., I argued that FBP-NUP-ists can account for the applicability of mathematics in much the same way that fictionalists can, and I argued that they can solve the problems of belief and reference in much the same way that they solve the epistemological problem). In this section, I would like to say just a few words about one of the remaining objections to platonism, in particular, an objection based on Ockham's razor (for my full response to this objection, see my book (chapter 7, section 4.2)).

I am trying to argue for the claim that fictionalism and FBP are both defensible and that they are equally well motivated. But one might think that such a stance cannot be maintained, because one might think that if both of these views are really defensible, then by Ockham's razor, fictionalism is superior to FBP, because it is more parsimonious, i.e., it doesn't commit to the existence of mathematical objects. To give a bit more detail here, one might think that Ockham's razor dictates that if *any* version of anti-platonism is defensible, then it is superior to platonism, regardless of whether the latter view is defensible or not. That is, one might think that in order to motivate platonism, one needs to refute every different version of anti-platonism.

This, I think, is confused. If *realistic* anti-platonists (e.g., Millians) could make their view work, then they could probably employ Ockham's razor against platonism. But we've already seen (section 2.2.3) that realistic anti-platonism is untenable. The only tenable version of anti-platonism is *anti-realistic* anti-platonism. But advocates of this view, e.g., fictionalists, cannot employ Ockham's razor against platonism, because they simply throw away the facts that platonists claim to be explaining. Let me develop this point in some detail.

One might formulate Ockham's razor in a number of different ways, but the basic idea behind the principle is the following: if

(1) theory A explains everything that theory B explains, and

(2) A is more ontologically parsimonious than B, and

(3) A is just as simple as B in all non-ontological respects,

then A is superior to B. Now, it is clear that fictionalism is more parsimonious than FBP, so condition (2) is satisfied here. But despite this, we cannot use Ockham's razor to argue that fictionalism is superior to FBP, because neither of the other two conditions is satisfied here.

With regard to condition (1), FBP-ists will be quick to point out that fictionalism does not account for everything that FBP accounts for. In particular, it doesn't account for facts such as that 3 is prime, that $2 + 2 = 4$, and that our mathematical theories are true in a face-value, non-factually-empty way. Now, of course, fictionalists will deny that these so-called "facts" really are facts. Moreover, if my response to the Quine-Putnam argument is acceptable, and if I am right that the Quine-Putnam argument is the only initially promising argument for the (face-value, non-factually-empty) truth of mathematics, then it follows that FBP-ists have no *argument* for the claim that their so-called "facts" really are facts. But unless fictionalists have an argument for the claim that these so-called "facts" really *aren't* facts — and more specifically, for the claim that our mathematical theories aren't true (in a face-value, non-factually-empty way) — we will be in a stalemate. And given the results that we've obtained so far, it's pretty clear that fictionalists *don't* have any argument here. To appreciate this, we need merely note that (a) fictionalists don't have any good *non-Ockham's-razor-based* argument here (for we've already seen that aside from the Ockham's-razor-based argument we're presently considering, there is no good reason for favoring fictionalism over FBP); and (b) fictionalists don't have any good *Ockham's-razor-based* argument here — i.e., for the claim that the platonist's so-called "facts" really aren't facts — because Ockham's razor cannot be used to settle disputes over the question of what the facts that require explanation *are*. That principle comes into play only after it has been agreed what these facts are. More specifically, it comes into play only in adjudicating between two explanations of an agreed-upon collection of facts. So Ockham's razor cannot be used to adjudicate between realism and anti-realism (whether in mathematics, or empirical science, or common sense) because there is no agreed-upon set of facts here, and in any event, the issue between realists and anti-realists is not which explanations we should accept, but whether we should suppose that the explanations that we eventually settle upon, using criteria such as Ockham's razor, are really *true*, i.e., provide us with accurate descriptions of the world.

Fictionalists might try to respond here by claiming that the platonist's appeal to the so-called "fact" of mathematical truth, or the so-called "fact" that $2+2=4$, is just a disguised assertion that platonism is true. But platonists can simply turn this argument around on fictionalists: if it is question begging for platonists simply to assert that mathematics is true, then it is question begging for fictionalists simply to assert that it's *not* true. Indeed, it seems to me that the situation here actually favors the platonists, for it is the fictionalists who are trying to mount a positive argument here and the platonists who are merely trying to defend their view.

Another ploy that fictionalists might attempt here is to claim that what we need to consider, in deciding whether Ockham's razor favors fictionalism over FBP, is not whether fictionalism accounts for all the *facts* that FBP accounts for, but whether fictionalism accounts for all the *sensory experiences*, or all the *empirical phenomena*, that FBP accounts for. I will not pursue this here, but I argue in my book (chapter 7, section 4.2) that fictionalists cannot legitimately respond to the above argument in this way.

Before we move on, it is worth noting that there is also a historical point to be made here. The claim that there are certain facts that fictionalism cannot account for is not an *ad hoc* device, invented for the sole purpose of staving off the appeal to Ockham's razor. Since the time of Frege, the motivation for platonism has always been to account for mathematical truth. This, recall, is precisely how I formulated the argument for platonism (or against anti-platonism) in section 2.2.1.

I now move on to condition (3) of Ockham's razor. In order to show that this condition isn't satisfied in the present case, I need to show that there are certain non-ontological respects in which FBP is simpler than fictionalism. My argument here is this: unlike fictionalism, FBP enables us to say that our scientific theories are true (or largely true) and it provides a uniform picture of these theories. As we have seen, fictionalists have to tell a slightly longer story here; in addition to claiming that our mathematical theories are fictional, they have to maintain that our empirical theories are, so to speak, *half* truths — in particular, that their nominalistic contents are true (or largely true) and that their platonistic contents are fictional. Moreover, FBP is, in this respect, more *commonsensical* than fictionalism, because it enables us to maintain that sentences like '$2+2=4$' and 'the number of Martian moons is 2' are true.

Now, I do not think that the difference in simplicity here between FBP and fictionalism is very substantial. But on the other hand, I do not think that the ontological parsimony of fictionalism creates a very substantial difference between the two views either. In general, the reason we try to avoid excess ontology is that ontological excesses tend to make our worldview more cumbersome, or less elegant, by adding unnecessary "loops and cogs" to the view. But we just saw in the preceding paragraph that in the case of FBP, this is not true; the immense ontology of FBP doesn't make our worldview more cumbersome, and indeed, it actually makes it less cumbersome. Moreover, the introduction of abstract objects is extremely uniform and non-arbitrary within FBP: we get *all* the abstract objects that there could possibly be. But, of course, despite these considerations, the fact remains that FBP does add a category to our ontology. Thus, it is less parsimonious than fictionalism, and so, in this respect, it is not as simple as fictionalism. Moreover, since the notion of an abstract object is not a commonsensical one, we can say that, in this respect, fictionalism is more commonsensical than FBP.

It seems, then, that FBP is simpler and more commonsensical than fictionalism in some ways but that fictionalism is simpler and more commonsensical in other ways. Thus, the obvious question is whether one of these views is simpler *overall*. But the main point to be made here, once again, is that there are no good *arguments* on either side of the dispute. What we have here is a matter of *brute intuition*: platonists are drawn to the idea of being able to say that our mathematical and empirical theories are straightforwardly true, whereas fictionalists are willing to give this up for the sake of ontological parsimony, but neither group has any *argument* here (assuming that I'm right in my claim that there are acceptable responses to all of the known arguments against platonism and fictionalism, e.g., the two Benacerrafian arguments and the Quine-Putnam argument). Thus, the dispute between FBP-ists and fictionalists seems to come down to a head-butt of intuitions. For my own part, I have *both* sets of intuitions, and overall, the two views seem *equally* simple to me.

3 CONCLUSIONS: THE UNSOLVABILITY OF THE PROBLEM AND A KINDER, GENTLER POSITIVISM

If the arguments sketched in section 2 are cogent, then there are no good arguments against platonism or anti-platonism. More specifically, the view I have been arguing for is that (a) there are no good arguments against FBP (although Benacerrafian arguments succeed in refuting all *other* versions of platonism); and (b) there are no good arguments against fictionalism (although Fregean arguments succeed in undermining all other versions of anti-platonism). Thus, we are left with exactly one viable version of platonism, *viz.*, FBP, and exactly one viable version of anti-platonism, *viz.*, fictionalism, but we do not have any good reason for favoring one of these views over the other. My first conclusion, then, is that we do not have any good reason for choosing between mathematical platonism and anti-platonism; that is, we don't have any good arguments for or against the existence of abstract mathematical objects. I call this the *weak epistemic conclusion.*

In the present section, I will argue for two stronger conclusions, which can be formulated as follows.

> *Strong epistemic conclusion*: it's not just that we *currently* lack a cogent argument that settles the dispute over mathematical objects — it's that we could *never* have such an argument.
>
> *Metaphysical conclusion*: it's not just that *we* could never settle the dispute between platonists and anti-platonists — it's that there is *no fact of the matter* as to whether platonism or anti-platonism is true, i.e., whether there exist any abstract objects.[38]

I argue for the strong epistemic conclusion in section 3.1 and for the metaphysical conclusion in section 3.2.

[38]Note that while the two epistemic conclusions are stated in terms of mathematical objects in particular, the metaphysical conclusion is stated in terms of abstract objects in general. Now, I actually think that generalized versions of the epistemic conclusions are true, but the arguments given here support only local versions of the epistemic conclusions. In contrast, my argument for the metaphysical conclusion is about abstract objects in general.

3.1 *The Strong Epistemic Conclusion*

If FBP is the only viable version of mathematical platonism and fictionalism is the only viable version of mathematical anti-platonism, then the dispute over the existence of mathematical objects comes down to the dispute between FBP and fictionalism. My argument for the strong epistemic conclusion is based on the observation that FBP and fictionalism are, surprisingly, very *similar* philosophies of mathematics. Now, of course, there is a sense in which these two views are polar opposites; after all, FBP holds that all logically possible mathematical objects exist whereas fictionalism holds that *no* mathematical objects exist. But despite this obvious difference, the two views are extremely similar. Indeed, they have much more in common with one another than FBP has with other versions of platonism (e.g., Maddian naturalized platonism) or fictionalism has with other versions of anti-platonism (e.g., Millian empiricism). The easiest way to bring this fact out is simply to list the points on which FBP-ists and fictionalists agree. (And note that these are all points on which platonists and anti-platonists of various other sorts do *not* agree.)

1. Probably the most important point of agreement is that according to both FBP and fictionalism, all consistent purely mathematical theories are, from a metaphysical or ontological point of view, equally "good". According to FBP-ists, all theories of this sort truly describe some part of the mathematical realm, and according to fictionalists, *none* of them do — they are all just fictions. Thus, according to both views, the only way that one consistent purely mathematical theory can be "better" than another is by being aesthetically or pragmatically superior, or by fitting better with our intentions, intuitions, concepts, and so on.[39]

2. As a result of point number 1, FBP-ists and fictionalists offer the same account of undecidable propositions, e.g., the continuum hypothesis (CH). First of all, in accordance with point number 1, FBP-ists and fictionalists both maintain that *from a metaphysical point of view*, ZF+CH and ZF+$\sim$CH are equally "good" theories; neither is "better" than the other; they simply characterize different sorts of hierarchies. (Of course, FBP-ists believe that there actually exist hierarchies of both sorts, and fictionalists do not, but in the present context, this is irrelevant.) Second, FBP-ists and fictionalists agree that the question of whether ZF+CH or ZF+$\sim$CH is correct comes down to the question of which is true in the intended parts of the mathematical realm (or for fictionalists, which would be true in the intended parts of the mathematical realm if there were sets) and that this, in turn, comes down to the question of whether CH or $\sim$CH is inherent in *our notion of set.* Third, both schools of thought allow that it *may* be that neither CH *nor* $\sim$CH is inherent in our notion of set and, hence, that there is no fact of the matter as

[39]In my book (chapter 8, note 3) I also argue that there's no important difference between FBP and fictionalism in connection with *inconsistent* purely mathematical theories.

to which is correct. Fourth, they both allow that even if there is no correct answer to the CH question, there could still be good pragmatic or aesthetic reasons for favoring one answer to the question over the other (and perhaps for "modifying our notion of set" in a certain way). Finally, FBP-ists and fictionalists both maintain that questions of the form 'Does open question Q (about undecidable proposition P) have a correct answer, and if so, what is it?' are questions for *mathematicians* to decide. Each different question of this form should be settled on its own merits, in the above manner; they shouldn't all be decided in *advance* by some metaphysical principle, e.g., platonism or anti-platonism. (See my [2001] and [2009] and my book (chapter 3, section 4, and chapter 5, section 3) for more on this.)[40]

3. Both FBP-ists and fictionalists take mathematical theory at face value, i.e., adopt a realistic semantics for mathematese. Therefore, they both think that our mathematical theories are straightforwardly *about* abstract mathematical objects, although neither group thinks they are about such objects in a metaphysically *thick* sense of the term 'about' (see note 17 for a quick description of the thick/thin distinction here). The reason FBP-ists deny that our mathematical theories are "thickly about" mathematical objects is that they deny that there are *unique* collections of objects that correspond to the totality of intentions that we have in connection with our mathematical theories; that is, they maintain that certain collections of objects just happen to satisfy these intentions and, indeed, that *numerous* collections of objects satisfy them. On the other hand, the reason *fictionalists* deny that our mathematical theories are "thickly about" mathematical objects is entirely obvious: it is because they deny that there are any such things as mathematical objects. (See my book (chapters 3 and 4) for more on this.)

4. I didn't go into this here, but in my book (chapter 3), I show that according to both FBP and fictionalism, mathematical knowledge arises directly out of logical knowledge and that, from an epistemological point of view, FBP and fictionalism are on all fours with one another.

5. Both FBP-ists and fictionalists accept the thesis that there are no causally efficacious mathematical objects and, hence, no causal relations between mathematical and physical objects. (See my book (chapter 5, section 6) for more on this.)

6. Both FBP-ists and fictionalists have available to them the same accounts of the applicability of mathematics and the same reasons for favoring and rejecting the various accounts. (In this essay I said only a few words about the account of applicability that I favor (section 2.2.4); for more on this account, as well as other accounts, see my book (chapters 5–7).)

7. Both FBP-ists and fictionalists are in exactly the same situation with respect to the dispute about whether our mathematical theories are contingent or necessary. My own view here is that both FBP-ists and fictionalists should maintain that (a) our mathematical theories are logically and conceptually contingent, because the existence claims of mathematics — e.g., the null set axiom — are neither logically nor conceptually true, and (b) there is no clear sense of metaphysical necessity on which such sentences come out metaphysically necessary. (For more on this, see my book (chapter 2, section 6.4, and chapter 8, section 2).)

8. Finally, an imprecise point about the "intuitive feel" of FBP and fictionalism: both offer a neutral view on the question of whether mathematical theory construction is primarily a process of invention or discovery. Now, *prima facie*, it seems that FBP entails a discovery view whereas fictionalism entails an invention view. But a closer look reveals that this is wrong. FBP-ists admit that mathematicians discover objective facts, but they maintain that we can discover objective facts about the mathematical realm by merely inventing consistent mathematical stories. Is it best, then, to claim that FBP-ists and fictionalists both maintain an invention view? No. For mathematicians do discover objective facts. For instance, if a mathematician settles an open question of arithmetic by proving a theorem from the Peano axioms, then we have discovered something about the natural numbers. And notice that *fictionalists* will maintain that there has been a discovery here as well, although, on their view, the discovery is not about the natural numbers; rather, it is about our *concept* of the natural numbers, or our *story* of the natural numbers, or what *would* be true if there were mathematical numbers.

I could go on listing similarities between FBP and fictionalism, but the point I want to bring out should already be clear: FBP-ists and fictionalists agree on almost everything. Indeed, in my book (chapter 8, section 2), I argue that there is only *one* significant disagreement between them: FBP-ists think that mathematical objects exist and, hence, that our mathematical theories are true, whereas fictionalists think that there are no such things as mathematical objects and, hence, that our mathematical theories are fictional. My argument for this — i.e., for the only-one-significant-disagreement thesis — is based crucially on points 1 and 3 above. But it is also based on point 5: because FBP-ists and fictionalists agree that mathematical objects would be causally inert if they existed, they both think that the question of whether or not there do exist such objects has no bearing on the physical world and, hence, no bearing on what goes on in the mathematical community or the heads of mathematicians. This is why FBP-ists and fictionalists can agree on so much — why they can offer the same view of mathematical practice

[40] I am not saying that every advocate of fictionalism holds this view of undecidable propositions. For instance, Field [1998] holds a different view. But his view is available to FBP-ists as well, and in general, FBP-ists and fictionalists have available to them the same views on undecidable propositions and the same reasons for favoring and rejecting these views. The view outlined in the text is just the view that *I* endorse.

— despite their bottom-level ontological disagreement. In short, both groups are free to say the same things about mathematical practice, despite their bottom-level disagreement about the existence of mathematical objects, because they both agree that it wouldn't matter to mathematical practice if mathematical objects existed.

If I'm right that the only significant disagreement between FBP-ists and fictionalists is the bottom-level disagreement about the existence of mathematical objects, then we can use this to motivate the strong epistemic conclusion. My argument here is based upon the following two sub-arguments:

(I) We could never settle the dispute between FBP-ists and fictionalists in a *direct* way, i.e., by looking *only* at the bottom-level disagreement about the existence of mathematical objects, because we have no epistemic access to the alleged mathematical realm (because we have access only to objects that exist within spacetime), and so we have no direct way of knowing whether any abstract mathematical objects exist.[41]

and

(II) We could never settle this dispute in an *indirect* way, i.e., by looking at the *consequences* of the two views, because they don't differ in their consequences in any important way, i.e., because the only significant point on which FBP-ists and fictionalists disagree is the bottom-level disagreement about the existence of mathematical objects.

This is just a sketch of my argument for the strong epistemic conclusion; for more detail, see chapter 8, section 2 of my book.

3.2 *The Metaphysical Conclusion*

In this section, I will sketch my argument for the metaphysical conclusion, i.e., for the thesis that there is no fact of the matter as to whether there exist any abstract objects and, hence, no fact of the matter as to whether FBP or fictionalism is true (for the full argument, see my book (chapter 8, section 3)). We can formulate the metaphysical conclusion as the thesis that there is no fact of the matter as to whether the sentence

(*) There exist abstract objects; i.e., there are objects that exist outside of spacetime (or more precisely, that do *not* exist *in* spacetime)

[41]This might seem similar to the Benacerrafian epistemological argument against platonism, but it is different: that argument is supposed to show that platonism is false by showing that even if we assume that mathematical objects exist, we could not know what they are *like*. I refuted this argument in my book (chapter 3), and I sketched the refutation above (section 2.1.1.5). The argument I am using here, on the other hand, is not directed against platonism or anti-platonism; it is aimed at showing that we cannot know (in any direct way) which of these views is correct, i.e., that we cannot know (in a direct way) whether there are any such things as abstract objects.

is true. Given this, my argument for the metaphysical conclusion proceeds (in a nutshell) as follows.

(i) We don't have any idea what a possible world would have to be *like* in order to count as a world in which there are objects that exist outside of spacetime.

(ii) If (i) is true, then there is no fact of the matter as to which possible worlds count as worlds in which there are objects that exist outside of spacetime, i.e., worlds in which (*) is true.

Therefore,

(iii) There is no fact of the matter as to which possible worlds count as worlds in which (*) is true — or in other words, there is no fact of the matter as to what the *possible-world-style truth conditions* of (*) are.

Now, as I make clear in my book, given the way I argue for (iii) — i.e., for the claim that there is no fact of the matter as to which possible worlds count as worlds in which (*) is true — it follows that there is no fact of the matter as to whether the *actual* world counts as a world in which (*) is true. But from this, the metaphysical conclusion — that there is no fact of the matter as to whether (*) is true — follows trivially.

Since the above argument for (iii) is clearly valid, I merely have to motivate (i) and (ii). My argument for (i) is based on the observation that we don't know — or indeed, have any idea — what it would be like for an object to exist outside of spacetime. Now, this is not to say that we don't know what *abstract objects* are like. That, I think, would be wrong. Of the number 3, for instance, we know that it is odd, that it is the cube root of 27, and so on. Thus, there is a sense in which we know what it is like. What I am saying is that we cannot imagine what *existence outside of spacetime* would be like. Now, it may be that, someday, somebody will clarify what such existence might be like; but what I think is correct is that no one has done this *yet*. There have been many philosophers who have advocated platonistic views, but I don't know of any who have said anything to clarify what non-spatiotemporal existence would really *amount* to. All we are ever given is a *negative* characterization of the existence of abstract objects — we're told that such objects do *not* exist in spacetime, or that they exist *non*-physically and *non*-mentally. In other words, we are told only what this sort of existence *isn't* like; we're never told what it *is* like.

The reason platonists have nothing to say here is that our whole conception of what existence *amounts* to seems to be bound up with extension and spatiotemporality. When you take these things away from an object, we are left wondering what its existence could *consist* in. For instance, when we say that Oliver North exists and Oliver Twist does not, what we *mean* is that the former resides at some particular spatiotemporal location (or "spacetime worm") whereas there is nothing in spacetime that is the latter. But there is nothing analogous to this in connection with abstract objects. Contemporary platonists do not think that the existence of 3 consists in there being something more encompassing than spacetime where 3 *resides*. My charge is simply that platonists have nothing substantive to say here, i.e., nothing substantive to say about what the existence of 3 consists in.

The standard contemporary platonist would respond to this charge, I think, by claiming that existence outside of spacetime is just like existence *inside* spacetime — i.e., that there is only *one* kind of existence. But this doesn't solve the problem; it just relocates it. I can grant that "there is only one kind of existence," and simply change my objection to this: we only know what certain *instances* of this kind are like. In particular, we know what the existence of concrete objects amounts to, but we do not know what the existence of abstract objects amounts to. The existence of concrete objects comes down to extension and spatiotemporality, but we have nothing comparable to say about the existence of abstract objects. In other words, we don't have anything more *general* to say about what existence amounts to than what we have to say about the existence of concrete objects. But this is just to say that we don't know what non-spatiotemporal existence amounts to, or what it might *consist* in, or what it might be *like*.

If what I have been arguing here is correct, then it would seem that (i) is true: if we don't have any idea what existence outside of spacetime could be like, then it would seem that we don't have any idea what a possible world would have to be like in order to count as a world that involves existence outside of spacetime, i.e., a world in which there are objects that exist outside of spacetime. In my book (chapter 8, section 3.3), I give a more detailed argument for (i), and I respond to a few objections that one might raise to the above argument.

I now proceed to argue for (ii), i.e., for the claim that if we don't have any idea what a possible world would have to be like in order to count as a world in which there are objects that exist outside of spacetime, then there is no fact of the matter as to which possible worlds count as such worlds — i.e., no fact of the matter as to which possible worlds count as worlds in which (*) is true, or in other words, no fact of the matter as to what the *possible-world-style truth conditions* of (*) are. Now, at first blush, (ii) might seem rather implausible, since it has an epistemic antecedent and a metaphysical consequent. But the reason the metaphysical consequent follows is that the ignorance mentioned in the epistemic antecedent is an ignorance of truth *conditions* rather than truth *value*. If we don't know whether some sentence is true or false, that gives us absolutely no reason to doubt that there is a definite fact of the matter as to whether it really *is* true or false. But when we don't know what the truth *conditions* of a sentence are, that is a very different matter. Let me explain why.

The main point that needs to be made here is that English is, in some relevant sense, *our* language, and (*) is *our* sentence. More specifically, the point is that *the truth conditions of English sentences supervene on our usage*. It follows from this that if our usage doesn't determine what the possible-world-style truth conditions of (*) are — i.e., doesn't determine which possible worlds count as worlds in which (*) is true — then (*) simply doesn't *have* any such truth conditions. In other words,

(iia) If our usage doesn't determine which possible worlds count as worlds in which (*) is true, then there is no fact of the matter as to which possible worlds count as such worlds.

Again, the argument for (iia) is simply that (*) is *our* sentence and, hence, could obtain truth conditions only from our usage.[42]

Now, given (iia), all we need in order to establish (ii), by hypothetical syllogism, is

(iib) If we don't have any idea what a possible world would have to be like in order to count as a world in which there are objects that exist outside of spacetime, then our usage doesn't determine which possible worlds count as worlds in which (*) is true.

But (iib) seems fairly trivial. My argument for this, in a nutshell, is that if the consequent of (iib) were false, then its antecedent couldn't be true. In a bit more detail, the argument proceeds as follows. If our usage *did* determine which possible worlds count as worlds in which (*) is true — i.e., if it determined possible-world-style truth conditions for (*) — then it would also determine which possible worlds count as worlds in which there are objects that exist outside of spacetime. (This is trivial, because (*) just *says* that there are objects that exist outside of spacetime.) But it seems pretty clear that if our usage determined which possible worlds count as worlds in which there are objects that exist outside of spacetime, then we would have at least *some idea* what a possible world would have to be like in order to count as a world in which there are objects that exist outside of spacetime. For (a) it seems that if we have *no idea* what a possible world would have to be like in order count as a world in which there are objects that exist outside of spacetime, then the only way our usage could determine which possible worlds count as such worlds would be if we "lucked into" such usage; but (b) it's simply not plausible to suppose that we have "lucked into" such usage in this way.

This is just a sketch of my argument for the metaphysical conclusion. In my book (chapter 8, section 3), I develop this argument in much more detail, and I respond to a number of different objections that one might have about the argument. For instance, one worry that one might have here is that it is illegitimate to appeal to possible worlds in arguing for the metaphysical conclusion, because possible worlds are themselves abstract objects. I respond to this worry (and a

[42]One way to think of a *language* is as a function from sentence types to meanings and/or truth conditions. And the idea here is that *every* such function constitutes a language, so that English is just one abstract language among a huge infinity of such things. But on this view, the truth conditions of English sentences do *not* supervene on our usage, for the simple reason that they don't supervene on *anything* in the physical world. We needn't worry about this here, though, because (a) even on this view, which abstract language is *our* language will supervene on our usage, and (b) I could simply reword my argument in these terms. More generally, there are lots of ways of conceiving of language and meaning, and for each of these ways, the supervenience point might have to be put somewhat differently. But the basic idea here — that the meanings and truth conditions of *our* words come from *us*, i.e., from our usage and intentions — is undeniable.

number of other worries) in my book, but I do not have the space to pursue this here.

3.3 My Official View

My official view, then, is distinct from both FBP and fictionalism. I endorse the FBP-fictionalist interpretation, or picture, of mathematical theory and practice, but I do not agree with either of the metaphysical views here. More precisely, I am in agreement with almost everything that FBP-ists and fictionalists say about mathematical theory and practice,[43] but I do not claim with FBP-ists that there exist mathematical objects (or that our mathematical theories are true), and I do not claim with fictionalists that there do not exist mathematical objects (or that our mathematical theories are not true).

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[43]That is, I am in agreement here with the kinds of FBP-ists and fictionalists that I've described in this essay, as well as in my book and my [2001] and my [2009].

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