<u>Problem:</u> Let $n \in \mathbb{Z}$ with $n \geq 2$. If n is not a perfect square, then $\sqrt{n}$ is irrational.

<u>proof:</u> Suppose $\sqrt{n}$ is rational.

Then $\sqrt{n} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $a \geq 1$, $b \geq 1$ and $\gcd(a,b) = 1$.

Then $n = \frac{a^2}{b^2}$.

So, $\boxed{b^2 n = a^2.}$ $(*)$

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where the p_i are distinct primes and $e_i \geq 1$ $\forall i$.

Equation $(*)$ then says that $\boxed{b^2 p_1^{e_1} \cdots p_k^{e_k} = a^2}$ $(**)$

Since n is not a perfect square, $\exists j$ with e_j odd.

By $(**)$ p_j divides a^2.

Since p_j is prime, $p_j \mid a$.

Hence $a^2 = p_j^{2f} q_1^{2f_1} \cdots q_m^{2f_m}$ where $p_j \neq q_i$ $\forall i, j$. and the q_i are primes, $f \geq 1$, $f_i \geq 1$.

[Here $a = p_j^{f} q_1^{f_1} \cdots q_m^{f_m}$ is the prime expansion of a.]

So, (**) becomes

$$b^2 p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} = p_j^{2f} q_1^{2f_1} \cdots q_m^{2f_m}$$

But p_j occurs to an even power on the right side (2f power) and an odd power on the left side

(even if p_j occurs in b, it's power will be an $\underbrace{\text{even}}_{(p_j\text{'s power in } b^2)}$ power + $\underbrace{\text{an odd power}}_{e_j}$)

Contradiction. $\blacksquare$