COMMUTATIVE NOETHERIAN SEMIGROUPS ARE FINITELY GENERATED

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Abstract. We provide a short proof that a commutative semigroup is finitely generated if its lattice of congruences is Noetherian.

1. Introduction

Let R be a unitary commutative ring and S a commutative monoid. Gilmer proves in [3] that the monoid ring $R[S]$ is Noetherian if and only if R is Noetherian and S is finitely generated. The proof consists of three parts:

(1) If $R[S]$ is Noetherian, then R is Noetherian and $\operatorname{Cong} S$, the lattice of congruences of S, is Noetherian.
(2) If $\operatorname{Cong} S$ is Noetherian, then S is finitely generated (as a semigroup or as a monoid).
(3) If R is Noetherian and S is finitely generated then $R[S]$ is Noetherian.

By far, the hardest part of this proof is the pure monoid theory represented by 2 in this list. We will say that a monoid (or semigroup) is Noetherian if its lattice of congruences is Noetherian. Then 2 says that any Noetherian monoid is finitely generated. The proof of this is due to Budach [1] and fills Chapter 5 of [3]. It depends on a primary decomposition theorem for congruences on Noetherian semigroups proved by Drbohlav in [2].

The purpose of this paper is to provide a shorter and more direct proof of this result. In fact, it is just as easy to show that any Noetherian semigroup is finitely generated, a result which Gilmer obtained in [4] by reducing to the monoid case.

2. Main Results

We begin with some definitions, notation and basic properties of partially ordered sets and semigroups.

Let $\mathcal{L}$ be a partially ordered set. Then $\mathcal{L}$ is **Artinian** if every nonempty subset of $\mathcal{L}$ has a minimal element (equivalently, $\mathcal{L}$ satisfies the descending chain condition), and $\mathcal{L}$ is **Noetherian** if every nonempty subset of $\mathcal{L}$ has a maximal element (equivalently, $\mathcal{L}$ satisfies the ascending chain condition). If $\sigma : \mathcal{K} \to \mathcal{L}$ is a strictly increasing (decreasing) map between partially ordered sets and $\mathcal{L}$ is Noetherian, then $\mathcal{K}$ is Noetherian (Artinian).

A **lower set** of $\mathcal{L}$ is a subset $D \subseteq \mathcal{L}$ such that for all $x, y \in \mathcal{L}$, if $x \leq y$ and $y \in D$, then $x \in D$. We write $\Downarrow\mathcal{L}$ for the set of lower sets of $\mathcal{L}$ ordered by inclusion. $\mathcal{L}$ embeds in $\Downarrow\mathcal{L}$ via the map $x \mapsto \{y \in \mathcal{L} \mid y \leq x\}$, hence if $\Downarrow\mathcal{L}$ is Artinian, then so is $\mathcal{L}$. The lower set generated by a subset $A \subseteq \mathcal{L}$ is $\{x \in \mathcal{L} \mid \text{ there exists } a \in A \text{ such that } x \leq a\}$.

The proof of the main theorem of this paper proceeds by reducing the question about finite generation of semigroups to the following purely order theoretic result:

Lemma 2.1. *Let $\mathcal{L}$ be a partially ordered set. If $\mathcal{L}$ is Noetherian and $\Downarrow\mathcal{L}$ is Artinian, then $\mathcal{L}$ is finite.*

Proof. Suppose to the contrary that $\mathcal{L}$ is infinite. Since $\mathcal{L}$ is Noetherian we can construct an infinite sequence $\{a_n \mid n \in \mathbb{N}\}$ of distinct elements of $\mathcal{L}$ such that a_1 is maximal in $\mathcal{L}$, and for all $n \geq 2$, a_n is maximal in $\mathcal{L} \setminus \{a_1, a_2, a_3, \ldots, a_{n-1}\}$. For $n \in \mathbb{N}$, let D_n be the lower set generated by $\{a_k \mid k \geq n\}$. Since $\Downarrow\mathcal{L}$ is Artinian and $D_{n+1} \subseteq D_n$ for all $n \geq 1$, there must be some n such that $D_n = D_{n+1}$. In particular, $a_n \in D_{n+1}$. This means that $a_n \leq a_m$ for some $m > n$. But $a_n = a_m$ is not possible because the elements in the sequence are distinct, and $a_n < a_m$ is not possible since a_n is maximal in $\mathcal{L} \setminus \{a_1, a_2, a_3, \ldots, a_{n-1}\}$, a set which also contains a_m. Thus we have a contradiction. □

This lemma follows also from the standard result [6], [7], [8, 1.4] that, if $\Downarrow\mathcal{L}$ is Artinian, then any infinite sequence in $\mathcal{L}$ contains an infinite strictly increasing subsequence. If, in addition, $\mathcal{L}$ is Noetherian, then no such infinite strictly increasing sequence exists, and so $\mathcal{L}$ cannot be infinite.

For the definitions and basic properties of commutative semigroups we refer the reader to [3]. If S is a commutative semigroup, we will write $\operatorname{Cong} S$ for the set of congruences of S ordered in the usual way: $\sim \leq \sim'$ if for all $x, y \in S$, $x \sim y$ implies $x \sim' y$. If $\operatorname{Cong} S$ is Noetherian, we say that S is a **Noetherian semigroup**. The smallest congruence in $\operatorname{Cong} S$ is equality, also known as the **identity congruence**. The largest congruence is the **universal congruence** defined by $x \sim y$ for all $x, y \in S$. For a fixed congruence $\sim$, $\operatorname{Cong}(S/\sim)$ is order isomorphic to the subset $\{\sim' \mid \sim' \geq \sim\}$ of $\operatorname{Cong} S$. In particular, if S is Noetherian then so is $S/\sim$. The subsemigroup generated by an element a or subset A of S will be written $\langle a \rangle$ or $\langle A \rangle$. In this paper "(finitely) generated" means "(finitely) generated as a semigroup".

Define a relation $\leq$ on S by $x \leq y$ if $x = y$ or $x + s = y$ for some $s \in S$. It is easy to see that $\leq$ is reflexive and transitive. Since it is possible to have $x \leq y \leq x$ but $x \neq y$, the relation $\leq$ is not, in general, a partial order on S.

One important case in which $\leq$ is a partial order on S is when every element is an idempotent, that is, $b = 2b$ for all $b \in S$. In this circumstance $(S, \leq)$ is a (join-)semilattice in which $+$ and $\vee$ coincide. See, for example, [5, 1.3.2].

In proving that Noetherian semigroups are finitely generated, certain congruences which behave well with respect to generating sets are the key: A congruence $\sim$ on a semigroup S satisfies $\star$ or is a **$\star$-congruence** if it has the following property: If Y is a subset of S whose image in $S/\sim$ generates $S/\sim$, then S is generated by Y and a finite set.

Note that the identity congruence satisfies $\star$, and that S is finitely generated if and only if the universal congruence satisfies $\star$.

Lemma 2.2. *Let S be a Noetherian semigroup. If the identity congruence is the only $\star$-congruence on S, then S is trivial.*

Proof.

(1) *$(S, \leq)$ is a partially ordered set.* Since $\leq$ is reflexive and transitive, it remains only to show that $a \leq b \leq a$ implies $a = b$ for $a, b \in S$. If $a \leq b \leq a$, then either $a = b$ or $a = b + t_1$ and $b = a + t_2$ for some $t_1, t_2 \in S$.

In the second case, set $T = \langle t_1, t_2 \rangle$ and define the congruence $\sim$ by $x \sim y$ if $x = y$ or there exist $t, t' \in T$ such that $x = y + t$ and $y = x + t'$. By construction, we have $a \sim b$, so to prove $a = b$ it suffices to show that $\sim$ is a $\star$-congruence.

Suppose that the image of $Y \subseteq S$ generates $S/\sim$, then for any element $x \in S$ we have $x \sim y$ for some $y \in \langle Y \rangle$. Either $x = y \in \langle Y \rangle$ or $x = y + t \in \langle t_1, t_2, Y \rangle$ for some $t \in T$. Thus $S = \langle t_1, t_2, Y \rangle$ and $\sim$ is a $\star$-congruence.

(2) *$\Downarrow(S, \leq)$ is Artinian. In particular, $(S, \leq)$ is Artinian.* For $D \in \Downarrow(S, \leq)$, define the congruence $\underset{D}{\sim}$ by $x \underset{D}{\sim} y$ if either $x = y$ or $x \notin D$ and $y \notin D$. It is easy to show that the map $D \mapsto \underset{D}{\sim}$ from $\Downarrow(S, \leq)$ to $\operatorname{Cong} S$ is decreasing.

This map is in fact strictly decreasing when restricted to proper lower sets of $(S, \leq)$: If $D, E \in \Downarrow(S, \leq)$ with $D \subset E \subset S$, then for any $x \in E \setminus D$ and $y \in S \setminus E$ we have $x \underset{D}{\sim} y$ but not $x \underset{E}{\sim} y$. Therefore $\underset{D}{\sim} > \underset{E}{\sim}$.

Since $\operatorname{Cong} S$ is Noetherian, this implies that the set of proper lower sets of S is Artinian. It follows immediately that $\Downarrow(S, \leq)$ is Artinian.

Remark: Since $(S, \leq)$ is partially ordered, the complement of a proper lower set is an ideal of S and vice versa. Moreover, for a proper lower set D, the congruence $\underset{D}{\sim}$ is the Rees congruence associated to the ideal $S \setminus D$. Hence we have also proved that the set of ideals of S ordered by inclusion is Noetherian, a fact which is true in any Noetherian semigroup. See [3, 5.1].

(3) *$(S, \leq)$ is a semilattice, that is, $b = 2b$ for all $b \in S$.* For $b \in S$ define the congruence $\sim$ by $x \sim y$ if either $x = y$ or $b \leq x, y$ and $x + mb = y + nb$ for some $m, n \in \mathbb{N}$. By construction we have $b \sim 2b$, so to prove $b = 2b$ it suffices to show that $\sim$ is a $\star$-congruence.

Suppose the image of $Y \subseteq S$ generates $S/\sim$. We will show that $S = \langle b, Y \rangle$.

If, to the contrary $S \neq \langle b, Y \rangle$, choose x minimal in $S \setminus \langle b, Y \rangle$. We have $x \sim y$ for some $y \in \langle Y \rangle$. Since $x \neq y \in \langle Y \rangle$, we must have $b \leq x, y$ and $x + mb = y + nb$ for some $m, n \in \mathbb{N}$. Since $x \neq b$, there is some x' such that $x = b + x'$. The element x' cannot be in $\langle b, Y \rangle$ since that would imply the same for x. By the minimality of x we have $x' = x$, that is, $x = x + b$. From this we get $x = x + mb = y + nb \in \langle b, Y \rangle$, a contradiction.

(4) *$(S, \leq)$ is Noetherian.* For an element $s \in S$ define the congruence $\sim_s$ by $x \sim_s y$ if $s + x = s + y$. It is easy to check that, since S is a semilattice, the map $s \mapsto \sim_s$ from $(S, \leq)$ to $\operatorname{Cong} S$ is strictly increasing. Since $\operatorname{Cong} S$ is Noetherian, so is $(S, \leq)$.

(5) *S is trivial.* We now have that $(S, \leq)$ is Noetherian and $\Downarrow(S, \leq)$ is Artinian, so from Lemma 2.1, we know that S is finite. But in this case, the universal congruence on S satisfies $\star$. Thus the universal congruence is also the identity congruence, meaning that S is trivial.

□

Theorem 2.3. *Any Noetherian semigroup is finitely generated.*

Proof. Let S be a Noetherian semigroup. Let $\approx$ be a maximal $\star$-congruence on S and $S' = S/\approx$. We show that the only $\star$-congruence on S' is the identity congruence. Any congruence on S' is represented by a congruence $\sim$ on S such that $\approx \leq \sim$. If

$Y \subseteq S$ generates $S/{\sim}$ and $\sim$ satisfies $\star$ with respect to S', then Y and a finite set generate $S/{\approx}$. But then, since $\approx$ satisfies $\star$, Y and a finite set generate S. Thus $\sim$ is a $\star$-congruence with respect to S. By the maximality of $\approx$, we have $\approx = \sim$, that is $\sim$ represents the identity congruence on S'.

Since S' is a Noetherian semigroup whose only $\star$-congruence is the identity congruence, Lemma 2.2 implies that S' is trivial. It follows immediately that $\approx$ is the universal congruence and hence S is finitely generated. □

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