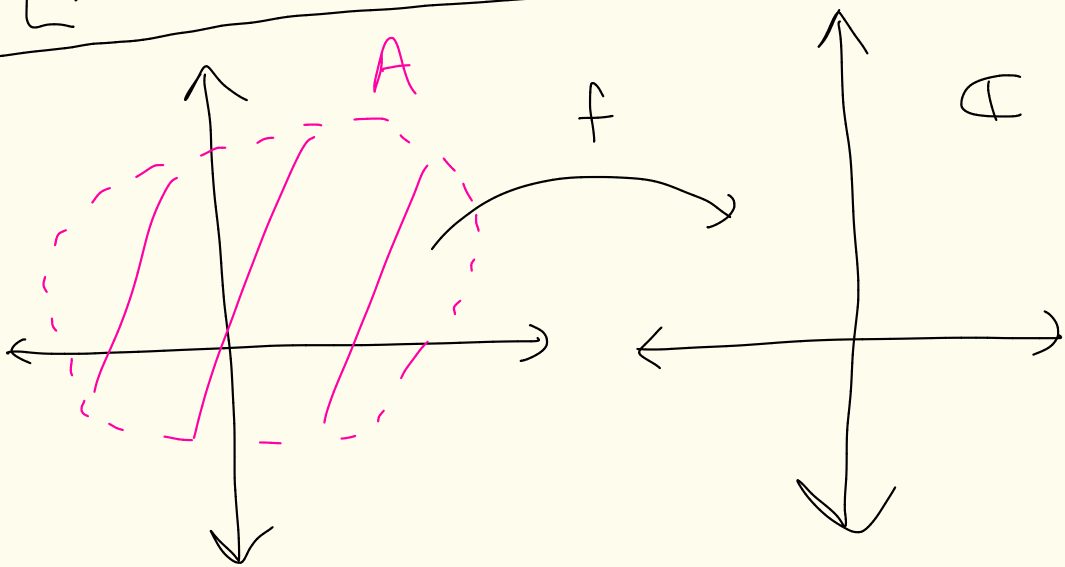
Topic 11 - More cool theorems

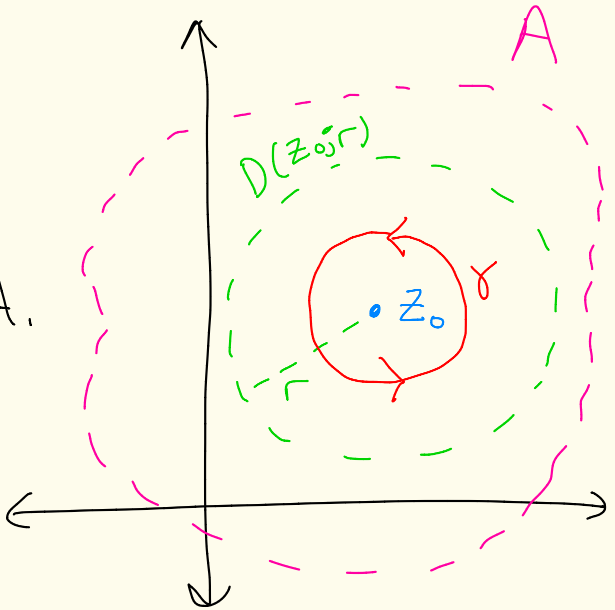
This theorem follows from what we've done.

<u>Theorem:</u> Let $A \subseteq \mathbb{C}$ be an open set and $f: A \to \mathbb{C}$ where f is analytic on A. Then $f^{(k)}$ exists and is also analytic on A for all $k \geq 1$.

[$f^{(k)}$ means the k-th derivative]



<u>proof:</u> We will show that $f^{(k)}$ exists at all points $z_0 \in A$.



Let $z_0 \in A$.
Since A is open there exists $r>0$ where $D(z_0;r) \subseteq A$.

Let $0<r'<r$.
Let γ be the circle of radius r' centered at z_0, oriented counter-clockwise.
So, γ is interior to A. Since f is analytic everywhere on γ and inside γ, by the Cauchy Integral theorem

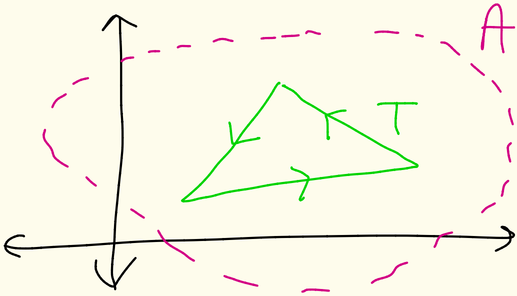
$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

Why is $f^{(k)}$ analytic at z_0?

Because $f^{(k+1)}$ exists on all of A and so $f^{(k+1)}$ on an open disk containing z_0.

$\blacksquare$

Morera's Theorem: Let $A \subseteq \mathbb{C}$ be a region (open & path-connected) and let $f: A \to \mathbb{C}$ be continuous on A. If $\int_T f = 0$ for every triangular path T in A, then f is analytic in A.



(4)

<u>proof:</u> First, observe that f will be shown to be analytic on A if it can be proven that f is analytic on each open disk contained in A. Henceforth, we will suppose that $A = D(a; R)$ where $a \in \mathbb{C}$ and $R > 0$.

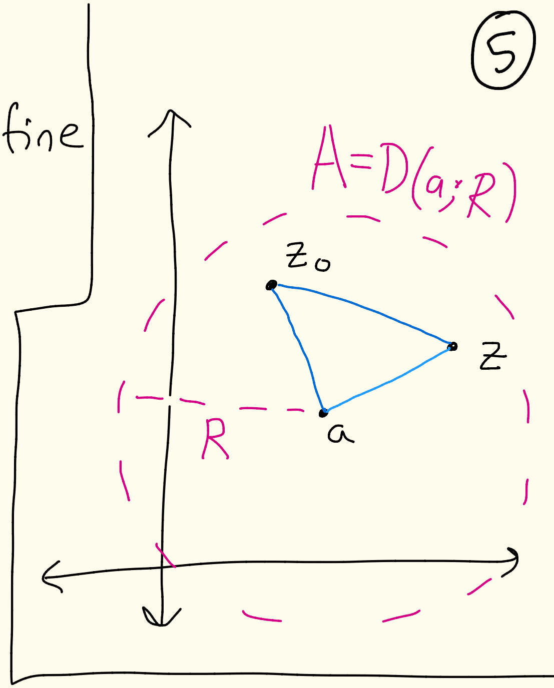
We will prove that f has an anti-derivative F on A. That is we will find $F : A \longrightarrow \mathbb{C}$ where $F' = f$ on A. So, F will be analytic on A and so by the previous theorem, $F' = f$ will be analytic on A.

Let's define F.

For any $z \in A$, define

$$F(z) = \int_{[a,z]} f$$

where $[a,z]$ is the line segment from a to z.



Given $z_0, z \in A$ we have that

$$0 = \int_{[a,z_0]} f + \int_{[z_0,z]} f + \int_{[z,a]} f$$

by assumption of theorem

and so

$$F(z) = \int_{[a,z]} f = -\int_{[z,a]} f = \int_{[a,z_0]} f + \int_{[z_0,z]} f$$

Thus, if $z, z_0 \in A$ then

cancel

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\left[\underbrace{\int_{[a,z_0]} f + \int_{[z_0,z]} f}_{F(z)} \underbrace{- \int_{[a,z_0]} f - \overbrace{\int_{[z_0,z_0]} f}^{\boxed{0}}}_{-F(z_0)}\right]$$

$$= \frac{1}{z-z_0}\int_{[z_0,z]} f$$

Let $\varepsilon > 0$. Since f is continuous at z_0, there exists $\delta > 0$ where if $z \in A$ and $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.

We will assume δ is small enough so that $D(z_0;\delta) \subseteq D(a;R) = A$. [Can do since A is open]

Then if $\underbrace{z \in D(z_0;\delta)}_{|z-z_0|<\delta}$ then

$$\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right|$$

$$= \left| \frac{1}{z-z_0} \int_{[z_0,z]} f - f(z_0) \frac{1}{(z-z_0)} \overbrace{\int_{[z_0,z]} 1}^{(z-z_0)} \right|$$

$$= \left| \frac{1}{z-z_0} \int_{[z_0,z]} \big(f(w)-f(z_0)\big)\, dw \right|$$

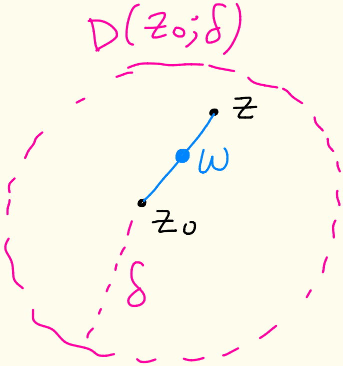
$$= \frac{1}{|z-z_0|} \left| \int_{[z_0,z]} \big(f(w)-f(z_0)\big)\, dw \right|$$

$$= \frac{1}{|z-z_0|} \left| \int_{[z_0,z]} (f(w)-f(z_0))\, dw \right|$$

$$\leq \frac{1}{|z-z_0|} \varepsilon \cdot \text{arclength}([z_0,z])$$

$$= \frac{1}{|z-z_0|} \cdot \varepsilon \cdot |z-z_0|$$

$$= \varepsilon.$$



Since w is on $[z_0,z]$ we have $w \in D(z_0;\delta)$ ie $|w-z_0|<\delta$ so $|f(w)-f(z_0)|<\varepsilon$ $\forall w$ on $[z_0,z]$

What have we done?

Given $\varepsilon > 0$, we found a $\delta > 0$ where if $|z-z_0| < \delta$, then

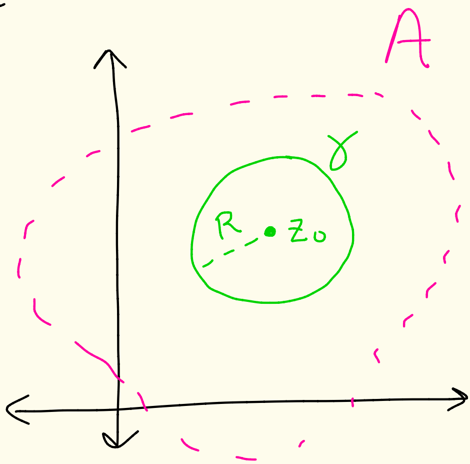
$$\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| < \varepsilon.$$

Thus, $\displaystyle\lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$.

That is $F'(z_0) = f(z_0)$ $\blacksquare$

Theorem (Cauchy Inequality)

Let f be analytic on a region A and let γ be a circle with radius $R>0$ and center $z_0 \in A$, so that γ and the interior of γ both lie in A.



Suppose that $|f(z)| \leq M$ for all z on γ.

Then,

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

for $k = 0, 1, 2, 3, \ldots$

proof: Orient γ counter-clockwise.

Then by the Cauchy Integral Formula

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

Note that if z is on γ, then

$$\left|\frac{f(z)}{(z-z_0)^{k+1}}\right| = \frac{|f(z)|}{|z-z_0|^{k+1}} = \frac{|f(z)|}{R^{k+1}} \leq \frac{M}{R^{k+1}}$$

if z is on γ then $|z-z_0| = R$

$|f(z)| \leq M$ when z on γ by assumption

Thus,

$$|f^{(k)}(z_0)| = \left|\frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right|$$

$$= \frac{k!}{2\pi}\left|\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right| \leq \frac{k!}{2\pi}\cdot\frac{M}{R^{k+1}}\cdot\underbrace{\left(\begin{matrix}\text{arclength}\\ \text{of } \gamma\end{matrix}\right)}_{2\pi R}$$

$|i| = 1$

$$= \frac{k!}{2\pi}\cdot\frac{M}{R^{k+1}}\cdot 2\pi R = \frac{k!}{R^k}\cdot M. \qquad \blacksquare$$

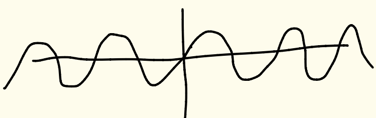
Liouville's Theorem

If f is entire (that is, f is analytic on all of $\mathbb{C}$) and f is bounded on $\mathbb{C}$ (that is, $\exists M > 0$ where $|f(z)| \leq M$ for all $z \in \mathbb{C}$), then f is a constant function.

[So, the only bounded, entire functions, are the constant functions]

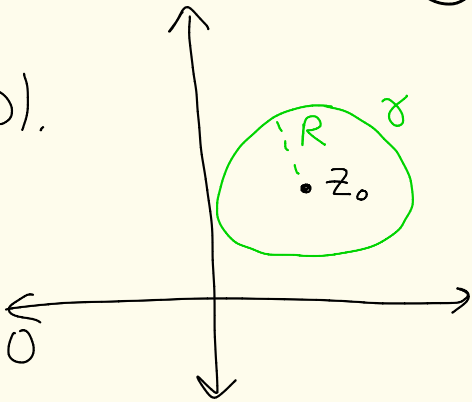
Totally different that analysis in $\mathbb{R}$!

Ex: $|\sin(x)| \leq 1$ and differentiable on all of $\mathbb{R}$, but not constant for example.



<u>Proof:</u> Let f be entire and $|f(z)| \le M$ for all $z \in \mathbb{C}$ (where $M > 0$).

Let $z_0 \in \mathbb{C}$.



Let γ be a circle of radius $R > 0$ centered at z_0.

Since $|f(z)| \le M$ for all z on γ, by the previous theorem

$$|f'(z_0)| \underset{\boxed{k=1}}{\le} \frac{1!}{R^1} \cdot M = \frac{M}{R}. \qquad (*)$$

This is true for any $R > 0$. So let $R \to \infty$, then $\frac{M}{R} \to 0$. So, by $(*)$

$|f'(z_0)| = 0$. Thus, $f'(z_0) = 0$.

Since z_0 was arbitrary, we know $f'(z_0) = 0 \ \forall z_0 \in \mathbb{C}$.

Since $\mathbb{C}$ is a region and $f'(z_0)=0$ $\forall z_0 \in \mathbb{C}$, by a theorem in class we proved after the FTOC, f is a constant function on $\mathbb{C}$. $\blacksquare$

<u>Theorem:</u> (Fundamental theorem of Algebra) Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

where $a_i \in \mathbb{C}$, $n \geq 1$, and $a_n \neq 0$.

Then, $P(z)$ has at least one zero in the complex plane. That is, there exists $z_0 \in \mathbb{C}$ where $P(z_0) = 0$.

proof by contradiction:

Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$.

Let $f(z) = \dfrac{1}{P(z)} = \dfrac{1}{a_0 + a_1 z + \cdots + a_n z^n}$

Since $P(z) \neq 0$ for all $z \in \mathbb{C}$, f is defined on all of $\mathbb{C}$ and is entire.

Let's now that f is bounded on $\mathbb{C}$.

Let
$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z}$$

so that $P(z) = (a_n + w) z^n$.

Note that if $|z| \geq R$ then

$$|w| \leq \left|\frac{a_0}{z^n}\right| + \left|\frac{a_1}{z^{n-1}}\right| + \cdots + \frac{|a_{n-1}|}{|z|}$$

$$\leq \frac{|a_0|}{R^n} + \frac{|a_1|}{R^{n-1}} + \cdots + \frac{|a_{n-1}|}{R}$$

$|z| \geq R$

$\frac{1}{|z|} \leq \frac{1}{R}$

Note that $\dfrac{|a_i|}{R^{n-i}} \to 0$ as $R \to \infty$ (for $0 \leq i \leq n-1$)

So we can find $R > 0$ where each of the terms $\dfrac{|a_i|}{R^{n-i}} < \underbrace{\dfrac{|a_n|}{2n}}_{\text{positive constant}}$ by letting R be large enough.

Pick such an $R > 0$.

Then, if $|z| \geq R$ we have

$$|w| \leq \frac{|a_0|}{R^n} + \frac{|a_1|}{R^{n-1}} + \dots + \frac{|a_{n-1}|}{R}$$

$$< \frac{|a_n|}{2n} + \frac{|a_n|}{2n} + \dots + \frac{|a_n|}{2n}$$

$$= n\left(\frac{|a_n|}{2n}\right) = \frac{|a_n|}{2}$$

$|w| < \frac{|a_n|}{2}$

So if $|z| \geq R$, then

$$|a_n + w| \geq \big||a_n| - |w|\big| = |a_n| - |w| > |a_n| - \frac{|a_n|}{2} = \frac{|a_n|}{2}$$

because $|w| < \frac{|a_n|}{2} < |a_n|$

Thus, if $|z| \geq R$ then

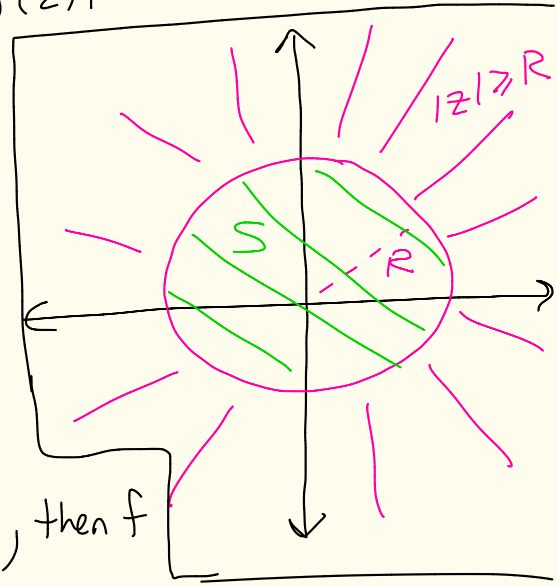
$$|P(z)| = |a_n + w||z^n| > \frac{|a_n|}{2}|z^n| \geq \frac{|a_n|}{2} R^n$$

previous page

$|z| \geq R$

Thus, if $|z| \geq R$ then

$$|f(z)| = \left|\frac{1}{P(z)}\right| = \frac{1}{|P(z)|} \leq \frac{2}{|a_n| R^n}$$



By analysis/topology results since f is continuous on $S = \{z \mid |z| \leq R\}$ and S is closed and bounded (compact), then f is bounded on S.

That is, $\exists K > 0$ where $|f(z)| \leq K$ for $z \in S$.

So,

$$|f(z)| \leq \max\left\{ \frac{2}{|a_n|R^n}, K \right\}$$

for all $z \in \mathbb{C}$.

So, f is entire and bounded, thus by Louiville's theorem $f(z) = c$ for some $c \in \mathbb{C}$. But then $\forall z \in \mathbb{C}$

$$P(z) = \frac{1}{f(z)} = \frac{1}{c}$$

which isn't true. $P(z)$ is not a constant function.

Contradiction.

Thus, P must have a zero in $\mathbb{C}$.

Fundamental Theorem of algebra as usually seen

So, given

$$p(z) = a_0 + a_1 z + \dots + a_n z^n$$

with $a_n \neq 0$, $n \geq 1$,

there exists $z_1 \in \mathbb{C}$ with $p(z_1) = 0$.

So, by polynomial division

$$p(z) = (z - z_1)\, Q_1(z)$$

where $Q_1(z)$ is a poly of degree $n-1$.

Repeat to get

$$p(z) = (z - z_1)(z - z_2)\, Q_2(z)$$

where $Q_2(z)$ is a poly of degree $n-2$

and $z_2 \in \mathbb{C}$.

Keep repeating to get

$$\boxed{p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)}$$

where $z_1, z_2, \dots, z_n \in \mathbb{C}$.

Lemma: Suppose f is analytic on a region A and that $|f(z)|$ is constant on A. Then $f(z)$ is constant on A.

proof: Suppose that $f(x+iy) = u(x,y) + i v(x,y)$.

We are assuming that on A we have

$$|f|^2 = \left(\sqrt{u^2+v^2}\right)^2 = u^2+v^2 = c$$

for some constant c. If $c=0$, then $|f|=0$ on A and thus $f=0$ on A. So now assume $c \neq 0$.

So, $[u(x,y)]^2 + [v(x,y)]^2 = c$ on A.

Differentiating we get

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0 \qquad (*)$$

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

on A.

Since f is analytic on A, by Cauchy-Riemann we know $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=\frac{-\partial u}{\partial y}$

Subbing these into $(*)$ and dividing $(*)$ by 2 we get:

$$u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=0 \qquad (**)$$

$$v\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=0$$

on A.

(**) becomes

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (***)$$

For any fixed input (x,y) the above is a linear system with two equations and two unknowns.

Since $\det\begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2+v^2 = c \neq 0.$

Thus, for each (x,y) there is a unique solution to $(***)$ which is $\frac{\partial u}{\partial x}(x,y) = \frac{\partial u}{\partial y}(x,y) = 0.$

Thus,

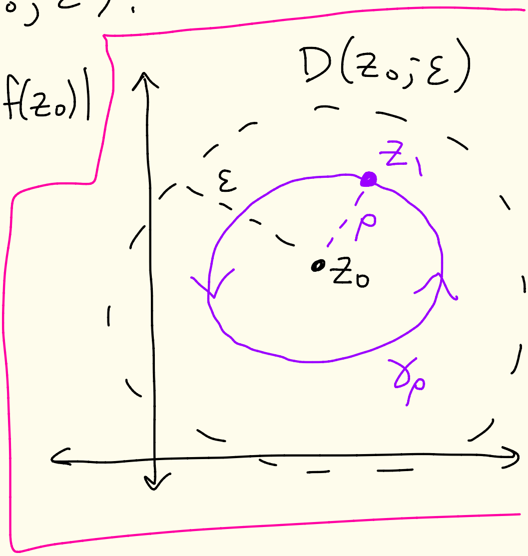
$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial u}{\partial y}(x,y)$$

$$= 0 + i\,0 = 0$$

for all $x+iy \in A$. Since A is a domain and $f'=0$ on A, f is constant on A. ▨

<u>Theorem:</u> Suppose that f is analytic in a neighborhood $D(z_0; \varepsilon)$ where $z_0 \in \mathbb{C}$ and $\varepsilon > 0$.
If $|f(z)| \leq |f(z_0)|$ for all $z \in D(z_0; \varepsilon)$, then f is constant on $D(z_0; \varepsilon)$.

<u>proof:</u> Suppose $|f(z)| \leq |f(z_0)|$ for all $z \in D(z_0; \varepsilon)$.
Let $z_1 \in D(z_0; \varepsilon)$ where $z_1 \neq z_0$.
Let $\rho = |z_0 - z_1|$.
Let γ_ρ be the circle centered at z_0 with radius ρ, oriented counter-clockwise.



By the Cauchy-integral theorem

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z - z_0} \, dz$$

Parameterize γ_ρ as

$$\gamma_\rho(t) = z_0 + \rho e^{it}, \quad 0 \le t \le 2\pi$$

and then $\gamma_\rho'(t) = i\rho e^{it}$.

So we get

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{z - z_0}\, dz$$

$$(*) \quad = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{(z_0 + \rho e^{it}) - z_0} \cdot i\rho e^{it}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt$$

[This result is called Gauss's mean value theorem]

From $(*)$ we get

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt \right|$$

see lemma on page 30

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0+\rho e^{it})|\, dt$$

Since $|f(z_0+\rho e^{it})| \leq |f(z_0)|$ for all t, by assumption, we get

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0+\rho e^{it})|\, dt$$

from above

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, dt = \frac{1}{2\pi}\left[2\pi |f(z_0)|\right]$$

$$= |f(z_0)|.$$

Thus, $\displaystyle \frac{1}{2\pi} \int_0^{2\pi} |f(z_0+\rho e^{it})|\, dt = |f(z_0)|.$

So,

$$-\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{it})|\,dt + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt}_{|f(z_0)|} = 0$$

Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} \Big[\underbrace{|f(z_0)| - |f(z_0+\rho e^{it})|}_{\geq 0}\Big]\,dt = 0$$

because $|f(z_0)| \geq |f(z_0+\rho e^{it})|$

We are integrating a continuous function that is ≥ 0 and the integral equals 0.

The only way this can happen is if $|f(z_0)| - |f(z_0+\rho e^{it})| = 0$

for all t.

So,

$$|f(z)| = |f(z_0)|$$

for all z on γ_ρ.

We can vary z_1 to get all curves γ_ρ inside on $D(z_0;\varepsilon)$.

So, $|f(z)| = |f(z_0)|$ for all $z \in D(z_0;\varepsilon)$.

So, $|f(z)|$ is constant on $D(z_0;\varepsilon)$.

By the lemma,

f is constant on $D(z_0;\varepsilon)$.

$\blacksquare$

(Max modulus theorem)

Theorem: Suppose that f is analytic on a domain A and f is not constant on A.

Then f does not have a max value on A.

That is, there does not exist $z_0 \in A$ where $|f(z_0)| \geq |f(z)|$ for all $z \in A$.

[We just proved this for $A = D(z_0; \varepsilon)$]

Proof in Churchill / Brown

probably also in Hoffman/Marsden

Def: Let $S \subseteq \mathbb{C}$ with $S \neq \emptyset$. We say that $z_0 \in \mathbb{C}$ is a **boundary point of S** if z_0 is not an interior point of S and z_0 is not an interior point of $\mathbb{C} - S$.

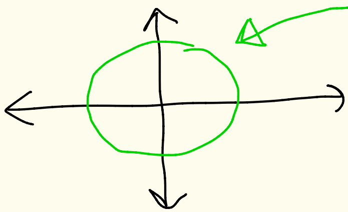
That is, a boundary point is a point where all of its neighborhoods contain points from S and $\mathbb{C} - S$.

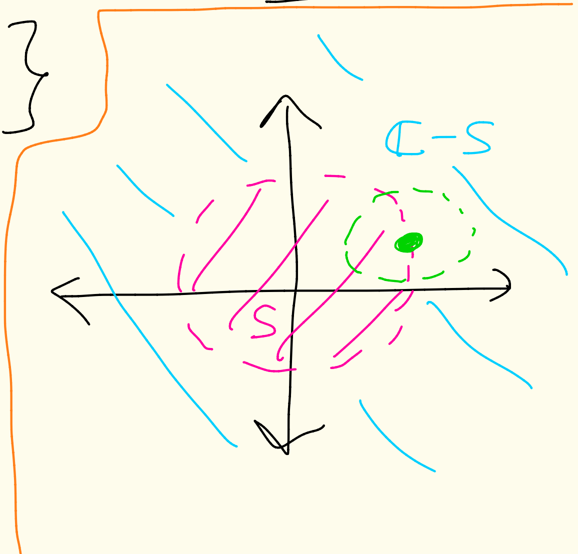
The **boundary of S** consists of all boundary points of S.

Ex: The boundary of

$$S = \{ z \mid |z| < 1 \}$$

is $\{ z \mid |z| = 1 \}$



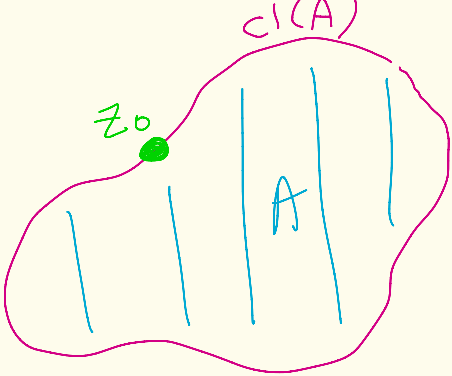


Def: Let $S \subseteq \mathbb{C}$, $S \neq \emptyset$. The closure of S is

$$cl(S) = S \cup (\text{boundary of } S).$$

Theorem (Max-Modulus Thm)

Let A be an open, connected, bounded set in $\mathbb{C}$. Suppose $f : cl(A) \to \mathbb{C}$ is analytic on A and continuous on $cl(A)$. Then $|f(z)|$ has a maximum value which lies on the boundary of A. That is, $\exists z_0$ on the boundary of A where $|f(z)| \leq |f(z_0)|$ $\forall z \in cl(A)$. If $|f(z_1)| = |f(z_0)|$ where z_1 is in the interior of A, then f is constant on $cl(A)$.



Lemma (for proof on pg 24)

Let $w(t)$ be a continuous complex-valued function defined on an interval $a \le t \le b$, then

$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt$$

proof: If $\left| \int_a^b w(t)\,dt \right| = 0$ then $\int_a^b w(t)\,dt = 0$. Since $w(t)$ is continuous, one can show this implies $w(t) = 0$ for all $a \le t \le b$. This gives $\int_a^b |w(t)|\,dt = 0$. And the theorem is proved.

Suppose $\left| \int_a^b w(t)\,dt \right| \ne 0$.

Then $\int_a^b w(t)\,dt = r_0 e^{i\theta_0}$ where $r_0 \ne 0$.

Thus, $r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt$

So, $r_0 = \mathrm{Re}(r_0) = \mathrm{Re}\left(\int_a^b e^{-i\theta_0} w(t)\,dt \right)$

$$= \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t) \right) dt$$

Note that

$$\mathrm{Re}\left(e^{-i\theta_0}w(t)\right) \le \left|e^{-i\theta_0}w(t)\right| = |w(t)|$$

$|e^{-i\theta_0}| = 1$

Thus,

$$\left|\int_a^b w(t)\,dt\right| = \left|r_0 e^{i\theta_0}\right| = r_0 =$$

$$= \int_a^b \mathrm{Re}\left(e^{-i\theta_0}w(t)\right)dt \le \int_a^b |w(t)|\,dt.$$

$\blacksquare$