TOPIC 1

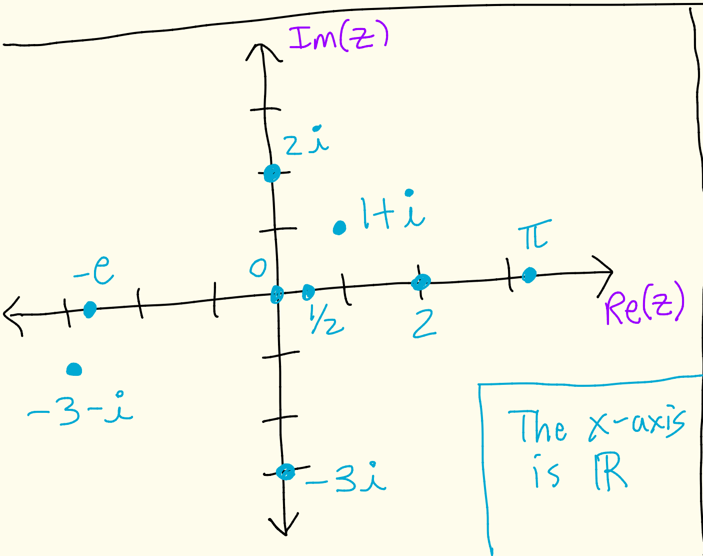
Complex Numbers

Complex Numbers

Def: We define the number i to be a root of the equation $x^2+1=0$. That is, $i^2=-1$.

The set of complex numbers, denoted by $\mathbb{C}$, is defined to be

$$\mathbb{C}=\left\{ x+iy \;\middle|\; x,y\in\mathbb{R} \right\}$$



Given $z=x+iy$ we call x the real part of z and y the imaginary part of z.

We write

$Re(z)=x$

$Im(z)=y$

Adding and multiplying in $\mathbb{C}$ is defined by

$$(x+iy)+(a+ib) = (x+a)+i(y+b)$$

and

$$\begin{aligned}(x+iy)(a+ib) &= xa + xib + iya + \overset{-1}{i^2}yb \\ &= (xa-yb)+i(xb+ya)\end{aligned}$$

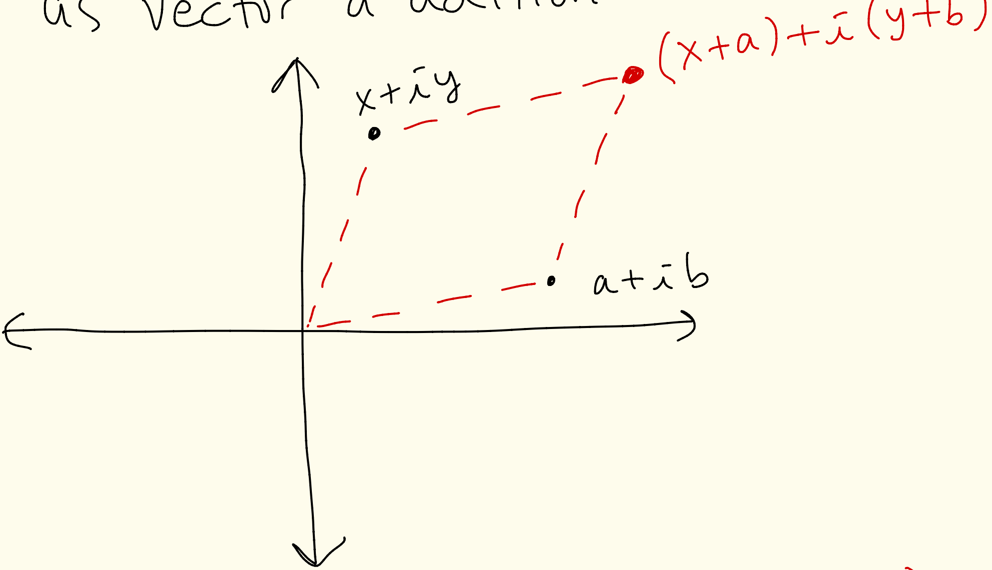
$i^2=-1$

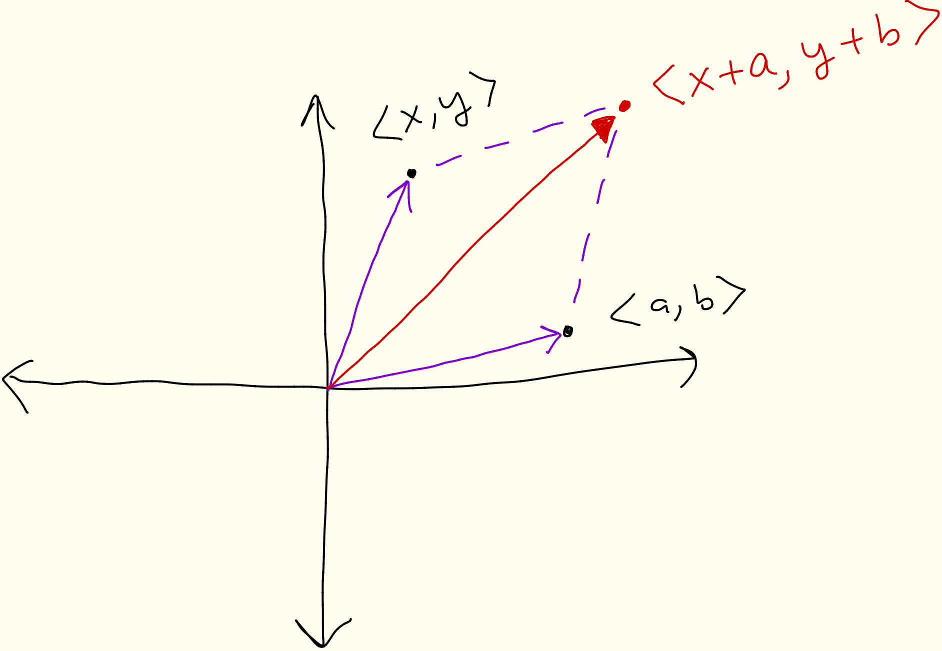
<u>Ex:</u> $\left(\frac{1}{2}-i\right)+(2+10i) = \frac{5}{2}+9i$

$$\begin{aligned}(2-i)(1+i) &= 2+2i-i-i^2 \\ &= 2+i+1 = 3+i\end{aligned}$$

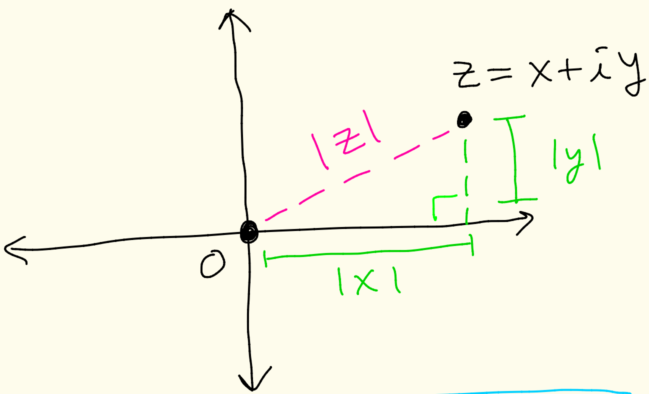
$i^2=-1$

One may think of addition in $\mathbb{C}$ as vector addition.





Def: Let $z = x + iy$ be a complex number. The **norm** or **absolute value** of z is the distance between 0 and z. The norm is denoted by $|z|$.



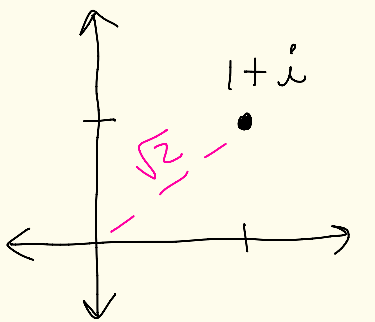
$$|x|^2 + |y|^2 = |z|^2$$

$x, y \in \mathbb{R}$
$|x| = \pm x$
$|y| = \pm y$

$$x^2 + y^2 = |z|^2$$

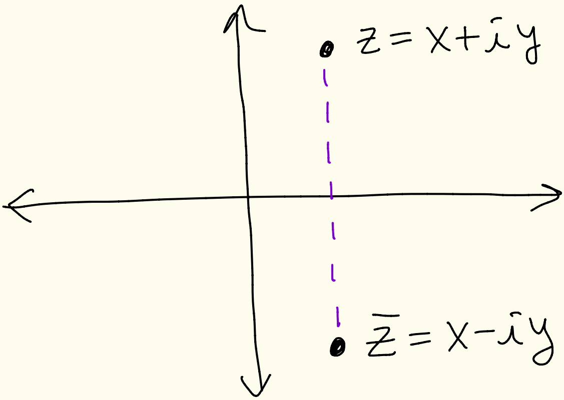
So, $\boxed{|z| = \sqrt{x^2 + y^2}}$

Ex: $|1+i| = \sqrt{1^2 + 1^2}$
$= \sqrt{2}$



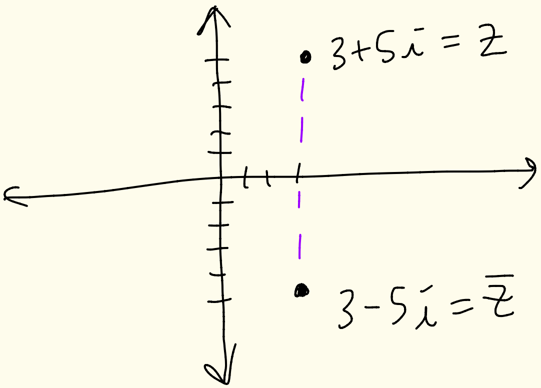
Def: Let $z = x + iy$ be a complex number. The <u>conjugate of z</u>, denoted by $\bar{z}$, is defined to be

$$\bar{z} = x - iy.$$

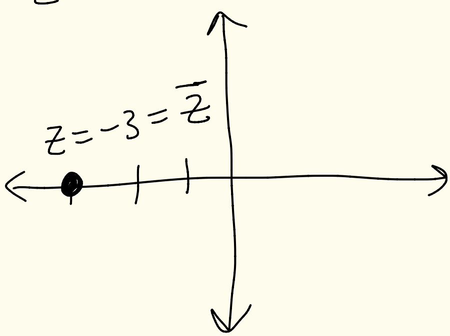


$w = 3 - 5i$
$\bar{w} = 3 + 5i$

Ex: $z = 3 + 5i$
$\bar{z} = 3 - 5i$



Ex:
$z = -3 = -3 + 0i$
$\bar{z} = -3$



Division in $\mathbb{C}$

To simplify $\frac{z}{w}$ (where $z, w \in \mathbb{C}$, $w \neq 0$)

into to form $a+bi$ then multiply by $\frac{\overline{w}}{\overline{w}}$.

Idea: $w = x+iy$

$w\overline{w} = (x+iy)(x-iy)$

$= x^2 + y^2$

which is a real #

Ex:

$$\frac{2+i}{3-2i} = \left(\frac{2+i}{3-2i}\right)\left(\frac{3+2i}{3+2i}\right)$$

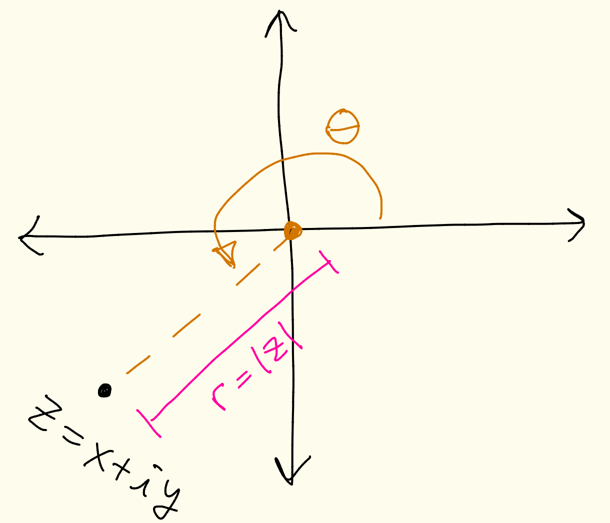
$$= \frac{6+4i+3i+2i^2}{9+6i-6i-4i^2}$$

$$= \frac{4+7i}{13}$$

$i^2 = -1$

$$= \frac{4}{13} + \frac{7}{13}i$$

Polar form of a complex number



Let $r = |z|$.

Consider the ray that starts at 0 and ends at z.

Let θ be the angle that this ray makes with positive x-axis.

If $z = x + iy$, then by trig

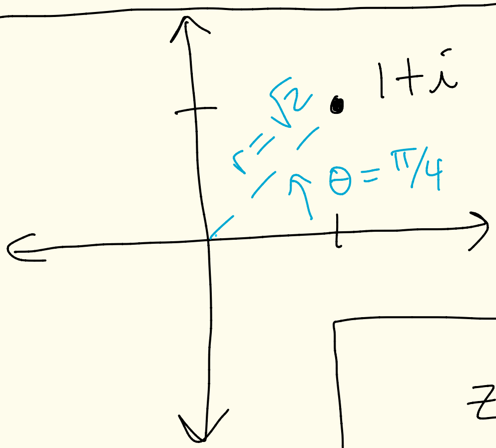
$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

So,
$$z = r\cos(\theta) + i\,r\sin(\theta)$$
$$= r\left[\cos(\theta) + i\sin(\theta)\right]$$

This is called the <u>polar form of z</u>.

θ is called an <u>argument of z</u> and we write $\theta = \arg(z)$.

Ex: $z = 1 + i$



$$r = |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \frac{\pi}{4}$$

$$z = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

verify: $\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

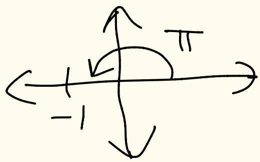
$$= \sqrt{2}\left[\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right] = 1+i$$

$$\arg(1+i) = \frac{\pi}{4} + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

$\arg(z)$ is a multi-valued function

Note that $\arg(z)$ is a multivalued function.
We can pick any $\boxed{2\pi\text{-range}}$ that makes it into a function.
This is called choosing a <u>branch of arg</u>.

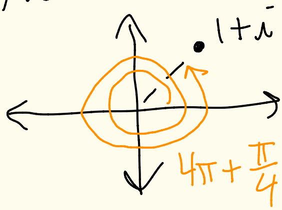
<u>Ex:</u> If we choose the branch of arg to be $[0, 2\pi)$, that is $0 \leq \arg(z) < 2\pi$, then

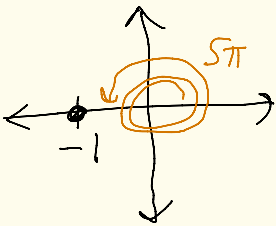


$\arg(1+i) = \frac{\pi}{4}$
$\arg(-1) = \pi$

<u>Ex:</u> If we choose the branch of arg to be $(3\pi, 5\pi]$, that is $3\pi < \arg(z) \leq 5\pi$,

$\arg(1+i) = 17\pi/4$
$\arg(-1) = 5\pi$





Proposition: Let $z, w \in \mathbb{C}$. Then: 10

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{zw} = \overline{z}\,\overline{w}$
3. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ if $w \neq 0$
4. $|z|^2 = z\overline{z}$ (or $|z| = \sqrt{z\overline{z}}$)
5. $z = \overline{z}$ iff z is real
6. $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$
7. $\overline{\overline{z}} = z$
8. $|zw| = |z||w|$
9. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ if $w \neq 0$
10. $|\overline{z}| = |z|$
11. $\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z|$
    $\mathrm{Im}(z) \leq |\mathrm{Im}(z)| \leq |z|$
12. $|z+w| \leq |z| + |w|$ (triangle inequality)
13. $|z+w| \geq \big||z|-|w|\big|$
14. $|z-w| \geq \big||z|-|w|\big|$

Proof: We will prove 11 – 14.

Proof of 11:

We have that

$$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| = \sqrt{(\mathrm{Re}(z))^2}$$

$\mathrm{Re}(z) \in \mathbb{R}$

$|x| = \sqrt{x^2}$

$$\leq \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}$$

$$= |z|$$

Similarly,

$$\mathrm{Im}(z) \leq |\mathrm{Im}(z)| = \sqrt{(\mathrm{Im}(z))^2} \leq \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}$$

$$= |z|$$

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proof of 12 $\left(|z+w| \le |z|+|w|\right)$ 12

We have that

$$
\begin{aligned}
|z+w|^2 &\overset{4}{=} (z+w)(\overline{z+w}) \\
&\overset{1}{=} (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&\overset{2/7}{=} z\overline{z} + \left(z\overline{w} + \overline{z\overline{w}}\right) + w\overline{w} \\
&\overset{6}{=} z\overline{z} + 2\,\mathrm{Re}(z\overline{w}) + w\overline{w} \\
&\overset{4}{=} |z|^2 + 2\,\mathrm{Re}(z\overline{w}) + |w|^2 \\
&\overset{11}{\le} |z|^2 + 2|z\overline{w}| + |w|^2 \\
&\overset{8}{=} |z|^2 + 2|z||\overline{w}| + |w|^2 \\
&\overset{10}{=} |z|^2 + 2|z||w| + |w|^2 \\
&= \left(|z|+|w|\right)^2.
\end{aligned}
$$

So, $|z+w|^2 \le \left(|z|+|w|\right)^2$.

Thus, $|z+w| \le |z|+|w|$.

Proof of 13 $\left(|z+w| \geq \big| |z|-|w| \big| \right)$ 13

If $a,b \in \mathbb{C}$, then

$$|a| = |a+b-b| \overset{12}{\leq} |a+b| + |-b| = |a+b| + |b|$$

So, $\boxed{|a|-|b| \leq |a+b|}$ $(*)$

Now back to the proof. Let $z,w \in \mathbb{C}$.

case i: Suppose $|z| \geq |w|$.

Then $|z|-|w| \geq 0$.

So, $\big| |z|-|w| \big| = |z|-|w|$.

since $|z|-|w| \geq 0$

In $(*)$ set $a=z$ and $b=w$.

We get $|z|-|w| \leq |z+w|$.

Now use $\big| |z|-|w| \big| = |z|-|w|$

to get

$$\big| |z|-|w| \big| \leq |z+w|.$$

<u>case ii</u> Suppose $|w| > |z|$.

Then $|w| - |z| > 0$.

So, $$\begin{aligned} \big||z| - |w|\big| &= \big|-(|z| - |w|)\big| \\ &= \big|\underbrace{|w| - |z|}_{>0}\big| \\ &= |w| - |z|. \end{aligned}$$

Now set $a = w$ and $b = z$ in $(*)$ and get
$$|w| - |z| \leq |w + z|.$$

Combine to get
$$\begin{aligned} \big||z| - |w|\big| &= |w| - |z| \\ &\leq |w + z|. \end{aligned}$$

Last time we finished the proof of Prop part 13 which was $|z+w| \geq ||z|-|w||$.

The proof of 14 is

$$|z-w| = |z+(-w)|$$

$$\overset{13}{\geq} ||z|-|-w||$$

$$= ||z|-|w||.$$

So,

$$|z-w| \geq ||z|-|w||.$$

$$\boxed{\boxed{\text{Prop 11 - 14 done}}}$$

De Moivre's Formula

If $z = r[\cos(\theta) + i\sin(\theta)]$
and n is a positive integer
then $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$.

Thm: Let $w = r[\cos(\theta) + i\sin(\theta)]$
where $w \neq 0$. The solutions to

$$z^n = w$$

are given by

$$z_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

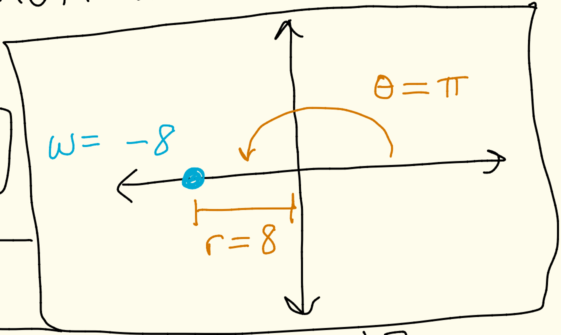
$k = 0, 1, 2, \ldots, n-1$

The proofs of these are in HW 1. It's optional if you want to do these proofs.

Ex: Find the solutions to $z^3 = -8$

$$w = -8$$
$$= 8[\cos(\pi) + i\sin(\pi)]$$

$$n = 3$$



$$z_k = 8^{1/3}\left[\cos\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right)\right]$$

$k = 0, 1, 2$

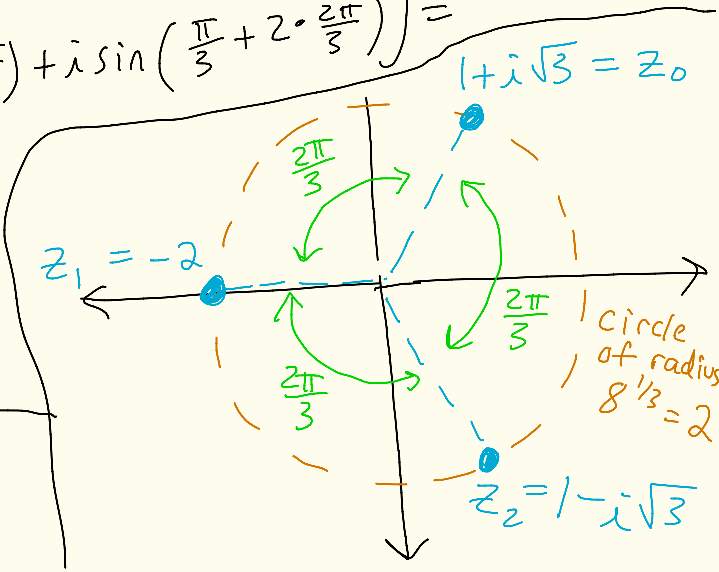
$$z_0 = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 1 + i\sqrt{3}$$

$$z_1 = 2\left[\cos\left(\underbrace{\frac{\pi}{3} + \frac{2\pi}{3}}_{\pi}\right) + i\sin\left(\underbrace{\frac{\pi}{3} + \frac{2\pi}{3}}_{\pi}\right)\right] = -2$$

$$z_2 = 2\left[\cos\left(\underbrace{\frac{\pi}{3} + 2\cdot\frac{2\pi}{3}}_{5\pi/3}\right) + i\sin\left(\frac{\pi}{3} + 2\cdot\frac{2\pi}{3}\right)\right] =$$

$$= 2\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

$$= 1 - i\sqrt{3}$$



Answers:

$z = -2,\ 1 \pm i\sqrt{3}$

Note:

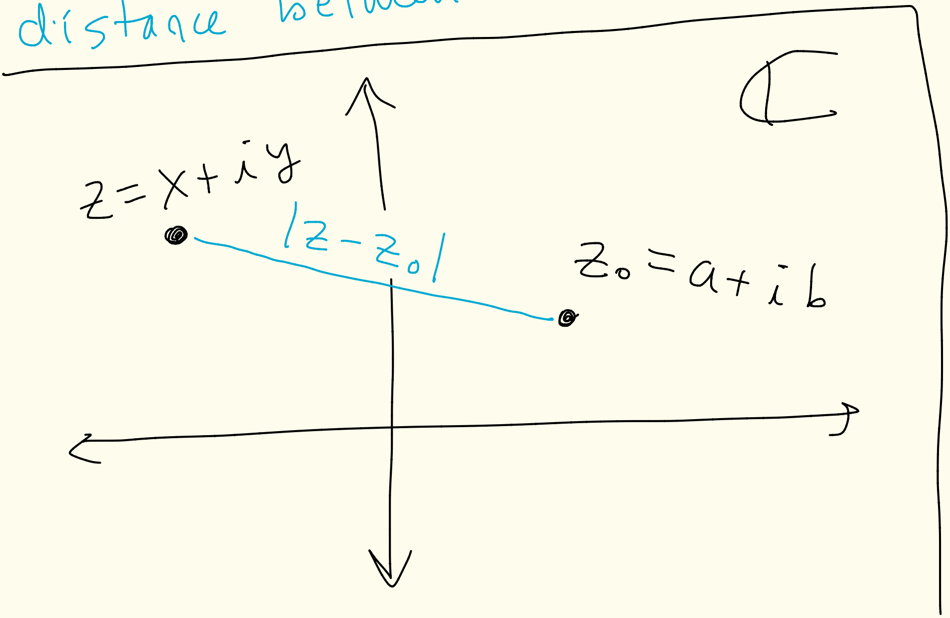
Let $z, z_0 \in \mathbb{C}$. Then $|z-z_0|$ is the distance between z & z_0.

Reason: $z = x+iy$, $z_0 = a+ib$

$$
\begin{aligned}
|z-z_0| &= |x+iy-a-ib| \\
&= |(x-a)+i(y-b)| \\
&= \sqrt{(x-a)^2+(y-b)^2}
\end{aligned}
$$

distance between (x,y) & (a,b)

So in $\mathbb{C}$, $|z-z_0|$ is the distance between $z=x+iy$ and $z_0=a+ib$



Do some HW problems.