# Topic 2 -

# Elementary Functions

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## The exponential function

We already have $e^x$ defined when $x$ is a real number. We want to extend this function to the complex numbers.

**Def:** Given $z = x + iy$, define

$$e^z = e^x\left[\cos(y) + i\sin(y)\right]$$

where $e^x$ is the usual real exponential function

**Ex:** $e^{2+\pi i} = e^2\left[\underbrace{\cos(\pi)}_{-1} + i\underbrace{\sin(\pi)}_{0}\right] = -e^2$

$e^{\pi i} = e^{0+\pi i} = \underbrace{e^0}_{1}\left[\cos(\pi) + i\sin(\pi)\right] = -1$

**Note:** If $z = x$ is real, then

$$e^{z} = e^{x+i0} = e^{x}\left[\underbrace{\cos(0)}_{1} + i\underbrace{\sin(0)}_{0}\right] = e^{x}$$

So our new exponential function agrees with the real exponential function on the real numbers. So we are extending the real valued exponential to all of $\mathbb{C}$.

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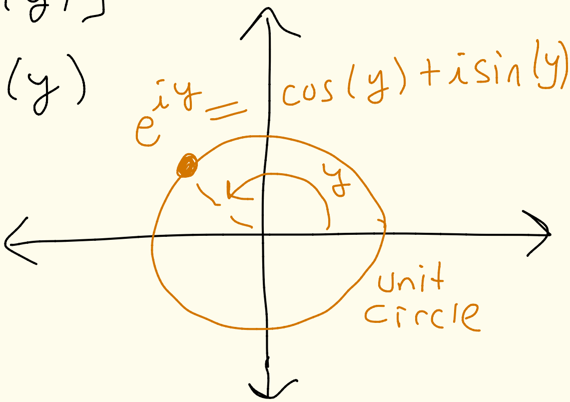
**Note:** Let $y \in \mathbb{R}$. Then

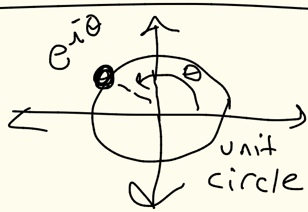
$$\begin{aligned} e^{iy} &= e^{0+iy} \\ &= e^{0}\left[\cos(y) + i\sin(y)\right] \\ &= \cos(y) + i\sin(y) \end{aligned}$$




So,

If $\theta \in \mathbb{R}$, then

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

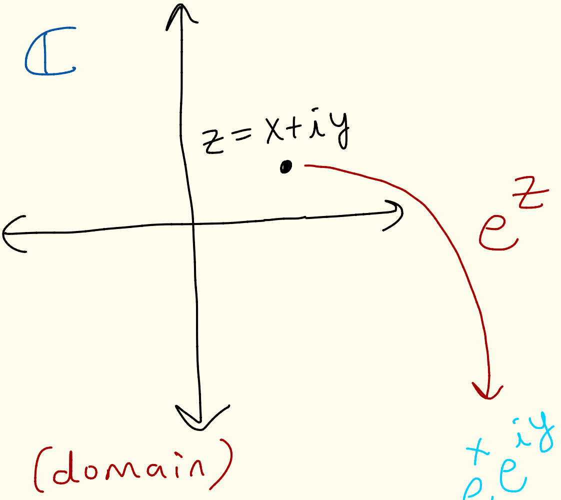


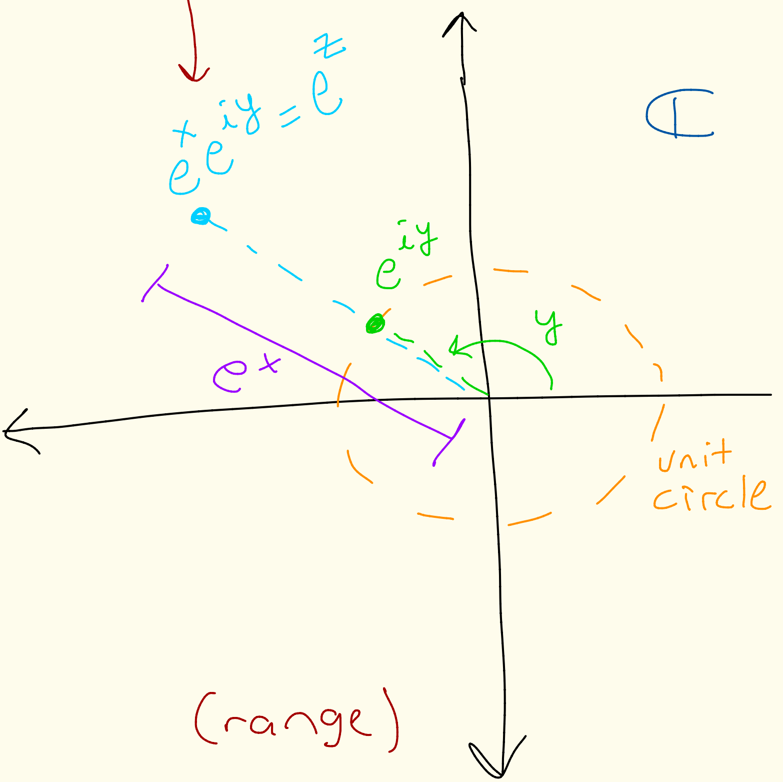
**Note:** $z = x + iy$

$$e^{z} = e^{x+iy} = e^{x}\Big[\underbrace{\cos(y) + i\sin(y)}_{e^{iy}}\Big]$$

$$\boxed{|e^{iy}| = 1}$$

its on the unit circle



Proposition:

1. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$
2. $|e^{x+iy}| = e^x$
3. $e^z \neq 0$ for all $z \in \mathbb{C}$
4. $e^z$ is $2\pi i$-periodic. That is, $e^{z+2\pi i n} = e^z$ for any integer $n$.

**Proof:**

1. Let $z, w \in \mathbb{C}$.
We want to show that $e^{z+w} = e^z e^w$.
Let $z = x+iy$ and $w = s+it$.
Then,

$$e^{z+w} = e^{(x+s)+i(y+t)}$$

$$= e^{x+s}\left[\cos(y+t) + i\sin(y+t)\right]$$

def of e

$$= e^x e^s \left[\cos(y)\cos(t) - \sin(y)\sin(t) + i\left(\sin(y)\cos(t) + \cos(y)\sin(t)\right)\right]$$

real-valued e satisfies $e^{x+s} = e^x e^s$ & trig formulas

$$= e^x\left[\cos(y) + i\sin(y)\right] e^s\left[\cos(t) + i\sin(t)\right]$$

$$= e^{x+iy} e^{s+it} = e^z e^w$$

2 Let $z = x+iy \in \mathbb{C}$.

Then,

$$|e^{x+iy}| \overset{1}{=} |e^x e^{iy}| = |e^x||e^{iy}|$$

$$= |e^x||\cos(y)+i\sin(y)|$$

$$= |e^x|\sqrt{(\cos(y))^2+(\sin(y))^2}$$

$\sin^2(y)+\cos^2(y)=1$
$y \in \mathbb{R}$

$$= |e^x| \cdot 1$$

$$= e^x$$

$e^x > 0$ for all $x \in \mathbb{R}$

---

3 Let $z = x+iy \in \mathbb{C}$.

Then $|e^z| \overset{2}{=} e^x \neq 0$.

So, $e^z \neq 0$. [Because if $e^z = 0$ then $|e^z| = |0| = 0$]

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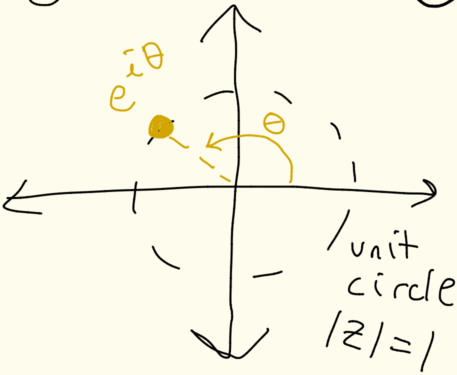
4 Let $z \in \mathbb{C}$ and $n$ be an integer.

Then $e^{z+i2\pi n} = e^z e^{i2\pi n} = e^z[\underbrace{\cos(2\pi n)}_{1} + i\underbrace{\sin(2\pi n)}_{0}]$

$$= e^z$$

■

**Note:** If $\theta \in \mathbb{R}$, then $e^{i\theta}$ is on the unit circle.




$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

We saw in the last proof this has absolute value 1.

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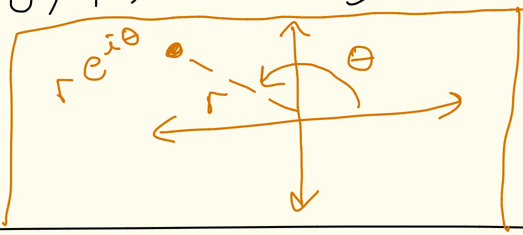
**Note:** (Polar form of $z$)

Suppose $z = r[\cos(\theta) + i\sin(\theta)]$.

Then, $z = re^{i\theta}$

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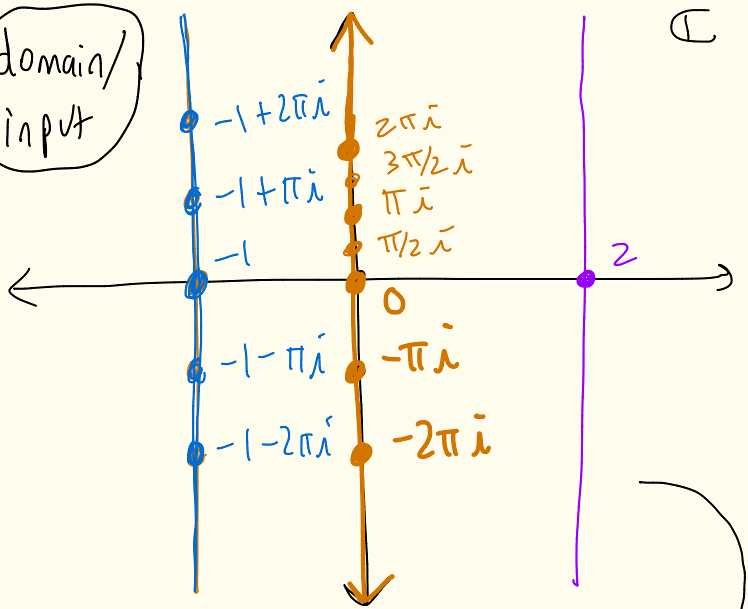
**Note:** One can show that $f(z) = e^z$ (as we defined it) is the unique function such that

1. $f(x) = e^x \quad \forall x \in \mathbb{R}$ ← here $e^x$ is the real valued $e^x$
2. $f$ is differentiable for all $z$
3. $f'(z) = f(z)$ for all $z$

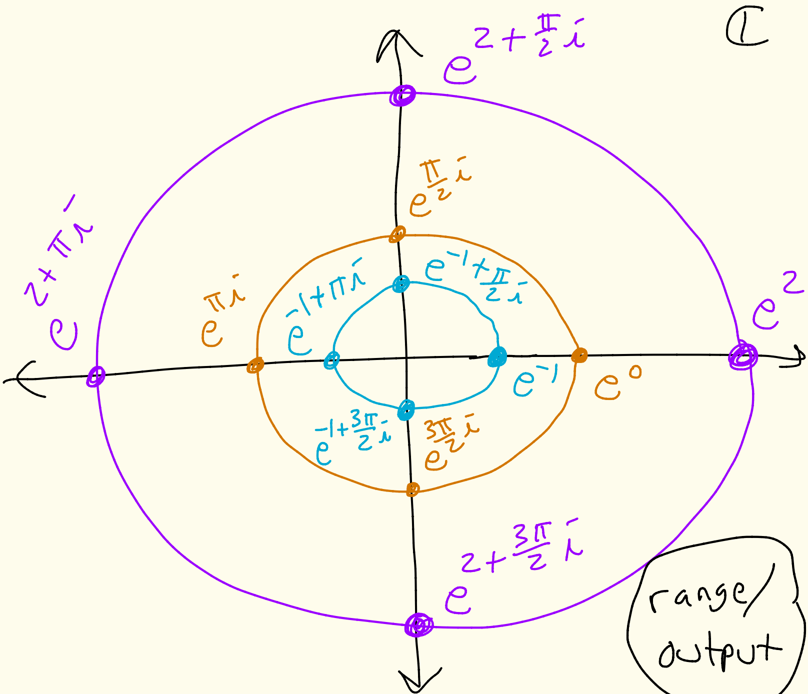
} We define derivate later

domain/input

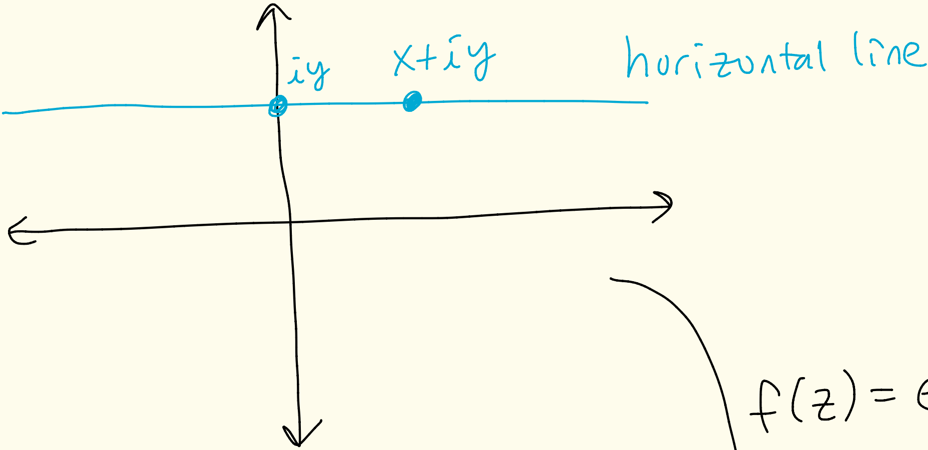



$$f(z) = e^z$$

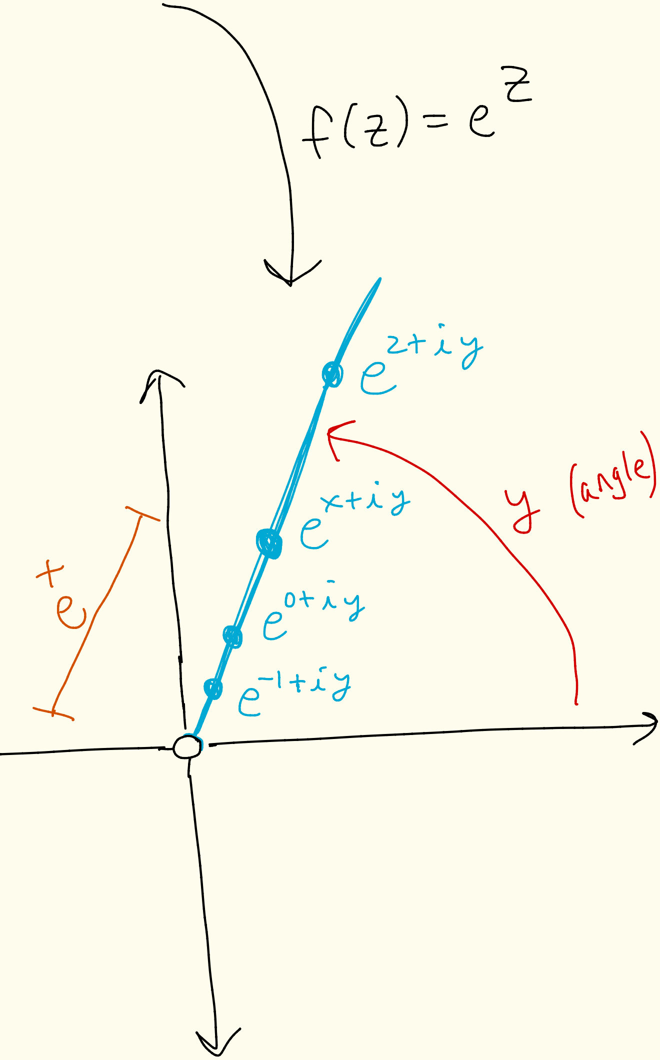
$$e^{-1+i\theta} = \underbrace{e^{-1}}_{\text{distance from } 0}\ \underbrace{e^{i\theta}}_{\text{angle}}$$




range/output



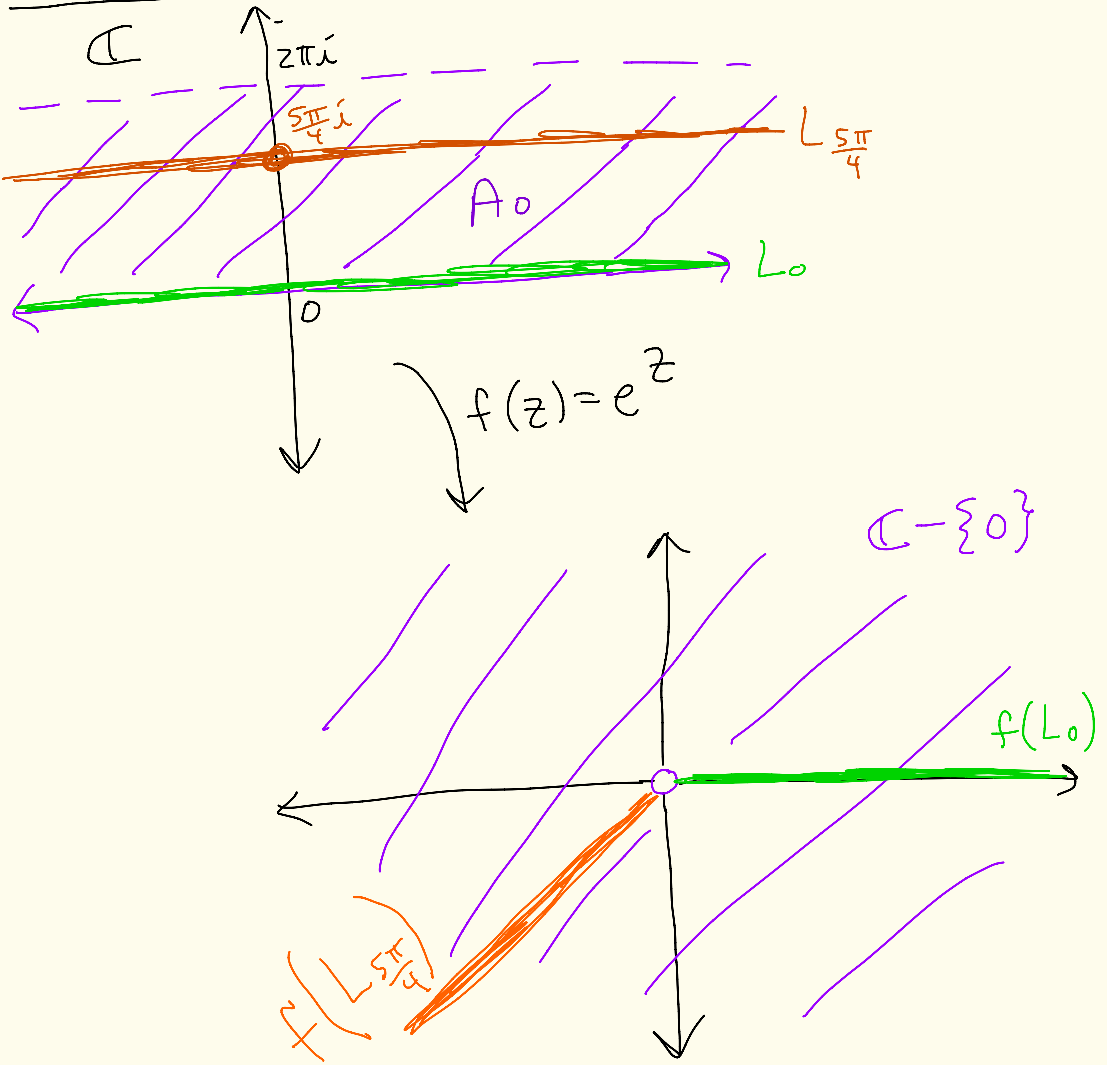

$$\boxed{e^{x+iy} = \underbrace{e^{x}}_{\text{distance from } 0}\,\underbrace{e^{iy}}_{\text{angle } y}}$$



Thus, $f(z) = e^z$ maps the set

$$A_0 = \{ x+iy \mid x \in \mathbb{R},\ 0 \leq y < 2\pi \}$$

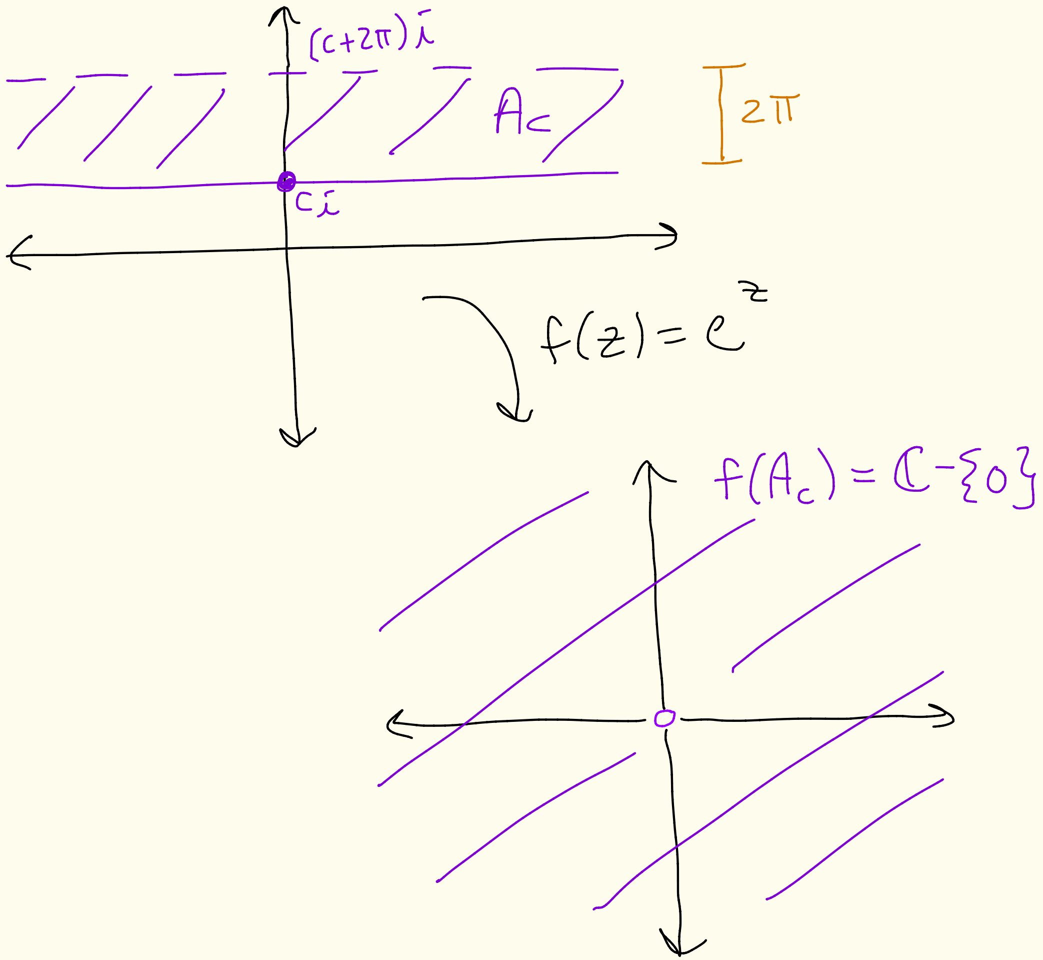
onto $\mathbb{C} - \{0\}$ in a 1-1 and onto way.



In general, if $c \in \mathbb{R}$, then (11)

$$A_c = \{ x + iy \mid x \in \mathbb{R} \text{ and } c \leq y < c + 2\pi \}$$

is mapped by $f(z) = e^z$ to $\mathbb{C} - \{0\}$ in a 1-1 and onto way.

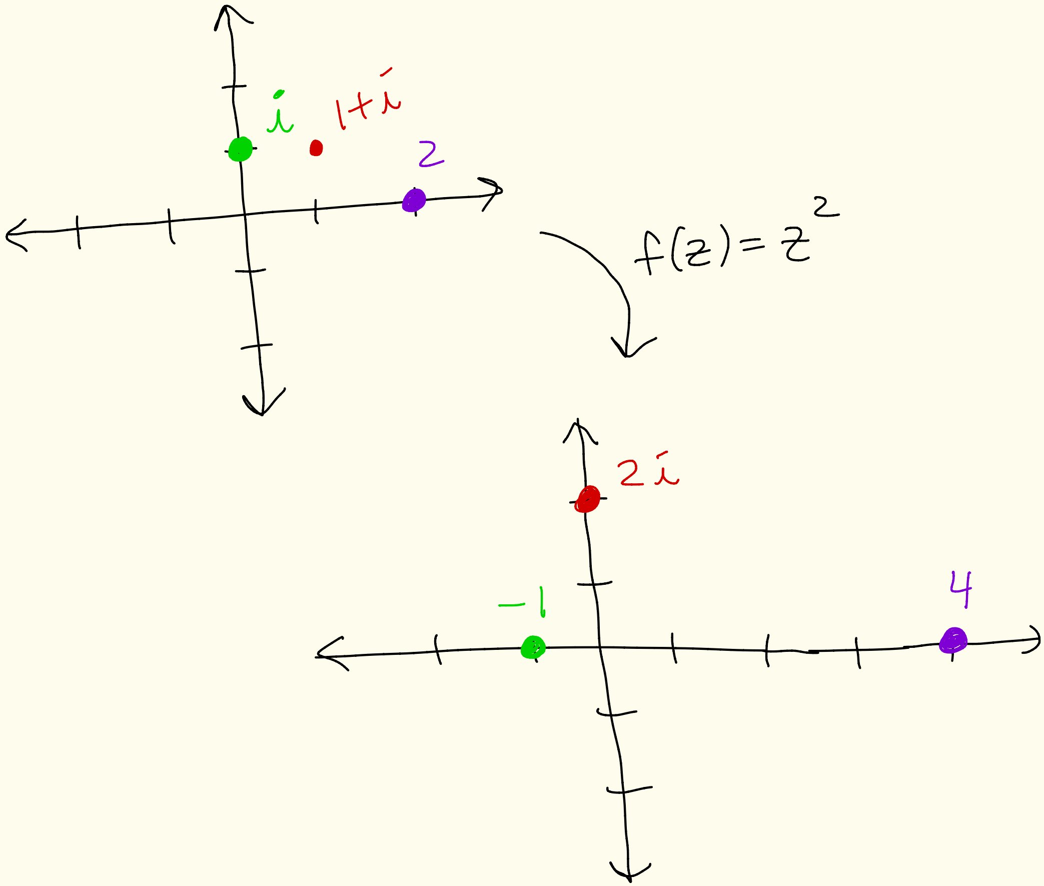


# Square function

Let $f:\mathbb{C}\to\mathbb{C}$ where $f(z)=z^2$

$f(2)=2^2=4 \qquad f(i)=i^2=-1$

$f(1+i)=(1+i)^2=1+2i+i^2=2i$



Let $z = re^{i\theta}$

Then, $f(z) = f(re^{i\theta}) = (re^{i\theta})^2$

$= r^2 e^{i2\theta}$

$f(z) = z^2$ squares the distance from the origin and doubles the angle.

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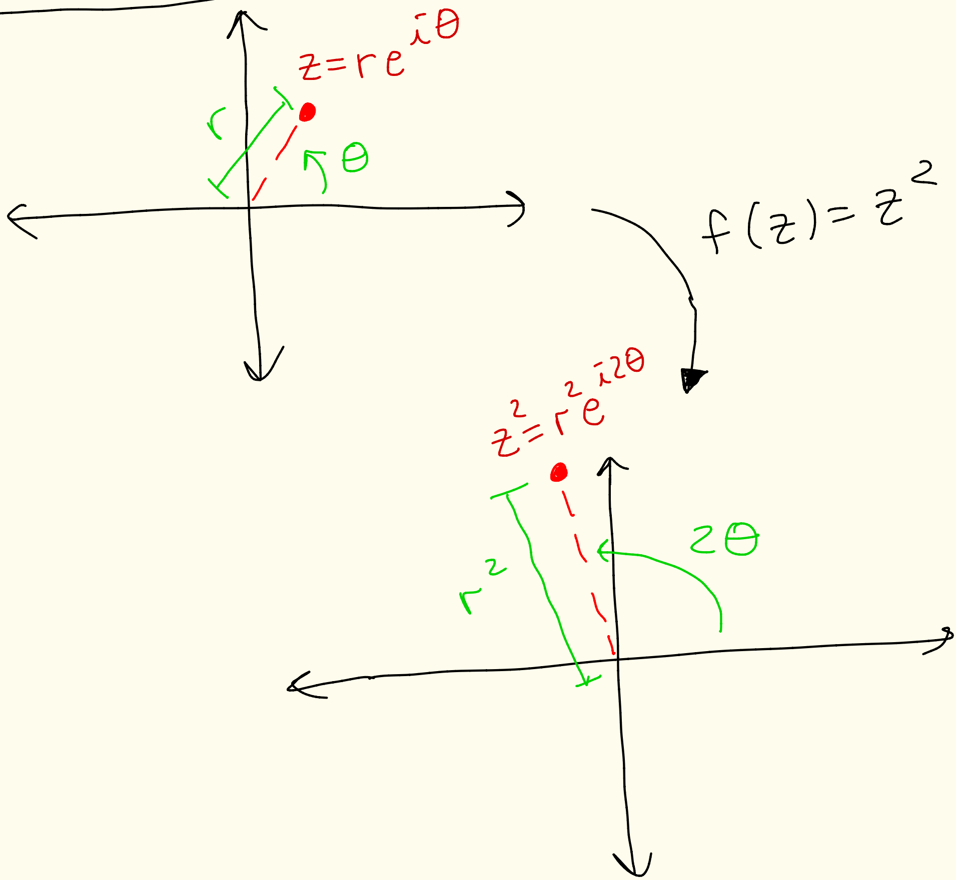
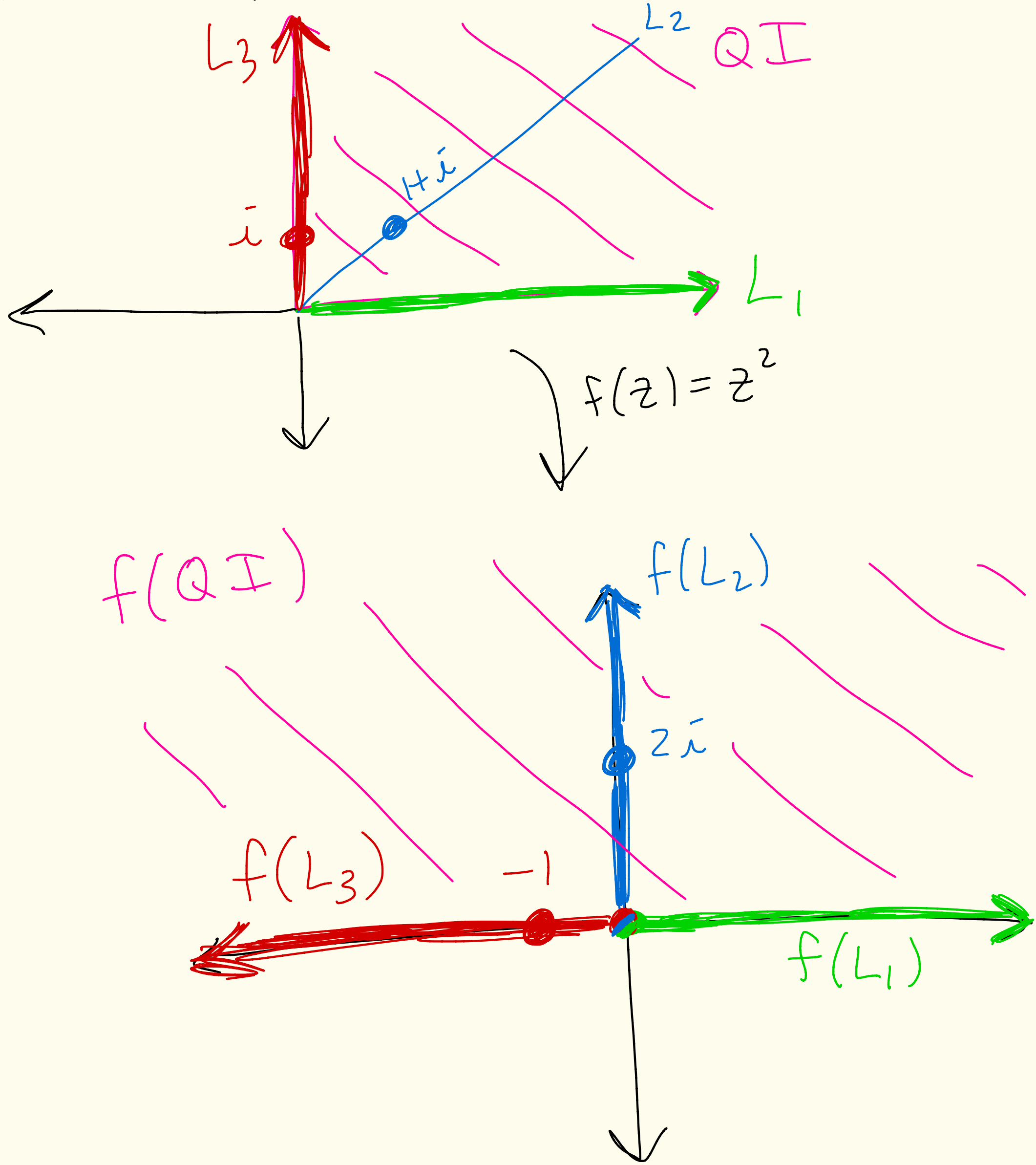


What does $f(z)=z^2$ map the 1st quadrant onto?



# Trig functions

We have functions $\sin(\theta)$ and $\cos(\theta)$ when $\theta \in \mathbb{R}$. Can we extend these functions to the complex plane?

Let $\theta \in \mathbb{R}$.

Then,

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)} \quad (1)$$

and

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$\boxed{e^{-i\theta} = \cos(\theta) - i\sin(\theta)} \quad (2)$$

Adding (1) and (2) and solving gives

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Computing (1) $-$ (2) and solving gives

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

This gives us a natural way to define sine and cosine for all of $\mathbb{C}$.

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<u>Def:</u> Given $z \in \mathbb{C}$ define

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$ and

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

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As we saw on the previous page if $z$ is real these definitions agree with the usual $\sin(z)$ & $\cos(z)$.

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So we have extended sine and cosine to the complex plane.

$\boxed{\text{Ex:}}$

$$\sin(\pi+i) = \frac{e^{i(\pi+i)} - e^{-i(\pi+i)}}{2i}$$

$$= \frac{e^{-1+\pi i} - e^{1-\pi i}}{2i}$$

$$= \frac{e^{-1}\left[\overbrace{\cos(\pi)}^{-1} + i\overbrace{\sin(\pi)}^{0}\right] - e^{1}\left[\overbrace{\cos(-\pi)}^{-1} + i\overbrace{\sin(-\pi)}^{0}\right]}{2i}$$

$$= \frac{-e^{-1} + e}{2i} = \frac{e - \frac{1}{e}}{2i}$$

$$= \frac{\left(e - \frac{1}{e}\right)}{2i} \cdot \frac{-i}{-i} = \frac{-i\left(e - \frac{1}{e}\right)}{2} = \frac{-i}{2}\left(e - \frac{1}{e}\right)$$

Thm: For all $z, w \in \mathbb{C}$ we have that:

1. $\sin(-z) = -\sin(z)$
2. $\cos(-z) = \cos(z)$
3. $\sin^2(z) + \cos^2(z) = 1$
4. $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
5. $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

pf: 1/2/3 are HW.
4/5 use def and algebra.

4 Note that

$$\sin(z+w) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i}$$

And
$\sin(z)\cos(w) + \cos(z)\sin(w) =$

$$= \underbrace{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)}_{\sin(z)} \underbrace{\left(\frac{e^{iw} + e^{-iw}}{2}\right)}_{\cos(w)} + \underbrace{\left(\frac{e^{iz} + e^{-iz}}{2}\right)}_{\cos(z)} \underbrace{\left(\frac{e^{iw} - e^{-iw}}{2i}\right)}_{\sin(w)}$$

$$= \frac{e^{i(z+w)} + \cancel{e^{i(z-w)}} - \cancel{e^{i(w-z)}} - e^{i(-w-z)}}{4i}$$

$$+ \frac{e^{i(z+w)} - \cancel{e^{i(z-w)}} + \cancel{e^{i(w-z)}} - e^{i(-w-z)}}{4i}$$

$$= \frac{2e^{i(z+w)} - 2e^{-i(z+w)}}{4i}$$

$$= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z+w)$$

$$\blacksquare$$

# Logarithm

The natural logarithm in real analysis is the inverse function of $e^x$. Can we do this in complex analysis?

Suppose

$$e^w = z \qquad (*)$$

where $z \neq 0$.

We want to solve for $w$ and define $\log(z) = w$.

Let $z = re^{i\theta}$ and $w = x + iy$.

Then $(*)$ becomes

$$e^x e^{iy} = re^{i\theta}$$

So, $e^x = r$ and $e^{iy} = e^{i\theta}$.

We have

$$r = e^x \quad \text{and} \quad e^{iy} = e^{i\theta}.$$

Thus, $x = \ln(r) = \ln|z|$

And, $y = \theta + 2\pi k$, where $k = 0, \pm 1, \pm 2, \ldots$

Thus

$$\begin{aligned} w &= x + iy \\ &= \ln(r) + i(\theta + 2\pi k) \\ &= \ln|z| + i\arg(z) \end{aligned}$$

where $\arg(z)$ can be any of the values $\theta + 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$

So, we could define

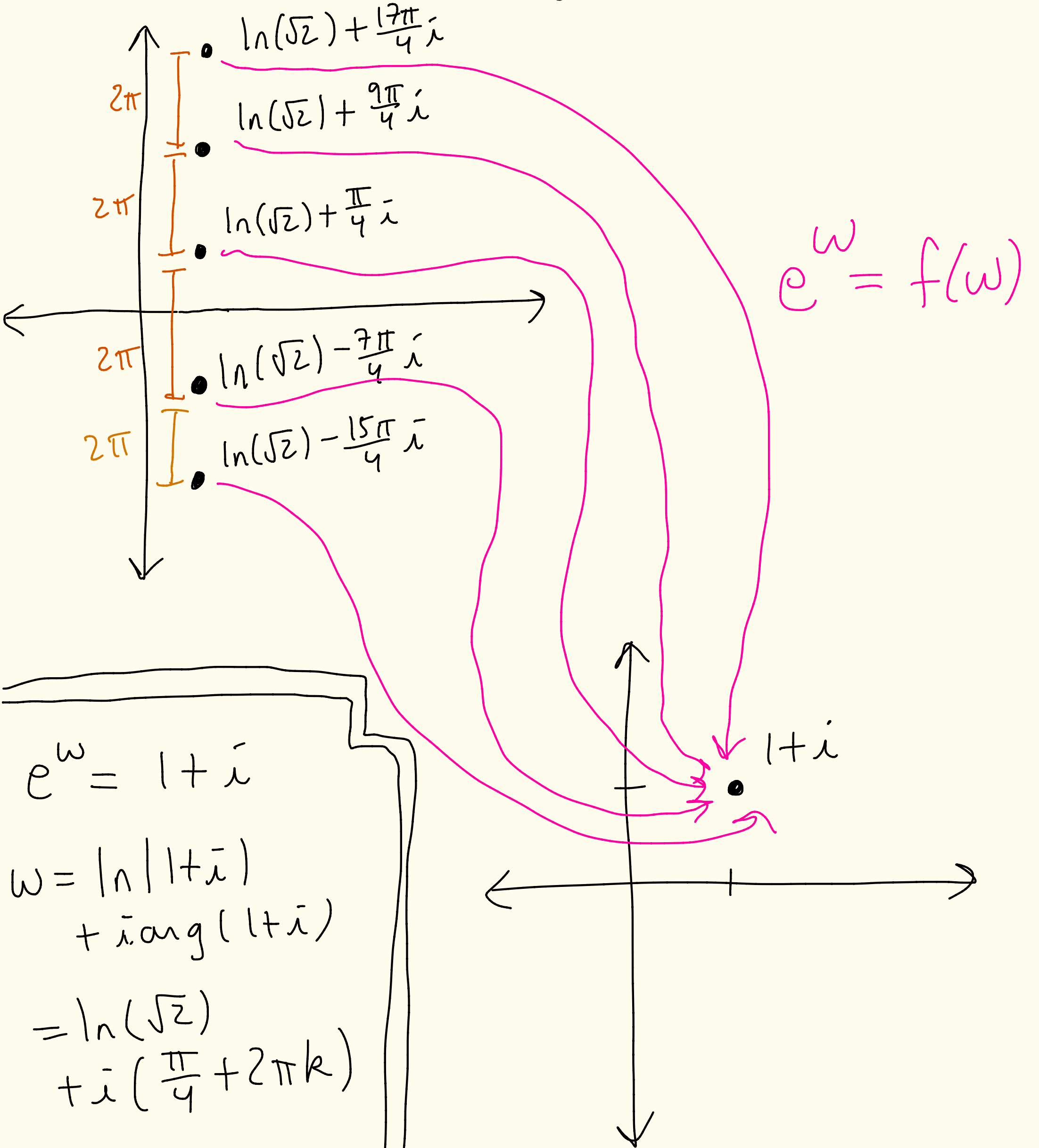
$$\log(z) = \ln|z| + i\arg(z)$$

where $\arg(z)$ can be any of the values $\theta + 2\pi k$ $k = 0, \pm 1, \pm 2, \ldots$

The issue here is this isn't a function since it has an infinite # of outputs.

picture of what we just did

Let's try to define $\log(1+i)$




$e^w = 1+i$

$w = \ln|1+i|$
$\quad + i\arg(1+i)$

$= \ln(\sqrt{2})$
$\quad + i\left(\frac{\pi}{4}+2\pi k\right)$

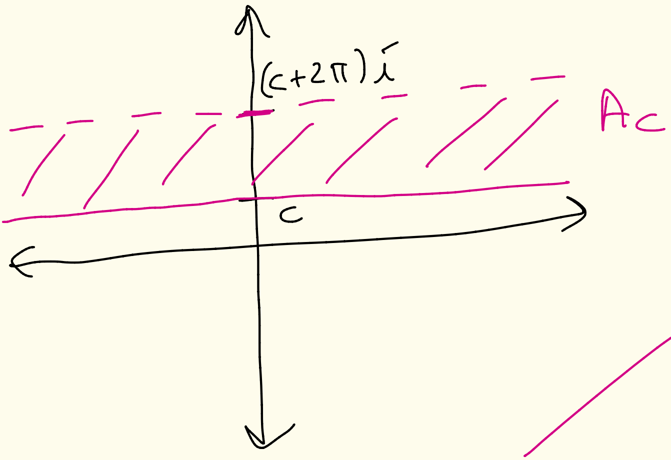
We need to find a domain where $e^z$ is 1-1 and then on that domain we can find its inverse.

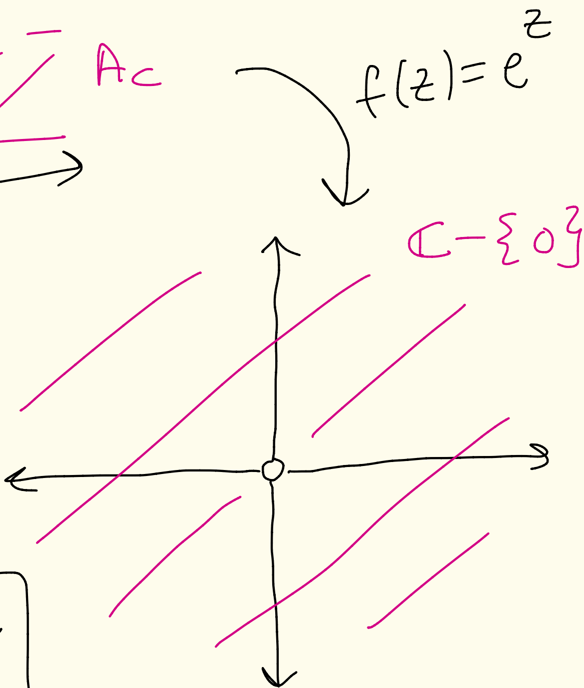
---

<u>Recall:</u> Let $c \in \mathbb{R}$. Define

$$A_c = \{ x+iy \mid x \in \mathbb{R},\ c \leq y < c+2\pi \}$$

Then $f(z) = e^z$ is 1-1 on $A_c$ and maps $A_c$ onto $\mathbb{C} - \{0\}$.




$$\boxed{\begin{aligned}&\mathbb{C} - \{0\} \\ &= \{ z \mid z \in \mathbb{C},\ z \neq 0 \}\end{aligned}}$$

<u>Def:</u> Let $c \in \mathbb{R}$.

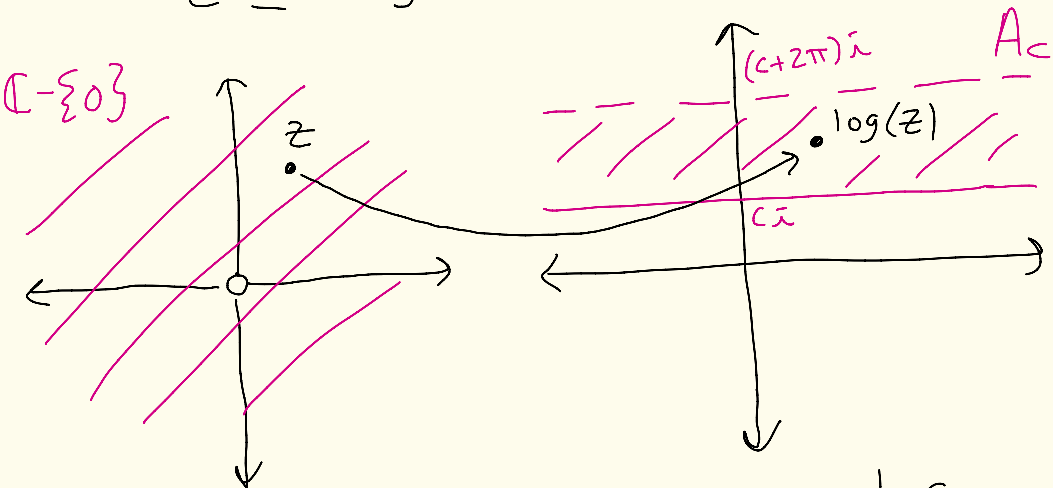
Define the <u>logarithm function</u>,

$\log: \mathbb{C} - \{0\} \to \mathbb{C}$ by

$$\log(z) = \ln|z| + i\arg(z)$$

where we choose $\arg(z)$ to satisfy

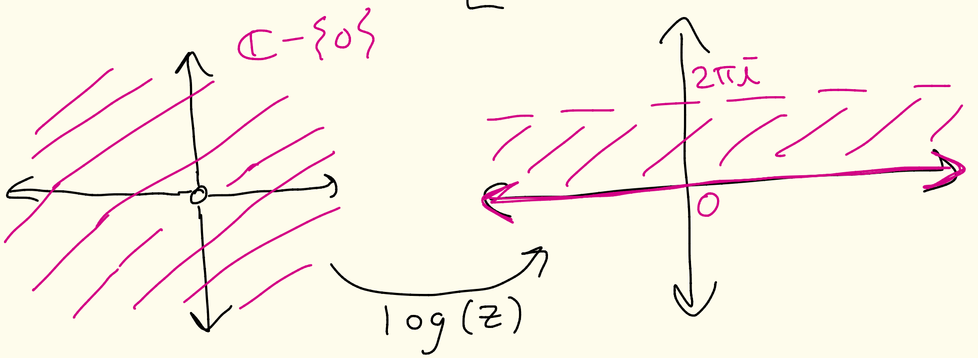
$$c \leq \arg(z) < c + 2\pi.$$




Note that the range of this log function is $A_c$.

This is called picking a <u>branch of the logarithm function</u>.

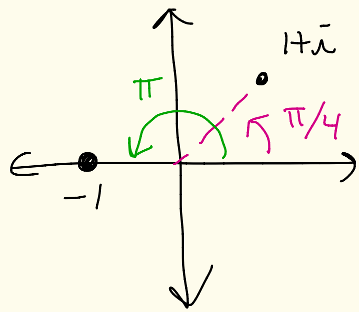
Ex: Pick $[0, 2\pi)$ as the branch of log $\left[\begin{array}{l} \text{ie } 0 \le \arg(z) < 2\pi \\ c = 0 \end{array}\right]$




Using this branch, calculate the following:

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln(\sqrt{2}) + i\,\pi/4$$

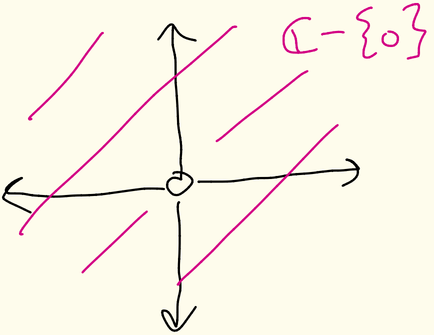
$$\log(-1) = \underbrace{\ln|-1|}_{\ln(1) = 0} + i\arg(-1) = 0 + \pi i = \pi i$$

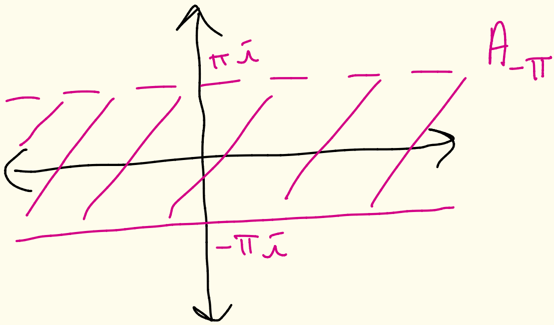



$$\boxed{|a+ib| = \sqrt{a^2 + b^2}}$$

Ex: Let's pick the branch of log corresponding to $[-\pi, \pi)$

[ie $-\pi \leq \arg(z) < \pi$, $c = -\pi$]

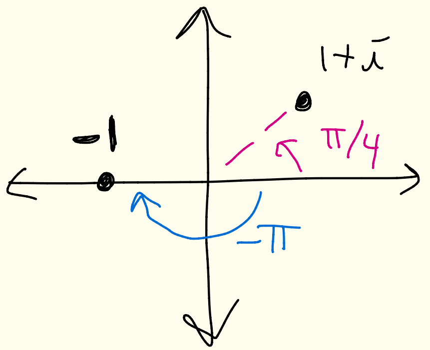


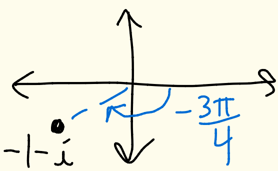




$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln(\sqrt{2}) + i\frac{\pi}{4}$$

$$\log(-1) = \ln|-1| + i\arg(-1) = 0 + i(-\pi)$$
$$= -\pi i$$

$$\log(-1-i) = \ln|-1-i| + i\left(-\frac{3\pi}{4}\right)$$
$$= \ln(\sqrt{2}) - i\frac{3\pi}{4}$$

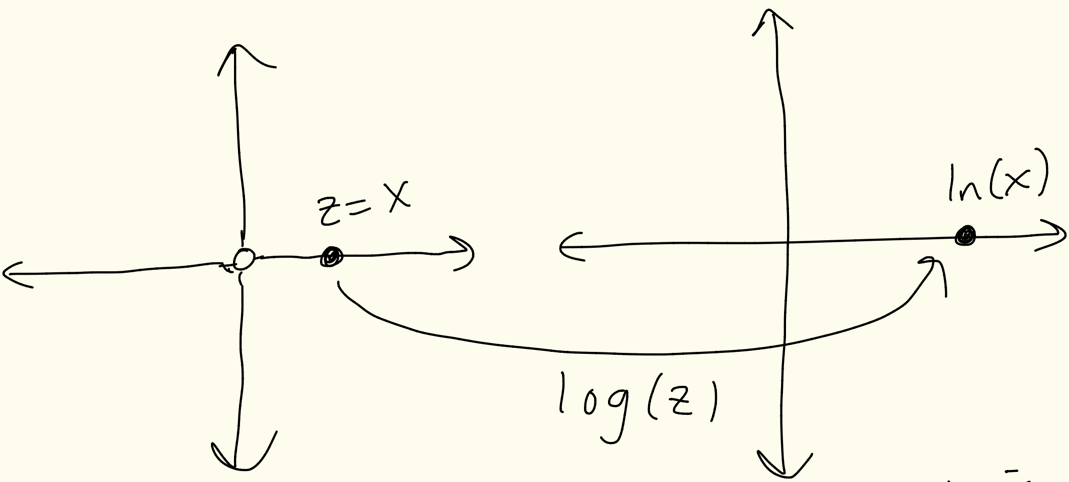


Note: If we choose a branch of the log that contains $0$ as an angle such as the branches $[0, 2\pi)$ or $[-\pi, \pi)$ then if $z = x + i0$ where $x > 0$ [ie $z$ is a positive real number]

we have

$$\log(z) = \ln|x| + i\arg(x)$$
$$= \ln(x) + i0$$
$$= \ln(x)$$




So our new log with such a branch is extending the old $\ln(x)$ to all of $\mathbb{C}$.

# Complex powers

**Motivation:** Let $a, b \in \mathbb{R}$ with $a > 0$. Then in real analysis we have

$$a^b = e^{\ln(a^b)} = e^{b\ln(a)}$$

For example, $2^3 = e^{\ln(2^3)} = e^{3\ln(2)}$.

---

**Def:** Let $a, b \in \mathbb{C}$ with $a \neq 0$. Define

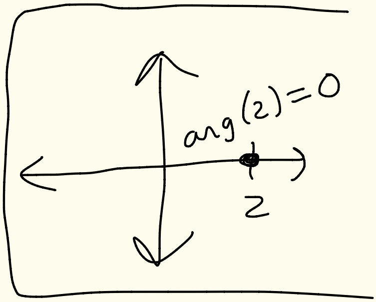
$$a^b = e^{b \log(a)}$$

where $\log$ is some branch of the logarithm function.

<u>Ex:</u> Choose the branch of $\log$ to be $[0, 2\pi)$.

Then

$$
\begin{aligned}
2^3 &= e^{3\log(2)} \\
&= e^{3[\ln|2| + i\arg(2)]} \\
&= e^{3[\ln(2) + i\,0]} \\
&= e^{3\ln(2)} \\
&= \underbrace{e^{\ln(2^3)} = 2^3 = 8}_{\text{real \# calculations}}
\end{aligned}
$$

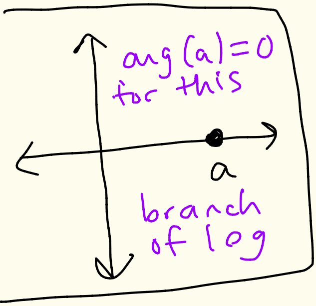


<u>Note:</u> Suppose we pick a branch of the logarithm such that $0$ is contained in the range of $\arg(z)$, such as $0 \le \arg(z) < 2\pi$ or $-\pi \le \arg(z) < \pi$. Then if $a, b \in \mathbb{R}$ and $a > 0$ then

$$\underbrace{a^b = e^{b\log(a)}}_{\text{complex analysis def of } a^b} = e^{b[\ln(a) + i\arg(a)]}$$

$$= e^{b[\ln(a) + i0]}$$

$$= \underbrace{e^{b\ln(a)} = e^{\ln(a^b)} = a^b}_{\text{real analysis version of } a^b}$$




Thus in this case our new definition of $a^b$ agrees with the real analysis def of $a^b$.

**Ex:** Let's calculate $(-1)^{1/2}$

For now let's wait to choose our branch of the logarithm.

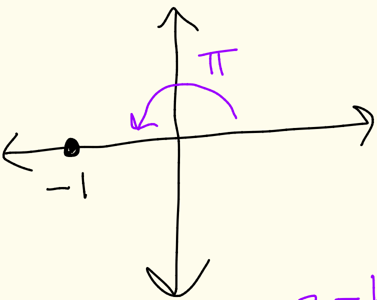
$$(-1)^{1/2} = e^{\frac{1}{2}\log(-1)} = e^{\frac{1}{2}\left[\underbrace{\ln|-1|}_{\ln(1)=0} + i\arg(-1)\right]}$$

$$= e^{\frac{1}{2}\left[0 + i(\pi + 2\pi k)\right]}$$

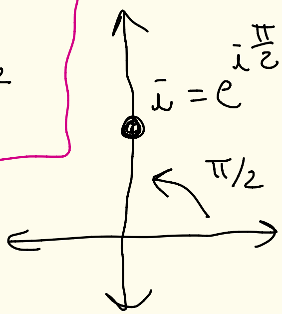
$$= e^{i\left(\frac{\pi}{2} + \pi k\right)}$$

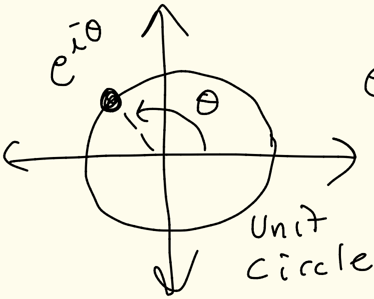
$$= e^{i\frac{\pi}{2}}\, e^{i\pi k}$$

$$= i\, e^{i\pi k}$$




$\arg(-1) = \pi + 2\pi k$
$k = 0, \pm 1, \pm 2, \ldots$





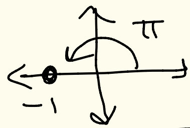

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$= ie^{i\pi k} = i\left[\underbrace{\cos(\pi k)}_{\pm 1} + i\underbrace{\sin(\pi k)}_{0}\right]$$

$$= \begin{cases} -i & \text{if } k \text{ is odd} \\ i & \text{if } k \text{ is even} \end{cases}$$

Choose the branch of log to be $[0, 2\pi)$

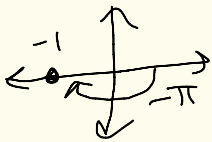
$$(-1)^{1/2} = e^{\frac{1}{2}[\underbrace{\ln|-1|}_{0} + i\arg(-1)]} = e^{\frac{1}{2}[0 + i\pi]}$$




$$= e^{\frac{\pi}{2}i} = i$$

Choose the branch of log to be $(-\pi, \pi)$

$$(-1)^{1/2} = e^{\frac{1}{2}[\underbrace{\ln|-1|}_{0} + i\arg(-1)]} = e^{\frac{1}{2}[i(-\pi)]}$$




$$= e^{-\frac{\pi}{2}i} = -i$$

**Def:** Let $n \geq 2$ be an integer. Define the n-th root function by

$$\sqrt[n]{z} = z^{1/n} = e^{1/n \log(z)}$$

where a specific choice of branch of the logarithm is chosen. This function is called a <u>branch of the n-th root function.</u>

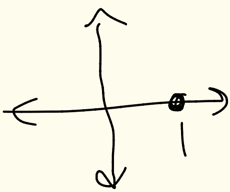
---

**Ex:** Consider $f(z) = z^{1/2} = e^{1/2 \log(z)}$ where the branch of the log is $[0, 2\pi)$

$$f(1) = 1^{1/2} = e^{1/2 \log(1)} = e^{1/2[\underbrace{\ln|1|}_{0} + i0]} = e^{0} = 1$$

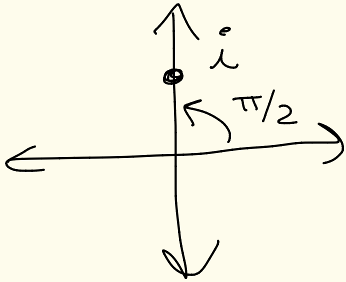
$f(-1) = i$ [from previous page]

$$f(i) = i^{1/2} = e^{\frac{1}{2}\log(i)} =$$



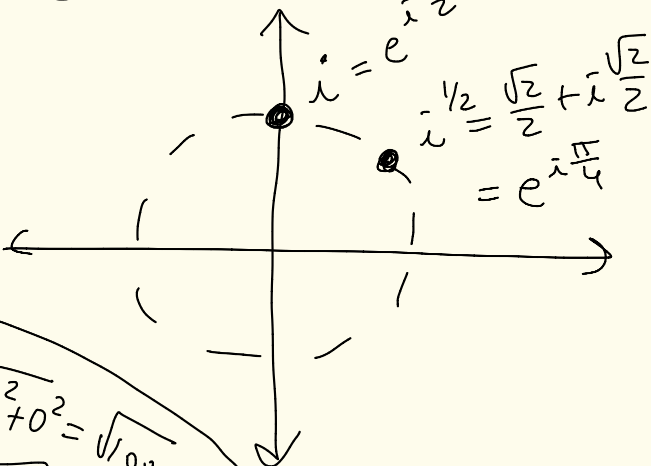
$$= e^{\frac{1}{2}\log(i)} = e^{\frac{1}{2}\left[\underbrace{\ln|i|}_{\ln(1)=0} + i\arg(i)\right]}$$

$$= e^{\frac{1}{2} i \frac{\pi}{2}}$$




$$= e^{i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$$




$$|-10| = |-10 + i0| = \sqrt{(-10)^2 + 0^2} = \sqrt{100} = 10$$

$$|a+ib| = \sqrt{a^2+b^2}$$

$$\log(z) = \ln|z| + i\arg(z)$$