TOPIC 2 –

Sequences and series of functions

Weierstrass M-Test

Analytic convergence theorem

Def: Suppose that $A \subseteq \mathbb{C}$ and $f_n : A \to \mathbb{C}$ for each $n \geq 1$.

1. Let $f : A \to \mathbb{C}$. We say that $(f_n)_{n=1}^{\infty}$ <u>converges pointwise to f on A</u> if for each $z \in A$ we have $\lim_{n\to\infty} f_n(z) = f(z)$.

<u>part 1 says:</u>
Given $\varepsilon > 0$ and $z \in A$, there exists $N > 0$ (N depends on both ε and z), where if $n \geq N$ then
$$|f_n(z) - f(z)| < \varepsilon$$

N is "local" to z

2 Let $f : A \to \mathbb{C}$. We say that $(f_n)_{n=1}^{\infty}$ <u>converges uniformly to f on A</u> if for every $\varepsilon > 0$ there is an $N > 0$ [N only depends on ε] where if $n \geq N$ then

$$|f_n(z) - f(z)| < \varepsilon$$

for all $z \in A$.

3 A series $\sum_{n=1}^{\infty} g_n(z)$

is said to <u>converge pointwise</u>

if the coorresponding partial

sums $S_k(z) = \sum_{n=1}^{k} g_n(z)$

converge pointwise.

A series $\sum_{n=1}^{\infty} g_n(z)$

<u>converges uniformly</u> if

the coorresponding partial

sums $S_k(z) = \sum_{n=1}^{k} g_n(z)$

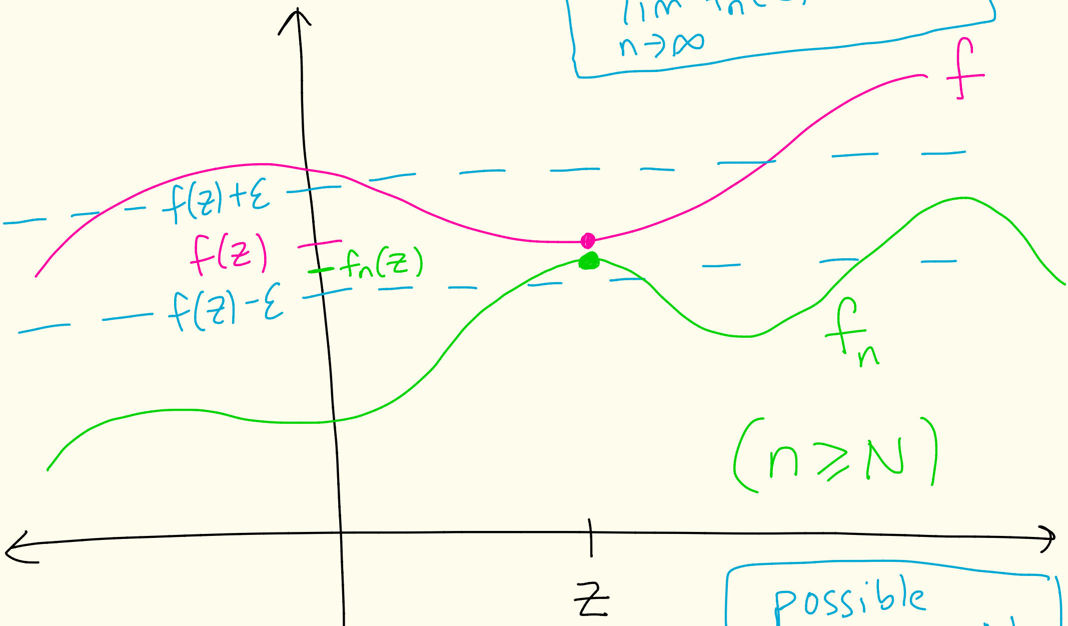
converge uniformly.

Some pictures in $\mathbb{R}$ to give idea

$\boxed{f_n \to f \text{ pointwise on } A}$

For the pic, let $A = \mathbb{R}$, suppose $f_n : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$

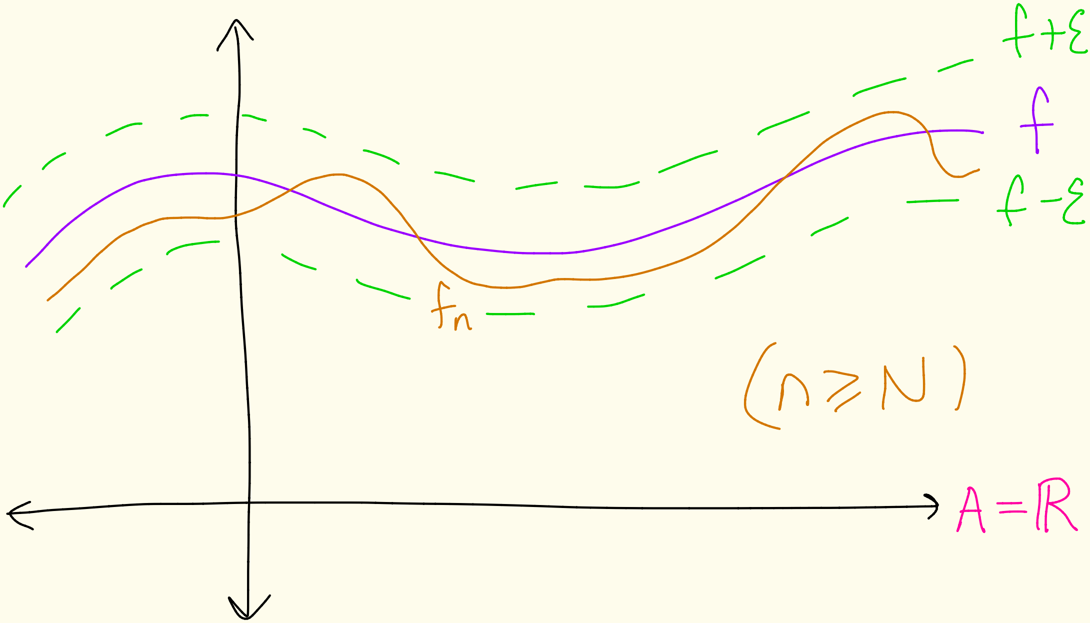
$\boxed{\text{For each } z \in A \ \lim_{n \to \infty} f_n(z) = f(z)}$



possible different N for each z & ε

Given $z \in A$ & $\varepsilon > 0$, $\exists N$ where if $n \geq N$ then $|f_n(z) - f(z)| < \varepsilon$

Same setup but $f_n \to f$ uniformly.



Let $\varepsilon > 0$.
There exists $N > 0$ where $n \geq N$ then $|f_n(z) - f(z)| < \varepsilon$ for all $z \in A$

Ex: Let $f_n(z) = z^n$, $n \geq 1$

sequence:

$z, z^2, z^3, z^4, z^5, z^6, \ldots$

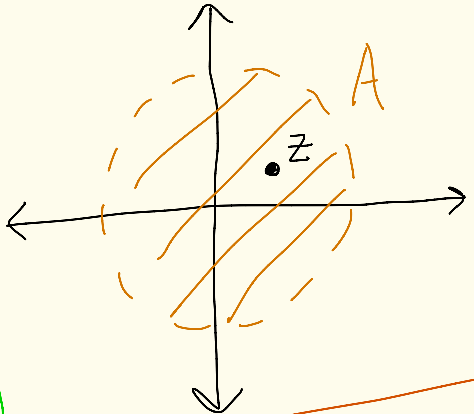
Let $A = D(0;1)$

$\left[D(z_0;r) = \{ z \mid |z - z_0| < r \} \right]$

Let $f : A \to \mathbb{C}$

be defined as

$f(z) = 0 \quad \forall z \in A.$



Claim: $f_n \to f$ pointwise on A

pf: Let $z \in A$.

Then, $|z| < 1$.

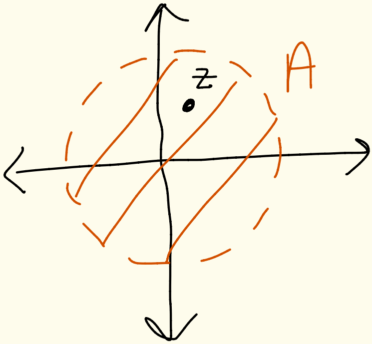
So, $\lim\limits_{n\to\infty} f_n(z) = \lim\limits_{n\to\infty} z^n = 0 = f(z)$ ▨

because $\lim\limits_{n\to\infty} |z^n| = \lim\limits_{n\to\infty} |z|^n = 0$ since $|z|<1$

Ex: Let $f_n(z) = \frac{z}{n}$, $n \geq 1$

Sequence is:

$z, \frac{z}{2}, \frac{z}{3}, \frac{z}{4}, \ldots$



Let $A = D(0;1)$

Let $f(z) = 0 \quad \forall z \in A$.

$$\boxed{\text{Claim: } f_n \to f \text{ uniformly on } A.}$$

Pf: Let $\varepsilon > 0$.

Pick $N > \frac{1}{\varepsilon}$.

Then if $n \geq N$ and $z \in A$, then

$$|f_n(z) - f(z)| = \left|\frac{z}{n} - 0\right| = \left|\frac{z}{n}\right|$$

$$= \frac{|z|}{n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$|z| < 1$ since $z \in A$

$n \geq N$

$\blacksquare$

Theorem: Let $A \subseteq \mathbb{C}$ be an open set.

(1) Suppose that $f_n : A \to \mathbb{C}$ for $n \geq 1$ and $f : A \to \mathbb{C}$

Suppose f_n is continuous on A for $n \geq 1$

If $f_n \to f$ uniformly on A, then f is continuous on A.

(2) Consequently if functions $g_k(z)$ are continuous on A and $g(z) = \sum_{k=1}^{\infty} g_n(z)$ converges uniformly on A, then g is continuous on A.

proof: (1) Let $z_0 \in A$.

We will show that f is continuous at z_0.

Let $\varepsilon > 0$.

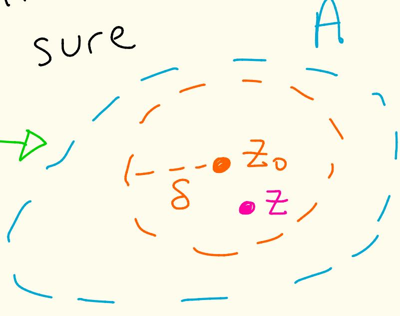
Since $f_n \to f$ uniformly on A, there exists $N > 0$ where $|f_N(z) - f(z)| < \frac{\varepsilon}{3}$ for all $z \in A$.

[We just found one function f_N that is $\varepsilon/3$-close to f on all of A. Ie we approximated f with a continuous function]

Since f_N is continuous at z_0 there exists $\delta > 0$ where if $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$

In the above we shrink δ if necessary to make sure that $D(z_0;\delta) \subseteq A$.

We can do this because A is open.



Then if $|z - z_0| < \delta$ then

$$
\begin{aligned}
&|f(z) - f(z_0)| \\
&= |f(z) - f_N(z) + f_N(z) - f_N(z_0) \\
&\qquad\qquad + f_N(z_0) - f(z_0)| \\
&\overset{\boxed{\triangle}}{\leq} |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| \\
&\qquad\qquad + |f_N(z_0) - f(z_0)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}
$$

So, f is continuous at z_0

2. We have that $g_k(z)$ are each continuous on A. Thus, $s_n(z) = \sum_{k=1}^{n} g_k(z)$ are each continuous on A.

The s_n are the f_n's.

Our sequence of functions on A is:

$$s_1(z) = g_1(z)$$
$$s_2(z) = g_1(z) + g_2(z)$$
$$s_3(z) = g_1(z) + g_2(z) + g_3(z)$$
$$\vdots \qquad \vdots$$

$$g(z) = \lim_{n\to\infty} s_n(z) = \lim_{n\to\infty} \sum_{k=1}^{n} g_k(z)$$

We are assuming that $s_n \to g$ uniformly on A.

Since each s_n is continuous on A, by part 1, $g(z) = \sum_{k=1}^{\infty} g_k(z)$ is continuous on A. ▨

Theorem: (Cauchy criterion)

Let $A \subseteq \mathbb{C}$.

① Let $f_n : A \to \mathbb{C}$, $n \geq 1$.

Then, f_n converges uniformly on A iff for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$ then

$$|f_n(z) - f_{n+p}(z)| < \varepsilon$$

for all $z \in A$ and $p \geq 1$.

$n+p$ takes place of m in the usual Cauchy def

② Let $g_k : A \to \mathbb{C}$, $k \geq 1$.

The series $\sum_{k=1}^{\infty} g_k$ converges uniformly on A iff for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$ then

$$\underbrace{\left| \sum_{k=n+1}^{n+p} g_k(z) \right|}_{\left| \sum_{k=1}^{n+p} g_k(z) - \sum_{k=1}^{n} g_k(z) \right|} < \varepsilon$$

for all $z \in A$ and $p \geq 1$.

proof:

1 ($\Rightarrow$) Suppose (f_n) converges uniformly on A.

Then there exists $f: A \to \mathbb{C}$ that (f_n) converges uniformly to.

Let $\varepsilon > 0$.

Then, there exists $N > 0$ where if $n \geq N$ then

$$|f_n(z) - f(z)| < \frac{\varepsilon}{2}$$

for all $z \in A$.

Then, if $n \geq N$ and $p \geq 1$ and $z \in A$ then

$$|f_n(z) - f_{n+p}(z)| = |f_n(z) - f(z) + f(z) - f_{n+p}(z)|$$

$\boxed{n+p \geq n \geq N}$

$$\leq |f_n(z) - f(z)| + |f(z) - f_{n+p}(z)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

($\Leftarrow$) We are assuming "for every $\varepsilon > 0$, there is a $N > 0$ where if $n \geq N$ then $|f_n(z) - f_{n+p}(z)| < \varepsilon$ for all $z \in A$ and $p \geq 1$."

This implies that for each $z \in A$, $(f_n(z))$ is a Cauchy sequence.

Thus, for each $z \in A$, we can define $f(z) = \lim_{n \to \infty} f_n(z)$.

We want to show that $f_n \to f$ uniformly on A.

Let $\varepsilon > 0$.
By our assumption, there is an $N > 0$ where if $n \geq N$ then
$|f_n(z) - f_{n+p}(z)| < \varepsilon/2$
for all $z \in A$ and $p \geq 1$.

For each $z \in A$, pick p_z large enough so that

$$|f_{n+p_z}(z) - f(z)| < \varepsilon/2$$

for all $n \geq 1$. [We can do this since $f_n \to f$ pointwise on A.]

If $n \geq N$, then

$$|f_n(z) - f(z)|$$

$$= |f_n(z) - f_{n+p_z}(z) + f_{n+p_z}(z) - f(z)|$$

$$\leq |f_n(z) - f_{n+p_z}(z)| + |f_{n+p_z}(z) - f(z)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $z \in A$. So, $f_n \to f$ uniformly on A.

(2) Apply part 1 to

$$S_n(z) = \sum_{k=1}^{n} g_k(z).$$

Then you'll get that

$$\sum_{k=1}^{\infty} g_k(z)$$ converges uniformly on A

iff for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$ then

$$\underbrace{|S_n(z) - S_{n+p}(z)|}_{\left| \sum_{k=1}^{n} g_k(z) - \sum_{k=1}^{n+p} g_k(z) \right| = \left| \sum_{k=n+1}^{n+p} g_k(z) \right|} < \varepsilon$$

for all $z \in A$ and $p \geq 1$

$\blacksquare$

Fact: Suppose $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Then, $\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$

State in class but refer to proof on website

Proof:

Let $s_n = \sum_{k=1}^{n} a_k$ and $\hat{s}_n = \sum_{k=1}^{n} |a_k|$

Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, both $(s_n)_{n=1}^{\infty}$ and $(\hat{s}_n)_{n=1}^{\infty}$ converge.

Note that $|s_n|$ and $\hat{s}_n$ are real numbers for $n \geq 1$.

By the Δ-inequality,

$$|s_n| = \left|\sum_{k=1}^{n} a_n\right| \leq \sum_{k=1}^{n} |a_n| = \hat{s}_n$$

Thus, $\lim_{n\to\infty} |S_n| \leq \lim_{n\to\infty} \hat{S}_n$,

So, $\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$ $\blacksquare$

Theorem (Weierstrass M-Test)

Let $A \subseteq \mathbb{C}$. Let $g_k : A \to \mathbb{C}$ for $k \geq 1$. Suppose there are real constants $M_k \geq 0$ such that

(i) $|g_k(z)| \leq M_k$ for all $z \in A$

and (ii) $\sum_{k=1}^{\infty} M_k$ converges.

Then, $\sum_{k=1}^{\infty} g_k(z)$ converges absolutely and uniformly on A.

proof: Let $\hat{S}_n = \sum_{k=1}^{n} |g_k(z)|$

and $t_n = \sum_{k=1}^{n} M_k = M_1 + M_2 + \dots + M_n$.

Let $\varepsilon > 0$.

We know that (t_n) converges because its the partial sums of $\sum_{k=1}^{\infty} M_k$

So, there exists $N > 0$ where if $n > m \geq N$ then $|t_n - t_m| < \varepsilon$

So, $$\sum_{k=m+1}^{n} M_k = \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k = \underbrace{t_n - t_m}_{>0} < \varepsilon$$

Hence, if $n > m \geq N$ and $z \in A$ we have

$$\left| \hat{S}_n(z) - \hat{S}_m(z) \right| = \left| \sum_{k=1}^{n} |g_k(z)| - \sum_{k=1}^{m} |g_k(z)| \right|$$

$$\overset{n>m}{=} \sum_{k=m+1}^{n} |g_k(z)| \overset{(i)}{\leq} \sum_{k=m+1}^{n} M_k < \varepsilon$$

So, $\sum_{k=1}^{\infty} |g_k(z)|$ converges for all $z \in A$.

Thus, $\sum_{k=1}^{\infty} g_k(z)$ converges absolutely for all $z \in A$.

Time for uniform convergence.

For each $z \in A$, let

$$S_n(z) = g_1(z) + \dots + g_n(z) = \sum_{k=1}^{n} g_k(z)$$

and $$s(z) = \lim_{n\to\infty} S_n(z) = \sum_{k=1}^{\infty} g_k(z).$$

For each $z \in A$ we have that

$$|s(z) - s_n(z)| = \left| \sum_{k=1}^{\infty} g_k(z) - \sum_{k=1}^{n} g_k(z) \right|$$

$$= \left| \sum_{k=n+1}^{\infty} g_k(z) \right|$$

$$\leq \sum_{k=n+1}^{\infty} |g_k(z)|$$

$$\leq \sum_{k=n+1}^{\infty} M_k$$

If $\sum a_k$ converges absolutely, then $\left|\sum a_k\right| \leq \sum |a_k|$

see website for proof

Since $\sum_{k=1}^{\infty} M_k$ converges, given $\varepsilon > 0$, there exists $N > 0$ where if $n \geq N$ we have

$$\underbrace{\left| \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \right|}$$

So, if $n \geq N$ then

$$\sum_{k=n+1}^{\infty} M_k = \left| \sum_{k=n+1}^{\infty} M_k \right| < \varepsilon$$

Thus, if $n \geq N$ and $z \in A$ then

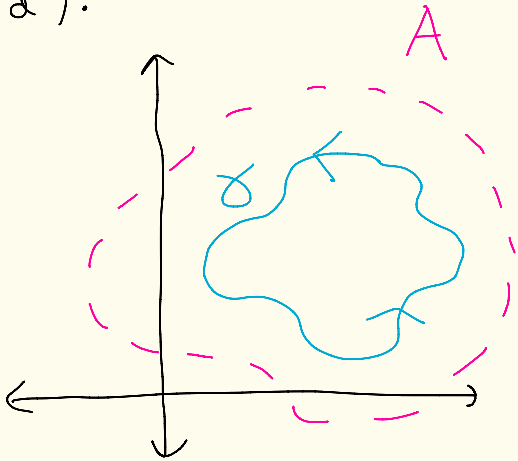
$$|s(z) - s_n(z)| \leq \sum_{k=n+1}^{\infty} M_k < \varepsilon$$

Thus, $s_n \to s$ uniformly on A.

That is, $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A. ▨

<u>Theorem</u>: Let $\gamma : [a,b] \to A$ be a piecewise-smooth curve where A is a region (open and path-connected).

Let $f_n : A \to \mathbb{C}$ be continuous functions on A where $n \geq 1$. Suppose $f_n \to f$ uniformly on A.



Then, $$\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma \lim_{n\to\infty} f_n = \int_\gamma f$$

<u>proof</u>: Since $f_n \to f$ uniformly on A, and each f_n is continuous on A, from a previous theorem we know that f is continuous on A.

So, $\int_\gamma f$ exists.

Let $\varepsilon > 0$.

Since $f_n \to f$ uniformly on A there exists $N > 0$ where if $n \geq N$ we have that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\text{length}(\gamma)}$$

for all $z \in A$.

Then, if $n \geq N$ we have

$$\left| \int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz \right|$$

$$= \left| \int_\gamma [f_n(z) - f(z)]\,dz \right|$$

$$= \left| \int_\gamma [f_n(z) - f(z)]\, dz \right|$$

$$< \underbrace{\frac{\varepsilon}{\text{length}(\gamma)}}_{M} \cdot \text{length}(\gamma)$$

$$= \varepsilon.$$

4680 Thm: If $|g(z)| \leq M$ for all z on γ then $\left| \int_\gamma g(z)\,dz \right| \leq M \cdot \text{length}(\gamma)$

Thus, $\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f$ ■

Corollary: Let A be a region and $\gamma : [a,b] \to A$ be a piecewise-smooth curve in A. Suppose $g_k : A \to \mathbb{C}$ is continuous on A for each $k \geq 1$. Suppose $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A. Then,
$$\int_\gamma \left(\sum_{k=1}^{\infty} g_k(z) \right) dz = \sum_{k=1}^{\infty} \left(\int_\gamma g_k(z)\,dz \right)$$

proof: Let $f_n(z) = \sum_{k=1}^{n} g_k(z)$ be the n-th partial sum of the series.

Let $f(z) = \sum_{k=1}^{\infty} g_k(z)$.

Since each g_k is continuous on A, we know each f_n is continuous on A.

By assumption $f_n \to f$ uniformly on A.

So we can use the previous thm which said

$$(*) \quad \lim_{n\to\infty} \int_\gamma \underbrace{\left[\sum_{k=1}^{n} g_k(z)\right]}_{f_n(z)} dz = \int_\gamma \underbrace{\left[\sum_{k=1}^{\infty} g_k(z)\right]}_{f(z)} dz$$

Thus, $\displaystyle\sum_{k=1}^{\infty}\left(\int_\gamma g_k(z)\,dz\right) = \lim_{n\to\infty} \sum_{k=1}^{n}\left(\int_\gamma g_k(z)\,dz\right)$

$$\underset{\text{finite sum}}{=} \lim_{n\to\infty} \int_\gamma \left(\sum_{k=1}^{n} g_k(z)\right) dz \overset{(*)}{=} \int_\gamma \left(\sum_{k=1}^{\infty} g_k(z)\right) dz$$

▨

Theorem: (Analytic Convergence Thm)

① Let A be an open set in $\mathbb{C}$. Let (f_n) be a sequence of analytic functions defined on A. If $f_n \to f$ uniformly on every closed disc contained in A, then f is analytic. Furthermore, $f_n' \to f'$ pointwise on A and uniformly on every closed disc in A.

② If (g_k) is a sequence of analytic functions defined on an open set $A \subseteq \mathbb{C}$ and $g(z) = \sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every closed disc in A, then $g(z)$ is analytic on A and $g'(z) = \sum_{k=1}^{\infty} g_k'(z)$ pointwise on A and uniformly on every closed disc contained in A.

Proof:

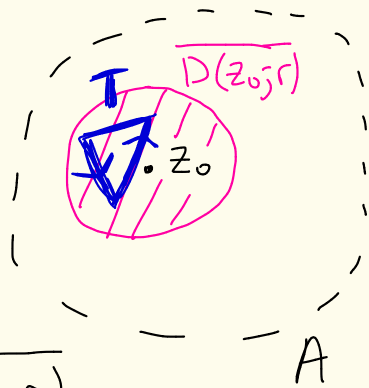
1. Let $z_0 \in A$.

We will show that f is analytic at z_0.

Let $r > 0$ so that

$$\overline{D(z_0;r)} = \{ z \mid |z - z_0| \le r \}$$

is contained in A.



Since by assumption $f_n \to f$ uniformly on $\overline{D(z_0;r)}$

we know $f_n \to f$ uniformly on

$$D(z_0;r) = \{ z \mid |z - z_0| < r \}.$$

By a previous thm, since each f_n is continuous on $D(z_0;r)$, f is also continuous on $D(z_0;r)$.

Let T be a triangular path located inside of $D(z_0;r)$.

Since each f_n is analytic inside and on T, by Cauchy's thm (4680) we know $\int_T f_n = 0$ for all n.

By the previous proposition, we have

$$0 = \lim_{n\to\infty} \int_T f_n = \int_T \lim_{n\to\infty} f_n = \int_T f$$

By Morera's thm (4680), f is analytic in $D(z_0; r)$.

So, f is analytic at z_0.

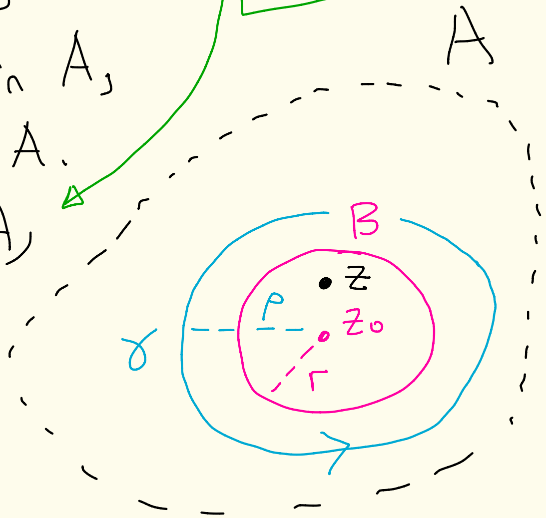
We now show that $f_n' \to f'$ uniformly on closed discs.

needs B closed to work

Let

$$B = \{ z \mid |z - z_0| \le r \}$$

be a closed disc in A, where $r > 0$ and $z_0 \in A$.



By HW 2 problem A, we can choose $\rho > r$ such that γ is a circle contained in A of radius ρ that contains B in its interior. Orient γ counter-clockwise.

For any $z \in B$, we have

$$f_n'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{(\zeta - z)^2} \, d\zeta$$

and

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

(4680) Cauchy Integral Thm

Let $\varepsilon > 0$.

Since $f_n \to f$ uniformly on

$$\overline{D(z_0;\rho)} = \{ z \mid |z - z_0| \leq \rho \}$$

inside and on γ

there exists $N > 0$ where if $n \geq N$ we have

$$|f_n(z) - f(z)| < \varepsilon \cdot \frac{(\rho - r)^2}{\rho}$$

for all $z \in \overline{D(z_0;\rho)}$.

If ς is on γ and $z \in B$, then $|\varsigma - z| \geq \rho - r$.

Thus, if $n \geq N$ and $z \in B$, then

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f_n(\varsigma) - f(\varsigma)}{(\varsigma - z)^2} dz \right|$$

$$< \frac{1}{2\pi} \cdot \frac{\left(\frac{\varepsilon (\rho - r)^2}{\rho} \right)}{(\rho - r)^2} \cdot \text{length}(\gamma)$$

$$= \frac{1}{2\pi} \cdot \frac{\varepsilon}{\rho} \cdot 2\pi\rho = \varepsilon.$$

$\left|\frac{1}{2\pi i}\right| = \frac{1}{2\pi}$

$|f_n(\varsigma) - f(\varsigma)| < \varepsilon \cdot \frac{(\rho - r)^2}{\rho}$

$|\varsigma - z| \geq \rho - r$

Thus, $f_n' \to f'$ uniformly on B.

② Set $f_n = \sum_{k=1}^{n} g_k$ and $f = \sum_{k=1}^{\infty} g_k$

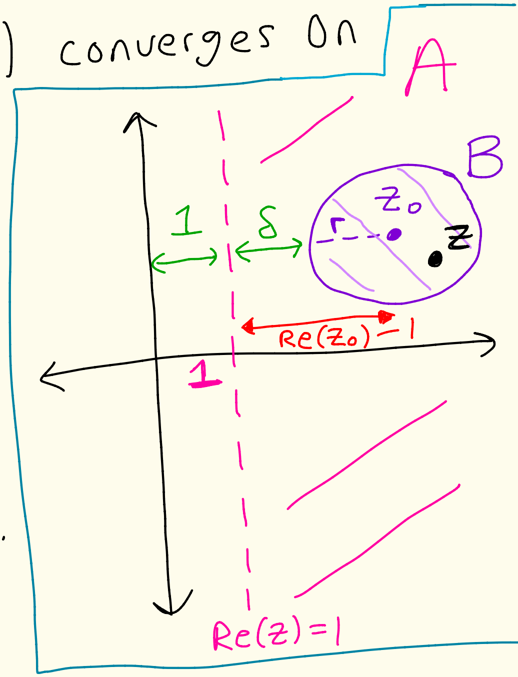
and use part ①

Ex: Let $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$

be the Riemann zeta function.

We know that $\zeta(z)$ converges on

$A = \{ z \mid \operatorname{Re}(z) > 1 \}$



We will now use the analytic convergence theorem to show that $\zeta(z)$ is analytic on A and find $\zeta'(z)$.

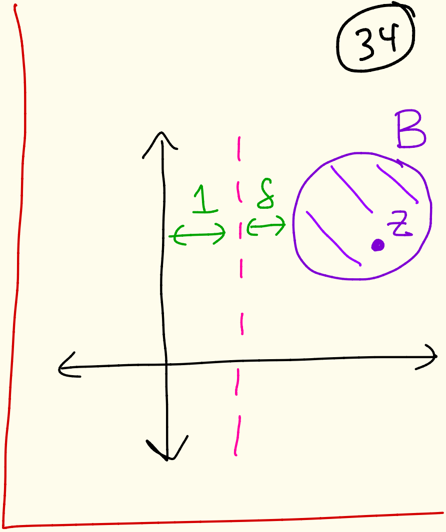
Let B be a closed disc in A with center z_0 and radius r.

Let δ be the distance from $\operatorname{Re}(z)=1$ to the disc B.

So, $\delta = \operatorname{Re}(z_0) - 1 - r$.

Let $z \in B$.

Then, $\mathrm{Re}(z) \geq 1 + \delta$.



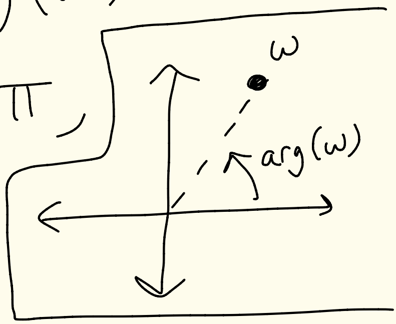
Let $z = x + iy$

Then $x \geq 1 + \delta$.

Then, if

$$\log(w) = \ln|w| + i\arg(w)$$

where $-\pi \leq \arg(w) < \pi$,



then

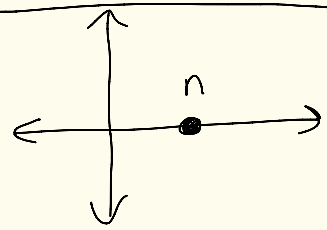
$$\left|\frac{1}{n^z}\right| = \left|n^{-z}\right| = \left|e^{-z\log(n)}\right| =$$

$$\boxed{a^b = e^{b\log(a)}}$$

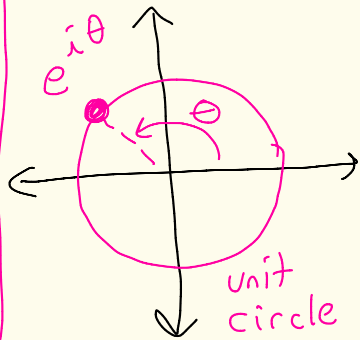
$$= \left| e^{-z\log(n)} \right| = \left| e^{-(x+iy)\left[\ln(n)+i\underbrace{\arg(n)}_{0}\right]} \right|$$

$$= \left| e^{-x\ln(n) - iy\ln(n)} \right|$$

$$= \left| e^{-x\ln(n)} \right| \left| e^{i(-y\ln(n))} \right|$$



$n = 1, 2, 3, 4, 5, \ldots$



$\theta \in \mathbb{R}$

$|e^{i\theta}| = 1$

$$= \left| e^{-x\ln(n)} \right| = e^{-x\ln(n)}$$

$$= e^{\ln(n^{-x})} = n^{-x}$$

$$= \frac{1}{n^{x}} \leq \frac{1}{n^{1+\delta}}$$

$$\boxed{\begin{array}{l} x \geq 1+\delta \\ n^{x} \geq n^{1+\delta} \end{array}}$$

So, if $z \in B$, then

$$\left| \frac{1}{n^{z}} \right| \leq \frac{1}{n^{1+\delta}}$$

Let $M_n = \dfrac{1}{n^{1+\delta}}$

Then, if $z \in B$, then $\left|\dfrac{1}{n^z}\right| \leq M_n$ for all $n \geq 1$.

And we know $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$

converges because $1+\delta > 1$.

p-series

Thus, by the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely and uniformly on B.

So, by the analytic convergence theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ is analytic on A

And, if $z \in A$ then

$$\zeta'(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n^z}\right)' = \sum_{n=1}^{\infty} \frac{-\log(n)}{n^z}$$

$$= \sum_{n=2}^{\infty} \frac{-\log(n)}{n^z}$$

4680

$$(a^z)' = \log(a) \cdot a^z$$

$$(n^{-z})' = \log(n) \cdot n^{-z} \cdot (-1)$$

$$= -\log(n) \cdot n^{-z}$$

$$\log(1) = \ln|1| + i\arg(1)$$

$$= 0 + i0$$

$$= 0$$