TOPIC 3 - POWER SERIES

<u>Def:</u> A <u>power series</u> is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where z_0, a_n are constants in $\mathbb{C}$.

We say that the power series is centered at z_0

<u>Ex:</u> $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$

$$= \sum_{n=0}^{\infty} (z - 0)^n$$

$a_0 = 1$
$a_1 = 1$
$a_2 = 1, \ldots$

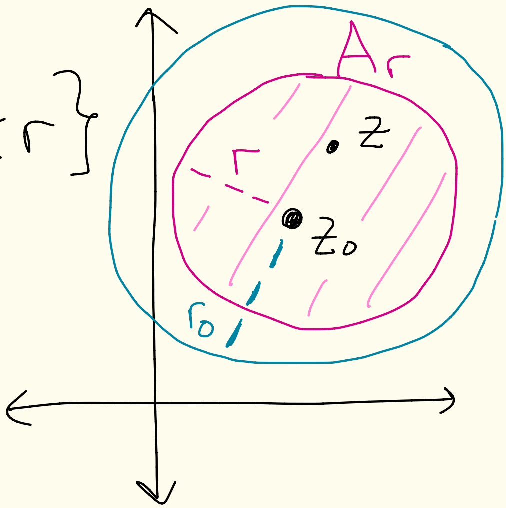
<u>Ex:</u> $\sum_{n=0}^{\infty} (-1)^n (z-1)^n$

$a_0 = 1,\ a_1 = -1,$
$a_2 = 1,\ a_3 = -1, \ldots$

$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

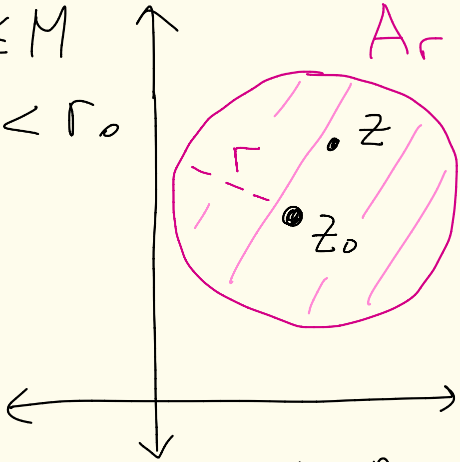
Lemma: (Abel – Weierstrass Lemma)

Let $r_0 \in \mathbb{R}$, $a_n \in \mathbb{C}$, $n \geq 0$.
Suppose that $r_0 > 0$ and that
$|a_n| r_0^n \leq M$ for all $n \geq 0$
where $M \in \mathbb{R}$. Then for
$r < r_0$, the series
$\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges
uniformly and absolutely on
the closed disc
$A_r = \{ z \mid |z - z_0| \leq r \}$



<u>proof:</u> Suppose that $r_0 > 0$ and $|a_n| r_0^n \leq M$ for all n. Let $r < r_0$

Let $z \in A_r$



Then,

$$|a_n(z-z_0)^n| = |a_n||z-z_0|^n \leq |a_n| r^n$$

$$= |a_n| \cdot r_0^n \left(\frac{r}{r_0}\right)^n$$

$$\leq M \cdot \left(\frac{r}{r_0}\right)^n$$

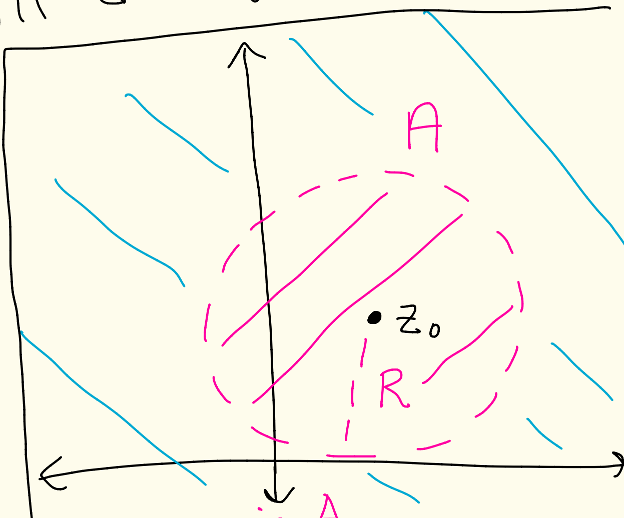
$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ $|z|<1$

Let $M_n = M \cdot \left(\frac{r}{r_0}\right)^n$.

Since $\frac{r}{r_0} < 1$, the series $\sum_{n=0}^{\infty} M_n = \overbrace{\sum_{n=0}^{\infty} M\left(\frac{r}{r_0}\right)^n}^{\text{geometric series } \left|\frac{r}{r_0}\right|<1} = M\left(\frac{1}{1-\frac{r}{r_0}}\right)$ converges. By the Weierstrass M-test, $\sum a_n(z-z_0)^n$ converges unif. and abs. on A_r ∎

Theorem (Power Series convergence) Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series. Then there is a unique number $R \geq 0$, possibly ∞, called the *radius of convergence*, such that if $|z-z_0| < R$ the series will converge and if $|z-z_0| > R$ the series will diverge.

Furthermore, the convergence is uniform and absolute on every closed disc contained in $A = \{z \mid |z-z_0| < R\}$



converges in A
diverges outside A
could converge or diverge at a point on boundary

<u>proof:</u> Let

$$S = \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} |a_n| r^n \text{ converges} \right\}$$

and $R = \sup(S)$.

sup = least upper bound

<u>Suppose $R=0$.</u>

Then the series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for all $z \in \mathbb{C}$ with $|z-z_0| < R = 0$ since there are no such z.

Suppose that $z_1 \in \mathbb{C}$ with $r_0 = |z_1 - z_0| > R$

We want to show that the series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ diverges at z_1.

Suppose instead that $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges.

Then $\lim_{n\to\infty} |a_n| r_0^n = \lim_{n\to\infty} |a_n| |z_1 - z_0|^n = 0$

Since $(|a_n| r_0^n)_{n=0}^{\infty}$ converges, it is bounded so $|a_n| r_0^n \leq M$ for some $M > 0$ and all n.

By the Abel-Weierstrass thm,

if $0<r<r_0$ then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

converges absolutely $\forall z \in A_r = \{z \mid |z-z_0| \leq r\}$

So, $\sum_{n=0}^{\infty} |a_n| |z-z_0|^n$ converges for all

$z \in A_r$ with $0<r<r_0$.

Then, $\sum_{n=0}^{\infty} |a_n| r^n$ converges for all r

with $R=0<r<r_0$.

This contradicts $R = \sup(S)$, since

then $r \in S$ for all r with

$R=0<r<r_0$.

Thus, $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ diverges at

z_1 if $|z_1 - z_0| > R$.

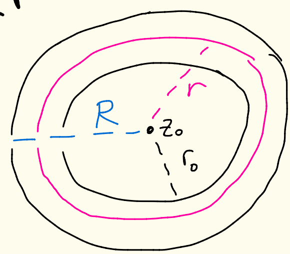
Assume that $R = \sup(S) > 0$ for the remainder of the proof.

Let $0 < r_0 < R$.

Since $R = \sup(S)$, there exists r where $0 < r_0 < r < R$ and $r \in S$ [otherwise we would have $R \leq r_0$.]

Then $\sum_{n=0}^{\infty} |a_n| r^n$ converges.

By the comparison test, since $0 < r_0 < r$ we have that $\sum_{n=0}^{\infty} |a_n| r_0^n$ converges.



Thus, if $0 < r_0 < R$, then $\sum |a_n| r_0^n$ converges.

Thus, $\lim_{n\to\infty} |a_n| r_0^n = 0$

Hence, $\left(|a_n| r_0^n\right)_{n=0}^{\infty}$ is convergent and thus bounded.

So, there exists $M > 0$ where $|a_n| r_0^n \leq M$ for all $n \geq 0$.

By the Abel-Weierstrass lemma,
$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely and uniformly on $A_r = \{z \mid |z-z_0| \leq r\}$ for any $0 < r < r_0 < R$.

Since r_0 can be any real number with $r_0 < R$, we have that:

Given $0 < r < R$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely and uniformly on $A_r = \{z \mid |z-z_0| \leq r\}$.

Given $r < R$ set $r_0 = \frac{R+r}{2}$ and use the above.

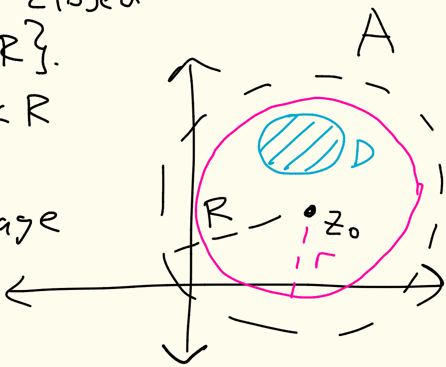
Hence, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z with $|z-z_0| < R$.

Given $|z-z_0| < R$, just set $r = |z-z_0| < R$ and use the above.

Moreover, suppose D is some closed disc in $A = \{z \mid |z - z_0| < R\}$.

Pick $r \in \mathbb{R}$ where $0 < r < R$ and $D \subseteq A_r \subseteq A$.

Then from the previous page $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely and uniformly on A_r and hence also on D.



Note: How do you find such an r?

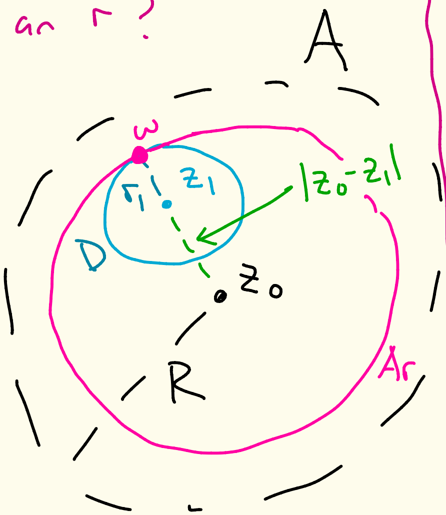
Let $D = D(z_1; r_1) \subseteq A$.

Let $r = r_1 + |z_1 - z_0|$.

Claim 1: $D \subseteq A_r$

Let $z \in D$.

Then, $|z - z_0| = |z - z_1 + z_1 - z_0|$
$\leq |z - z_1| + |z_1 - z_0| \leq r_1 + |z_1 - z_0| = r$



Claim 2: $A_r \subseteq A$.

If $z_1 = z_0$, then $r = r_1$ and thus $A_r = D \subseteq A$.

Suppose $z_1 \neq z_0$. Set $w = z_0 + (z_1 - z_0) + r_1 \dfrac{(z_1 - z_0)}{|z_1 - z_0|}$

Then, $w \in D$ since $|w - z_1| = \left| r_1 \dfrac{(z_1 - z_0)}{|z_1 - z_0|} \right| = r_1$. So, $w \in A$.

So, $|w - z_0| < R$. Thus,

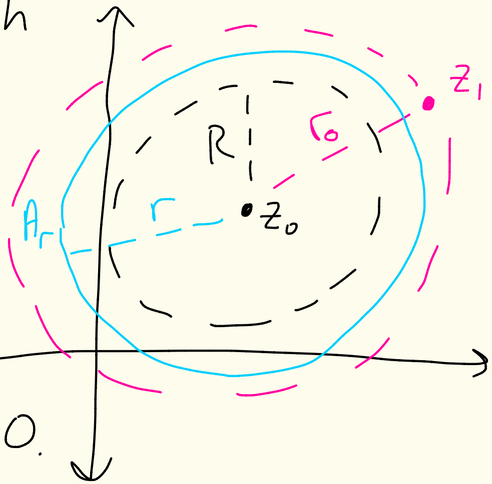
$$R > |w - z_0| = \left| (z_1 - z_0) + r_1 \frac{(z_1 - z_0)}{|z_1 - z_0|} \right| = \left| \frac{z_1 - z_0}{z_1 - z_0} \right| \left| (z_1 - z_0) + r_1 \right|$$
$$= |r_1 + (z_1 - z_0)| \geq \big| |r_1| - |z_1 - z_0| \big| = |r_1 + |z_0 - z_1|| = r.$$

Thus, $r < R$ and $A_r = \{z \mid |z - z_0| \leq r\} \subseteq \{z \mid |z - z_0| < R\} = A$ $\blacksquare$

Now we switch gears.

Suppose $z_1 \in \mathbb{C}$ with $r_0 = |z_1 - z_0| > R$ and $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges.

This is a proof by contradiction since we want divergence



Then, $\lim_{n \to \infty} a_n (z_1 - z_0)^n = 0$.

So, the sequence $\left(a_n (z_1 - z_0)^n \right)_{n=0}^{\infty}$ is bounded.

Thus, $|a_n| |z_1 - z_0|^n = |a_n| r_0^n \leq M$ for some $M > 0$ and all $n \geq 0$.

Thus, the Abel-Weierstrass thm tells us that if $R < r < r_0$ then $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely if $z \in A_r = \{ z \mid |z - z_0| \leq r \}$

So, $\sum_{n=0}^{\infty} |a_n||z-z_0|^n$ converges for all $z \in A_r$

Thus, $\sum_{n=0}^{\infty} |a_n| t^n$ converges for all $R < t < r$.

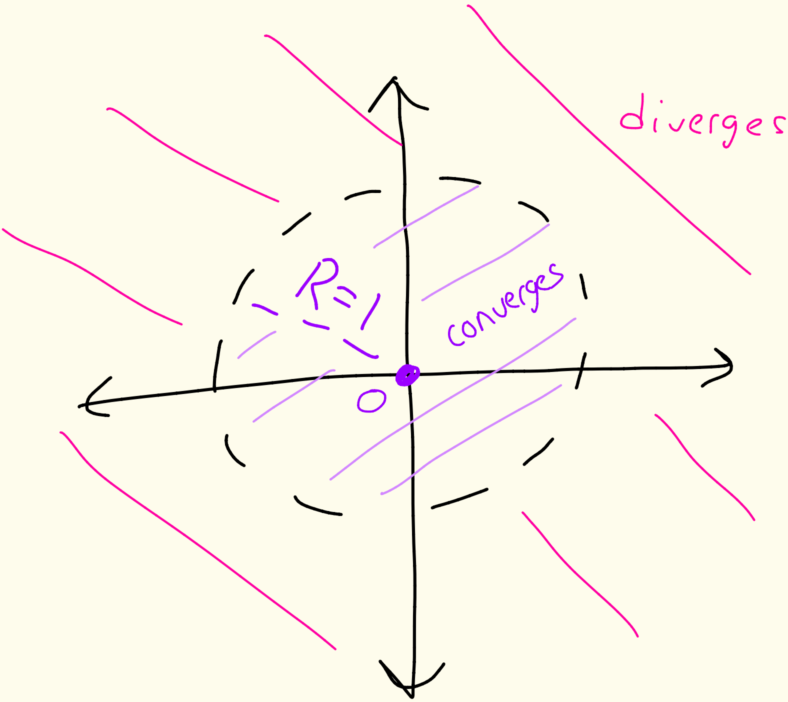
But then $t \in S$ and $R < t$.

This contradicts, $R = \sup(S)$.

So, $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ diverges for all z_1 such that $|z_1 - z_0| > R$. $\blacksquare$

Ex: $\displaystyle\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$

We showed early in the class that this series converges for $|z|<1$ and diverges for $|z|>1$.



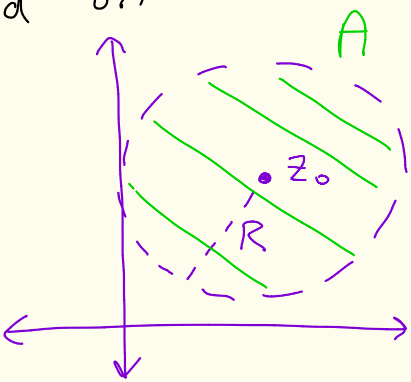
<u>Theorem:</u> Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

be a power series defined on

$A = D(z_0; R)$ where

R is the radius of

convergence of f.



Then,

① f is analytic in A

② $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$

and this series has the same radius of convergence R

and ③ $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

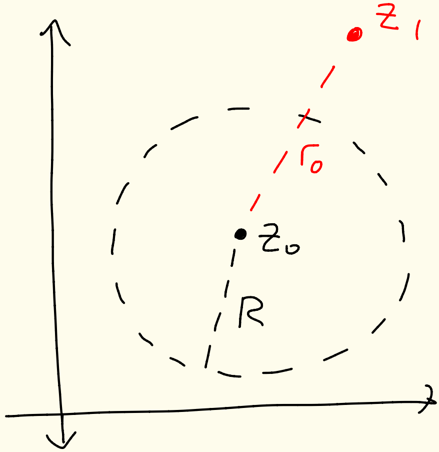
<u>proof:</u> By the theorem from last class and the analytic convergence theorem we know f is analytic in A, and f' exists in A and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}, \ \forall z \in A$$

Let's show that $\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ has radius of convergence R.

Let $z_1 \in \mathbb{C}$ with $r_0 = |z_1 - z_0| > R$.

We will show $\sum_{n=1}^{\infty} n a_n (z_1 - z_0)^{n-1}$ diverges.



Suppose $\sum_{n=1}^{\infty} n a_n (z_1 - z_0)^{n-1}$ converged.

Then, $\lim_{n\to\infty} n a_n (z_1 - z_0)^{n-1} = 0$

So, $\lim_{n\to\infty} |n a_n r_0^{n-1}| = \lim_{n\to\infty} |n a_n (z_1 - z_0)^{n-1}| = 0$

Since $(n|a_n|r_0^{n-1})_{n=1}^{\infty}$ converges, we know $|n a_n r_0^{n-1}| \leq M$ for some $M > 0$ and $n \geq 1$.

[HW 0-convergent sequences are bounded]

Let $M' = \max\{M, |na_0 r_0^{n-1}|\}$.

Thus,

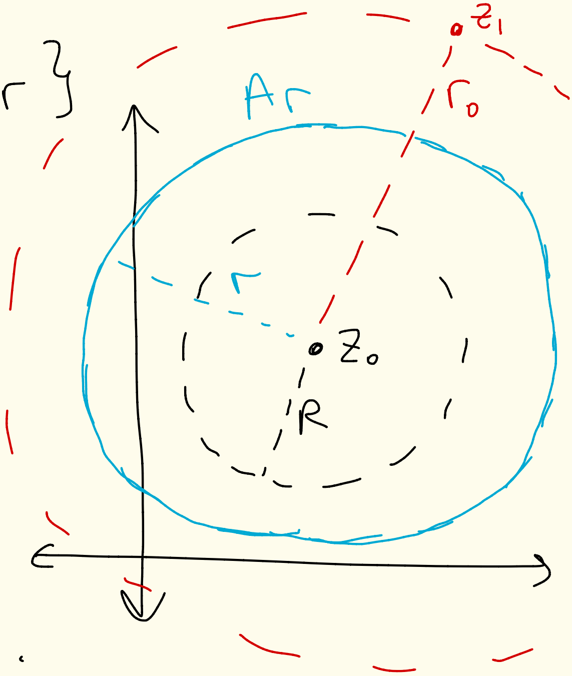
$$|a_n| r_0^n = |a_n r_0^n| = |n a_n r_0^{n-1}| \left|\frac{r_0}{n}\right| \le M' \cdot r_0$$

for all $n \geq 0$.

By Abel-Weierstrass theorem $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges on

$A_r = \{z \mid |z-z_0| \le r\}$

for any r with $0 < r < r_0$



This contradicts R being the radius of convergence of $f(z)$ if you pick some r with $R < r < r_0$.

Thus, $\sum_{n=1}^{\infty} n a_n (z_1 - z_0)^n$ diverges.

So, R is its radius of convergence.

You can keep applying the analytic convergence theorem and the above arguments to get the power series for $f^{(n)}(z)$ for $n \geq 1$.

They will all have radius of convergence R.

So,

$$f(z) = a_0 + a_1(z - z_0) + a_2 (z - z_0)^2 + \cdots$$

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots$$

$$f''(z) = 2a_2 + 3 \cdot 2 a_3 (z - z_0) + 4 \cdot 3 \cdot a_4 (z - z_0)^2 + \cdots$$

By induction you could show that

$$f^{(k)}(z) = k! a_k + \sum_{n=k+1}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n (z - z_0)^{n-k}$$

and has radius of convergence R.

So,

$$f^{(k)}(z_0) = k!\, a_k + \sum_{n=k+1}^{\infty} 0$$

Thus,

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$ ▨

Theorem (Uniqueness of Power Series)

If

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

for all $z \in D(z_0; r)$ with $r>0$ then $a_n = b_n$ for all $n = 0, 1, 2, \ldots$

pf: $a_n = \dfrac{f^{(n)}(z_0)}{n!} = b_n$ ▨

Ratio Test Let $\sum_{k=1}^{\infty} b_k$ be a series of complex numbers. Suppose that

$$r = \lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right|$$

exists.

(1) If $r < 1$, then $\sum_{k=1}^{\infty} b_k$ converges absolutely.

(2) If $r > 1$, then $\sum_{k=1}^{\infty} b_k$ diverges.

(3) If $r = 1$, then the test is inconclusive, the series may converge or diverge.

<u>proof:</u>

<u>case 1:</u> Suppose $0 \leq r < 1$.

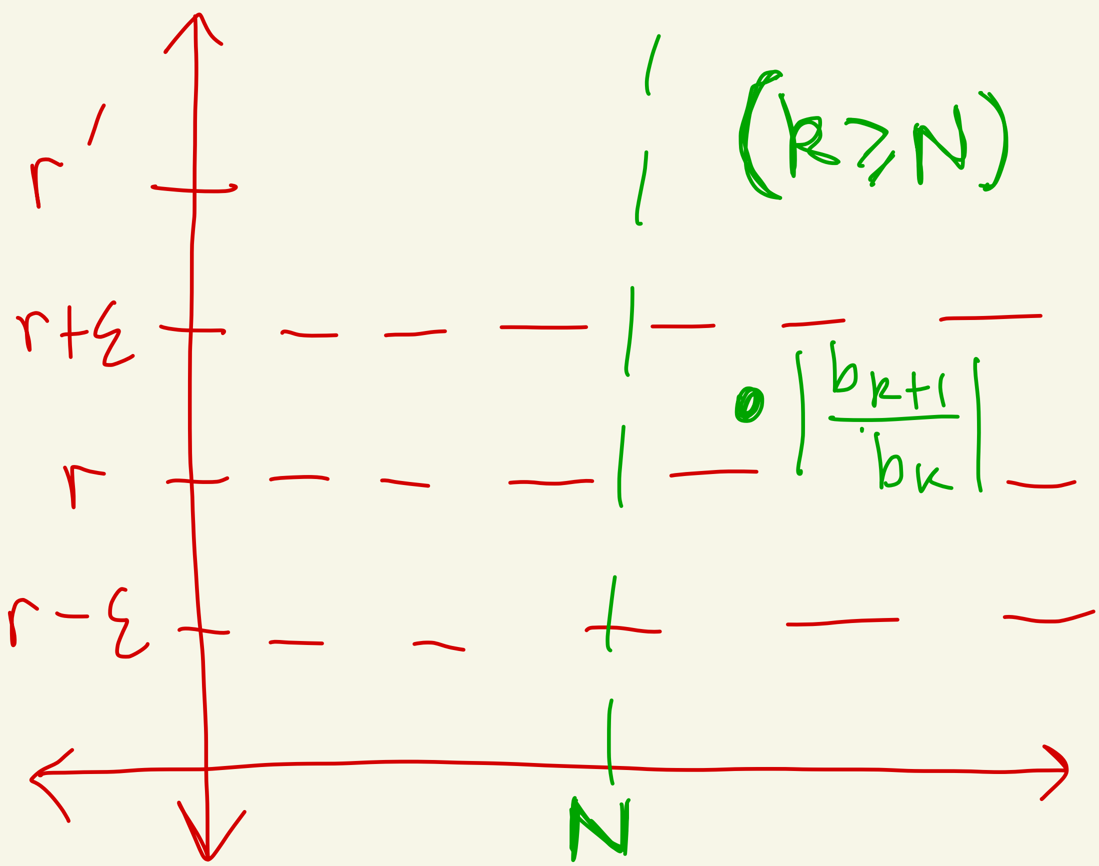
Let $r' \in \mathbb{R}$ with $r < r' < 1$.

Since we have a sequence of real numbers $\left|\frac{b_{k+1}}{b_k}\right|$ converging to r, there must exist $N > 0$ where if $k \geq N$, then $\left|\frac{b_{k+1}}{b_k}\right| < r'$.

<u>Why?</u>

Set $\varepsilon = \frac{r' - r}{2}$

Then $\exists N > 0$ where if $k \geq N$ then $\left|\frac{b_{k+1}}{b_k}\right| < r + \varepsilon$



$$= \frac{r'}{2} + \frac{r}{2} < \frac{r'}{2} + \frac{r'}{2} = r'$$

Thus, if $k \geq N$ then

$$|b_k| < r'|b_{k-1}| < (r')^2 |b_{k-2}|$$

$$< \cdots < (r')^{k-N} |b_N|$$

The series

$$\sum_{k=N}^{\infty} (r')^{k-N} |b_N|$$

$$= |b_N| \sum_{k=N}^{\infty} (r')^{k-N}$$

$$= |b_N| \left(1 + r' + (r')^2 + (r')^3 + \cdots \right)$$

$$= |b_N| \frac{1}{1-r'} \quad \left(\text{geometric sum}\right)$$

Since $0 < r' < 1$,

Since $|b_k| < (r')^{k-N} |b_N|$

for all $k \geq N$ and $\sum_{k=N}^{\infty} (r')^{k-N} |b_N|$

converges, by the comparison test

(HW 1 #5), we know

$$\sum_{k=N}^{\infty} |b_k| \text{ converges.}$$

By HW 1 #2, this implies

$$\sum_{k=1}^{\infty} |b_k| \text{ converges.}$$

Thus, $\sum_{k=1}^{\infty} b_k$ converges absolutely.

<u>Case 2:</u> Suppose $r > 1$.

Choose $r' \in \mathbb{R}$ with $1 < r' < r$.

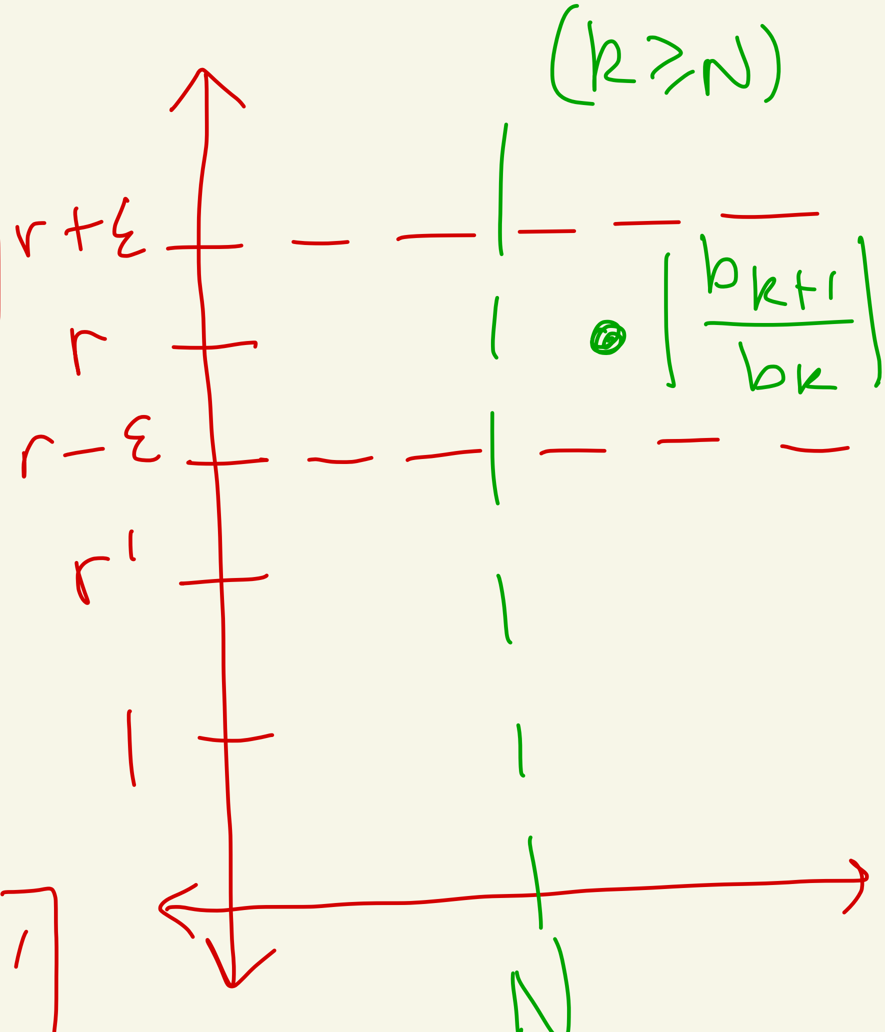
There must exist $N > 0$ where if $k \geq N$ then $\left|\frac{b_{k+1}}{b_k}\right| > r'$.

Why?

Let $\varepsilon = \frac{r - r'}{2}$

There exists $N > 0$ where if $k \geq N$

$$\left|\frac{b_{k+1}}{b_k}\right| > r - \varepsilon = \frac{r}{2} + \frac{r'}{2} > \frac{r'}{2} + \frac{r'}{2} = r'$$



Then,

$$|b_{N+p}| > r'|b_{N+p-1}| > (r')^2|b_{N+p-2}|$$

$$> \cdots > (r')^p |b_N|$$

Thus,

$$\lim_{k\to\infty} |b_k| = \lim_{p\to\infty} |b_{N+p}|$$

$$> \lim_{p\to\infty} (r')^p \underbrace{|b_N|}_{\text{fixed \#}}$$

$$= \infty \quad (\text{since } 1 < r')$$

Thus, $\lim_{k\to\infty} b_k \neq 0$.

By the divergence test $\sum_{k=1}^{\infty} b_k$ diverges.

<u>Case 3:</u> Suppose $r=1$.

The test is inconclusive.

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ has

$$\lim_{k\to\infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim \left| \frac{k}{k+1} \right| = 1 \leftarrow r$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>

For example, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has

$$\lim_{k\to\infty}\left|\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}}\right| = \lim_{k\to\infty}\left|\frac{k^2}{(k+1)^2}\right| = 1 \leftarrow r$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. $\blacksquare$

Ex: Find the radius of convergence of

$$\sum_{n=0}^{\infty} \underbrace{\boxed{\frac{(-1)^{n-1}}{2^n} z^n}}_{b_k} = -1 + \frac{1}{2} z - \frac{1}{2^2} z^2 + \cdots$$

Note that

$$\lim_{k\to\infty} \left| \underbrace{\frac{(-1)^{k+1-1} z^{k+1}}{2^{k+1}}}_{b_{k+1}} \cdot \underbrace{\frac{2^k}{(-1)^{k-1} z^k}}_{1/b_k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{z}{2} \right| = \frac{|z|}{2}$$

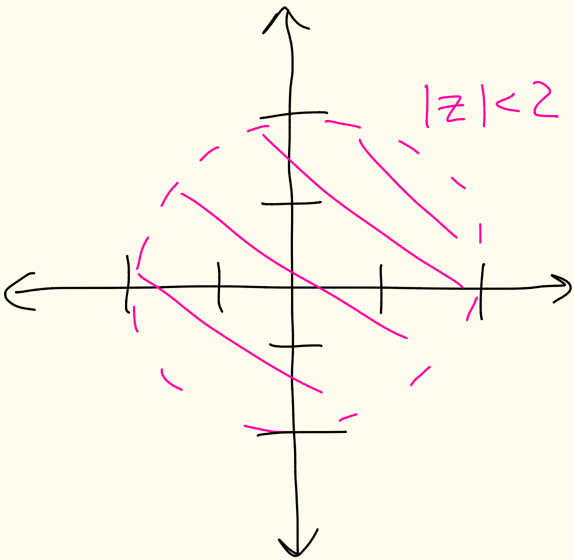
The ratio test says that

The series will converge when $\left|\frac{z}{2}\right| < 1$

and diverge when $\left|\frac{z}{2}\right| > 1$.

So the series converges when $|z| < 2$ and diverges when $|z| > 2$.

$R = 2$ is the radius of convergence.



Theorem (Taylor's Theorem)

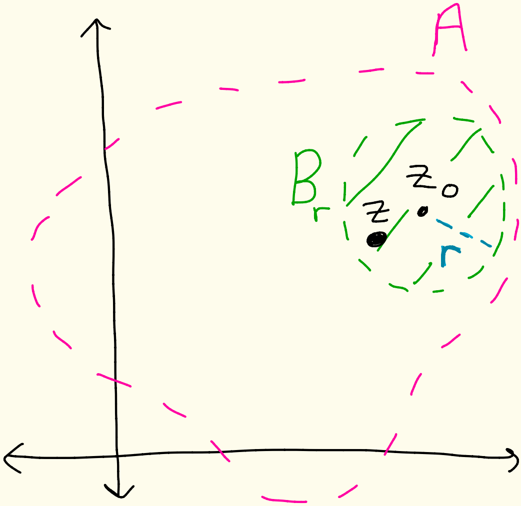
Let f be analytic on an open set $A \subseteq \mathbb{C}$.

Let $z_0 \in A$.

Let

$$B_r = \{ z \mid |z - z_0| < r \} = D(z_0; r)$$

be contained in A.



Then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

converges to $f(z)$ for all z in B_r.

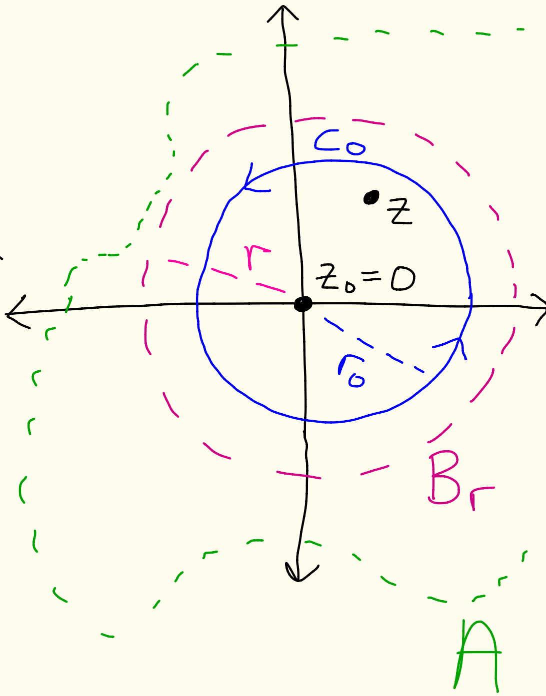
Called the Taylor series of f centered at z_0

Proof: We first prove the theorem when $z_0 = 0$.

Let $B_r = \{ z \mid |z - z_0| < r \} \subseteq A$.

Let $z \in B_r$.

Let C_0 denote some circle of radius r_0, centered at $z_0 = 0$, oriented counter-clockwise that is contained in the disc B_r but is large enough so that z is interior to it.



Since $f(z)$ is analytic inside and on C_0, by the Cauchy-integral formula

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(\xi)}{\xi - z} \, d\xi$$

Recall that if $w \neq 1$ then

$$\sum_{n=0}^{N-1} w^n = 1 + w + w^2 + \cdots + w^{N-1} = \frac{1 - w^N}{1 - w}$$

$$= \frac{1}{1-w} - \frac{w^N}{1-w}$$

Thus, if $w \neq 1$, then

$$\frac{1}{1-w} = \sum_{n=0}^{N-1} w^n + \frac{w^N}{1-w}$$

where $N \geq 1$.

$w = \frac{z}{\varsigma} \neq 1$ because $z \neq \varsigma$ when ς is on C_0

Hence,

$$\frac{1}{\varsigma - z} = \left(\frac{1}{\varsigma}\right)\left(\frac{1}{1 - \frac{z}{\varsigma}}\right)$$

$$= \left(\frac{1}{\varsigma}\right)\left(\left[\sum_{n=0}^{N-1}\left(\frac{z}{\varsigma}\right)^n\right] + \frac{\left(\frac{z}{\varsigma}\right)^N}{1 - z/\varsigma}\right)$$

$$= \left[\sum_{n=0}^{N-1}\left(\frac{1}{\varsigma^{n+1}}\right) z^n\right] + z^N \frac{1}{(\varsigma - z)\varsigma^N}$$

Thus,

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \left(\int_{C_0} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \right) z^n$$

$$+ \frac{z^N}{2\pi i} \int_{C_0} \frac{f(\zeta)}{(\zeta - z)\zeta^N}\, d\zeta$$

By Cauchy's integral formula

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta = \frac{f^{(n)}(0)}{n!}$$

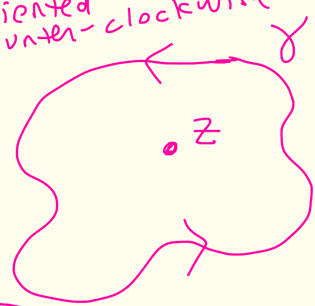
4680

γ is simple, closed, piecesmooth curve oriented counter-clockwise

f analytic in and on γ.

z interior to γ

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$



Cauchy integral formula

Thus,

$$\boxed{f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z)}$$

where

$$\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(\zeta)}{(\zeta - z)\zeta^N} d\zeta$$

We will now show that

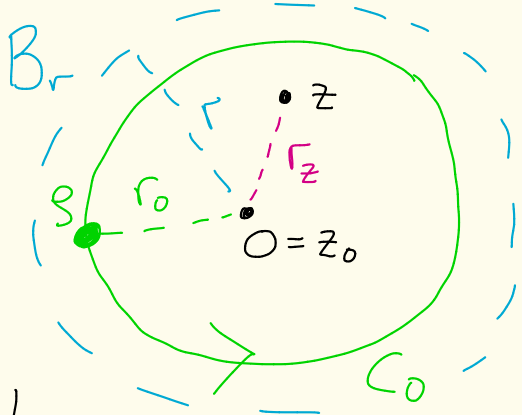
$$\rho_N(z) \to 0 \quad \text{as} \quad N \to \infty.$$

If we can show that then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

Let's show $P_N(z) \to 0$ as $N \to \infty$

Let $r_z = |z|$.



If ρ is on C_0, then

$$|\rho - z| \overset{\boxed{\text{4680 formula}}}{\geq} \big| |\rho| - |z| \big| = |r_0 - r_z| \underset{\boxed{r_0 > r_z}}{=} r_0 - r_z$$

By max-modulus thm (4680) or topology (since f is continuous on the compact set C_0) there exists $M > 0$ where

$$|f(\rho)| \leq M$$

for all ρ on C_0.

Then,

$$|\rho_N(z)| = \left| \frac{z^N}{2\pi i} \int_{C_0} \frac{f(\xi)}{(\xi - z)\xi^N}\, d\xi \right|$$

$$\underset{\uparrow\ \boxed{|i|=1}}{=} \frac{|z|^N}{2\pi} \left| \int_{C_0} \frac{f(\xi)}{(\xi - z)\xi^N}\, d\xi \right|$$

$$\leq \frac{r_z^N}{2\pi} \cdot \frac{M}{(r_0 - r_z) r_0^N} \cdot \underbrace{2\pi r_0}_{\text{arclength of } C_0}$$

$$\boxed{\begin{gathered} |f(\xi)| \leq M \\ \frac{1}{|\xi - z|} \leq \frac{1}{r_0 - r_z} \\ |\xi|^N = r_0^N \\ \left| \frac{f(\xi)}{(\xi - z)\xi^N} \right| \leq \frac{M}{(r_0 - r_z) r_0^N} \end{gathered}}$$

$$= \left(\frac{M r_0}{r_0 - r_z} \right) \underbrace{\left(\frac{r_z}{r_0} \right)^N}_{\text{goes to } 0} \longrightarrow 0$$

as $N \to \infty$ because $0 < \frac{r_z}{r_0} < 1$.

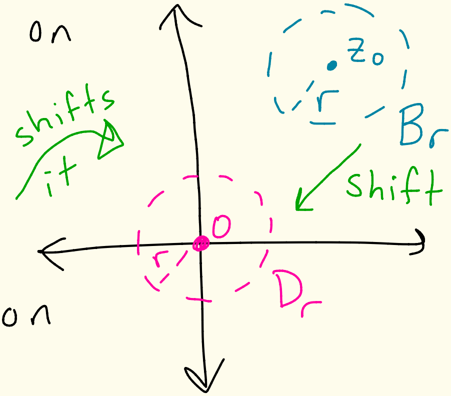
Thus, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$.

This concludes $z_0 = 0$ case.

Now we prove the thm when z_0 is arbitrary.

Suppose f is analytic on

$$B_r = \{ z \mid |z - z_0| < r \}.$$



Let $\boxed{g(z) = f(z + z_0)}$

Then, g is analytic on

$$D_r = \{ z \mid |z| < r \}.$$

Thus, by the $z_0 = 0$ case we know

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \quad \text{for all } z \in D_r$$

Then,

$$f(z + z_0) = g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

for all $z \in D_r$

$$\boxed{\begin{aligned} g^{(n)}(z) &= f^{(n)}(z + z_0) \\ g^{(n)}(0) &= f^{(n)}(0 + z_0) \end{aligned}}$$

Now sub $z - z_0$ for z to get

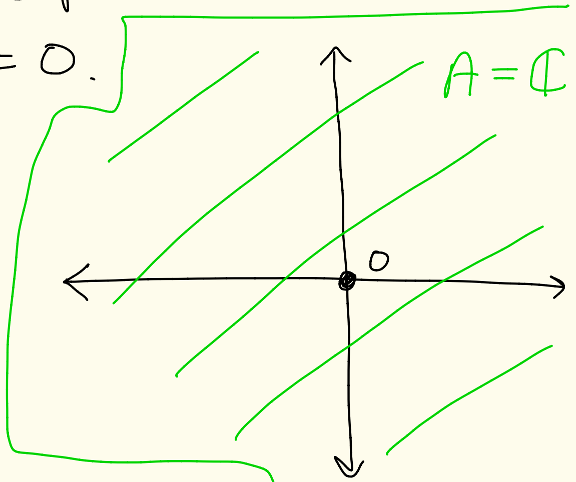
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad \text{for all } z \in B_r$$

▨

Ex: $f(z) = e^z$

f is analytic everywhere

Let's look at the power series centered at $z_0 = 0$.



$f(z) = e^z$

$f'(z) = e^z$

$f''(z) = e^z$

$\vdots \quad \vdots$

$f^{(n)}(z) = e^z$

$f^{(n)}(0) = e^0 = 1$

f analytic on $A = \mathbb{C}$

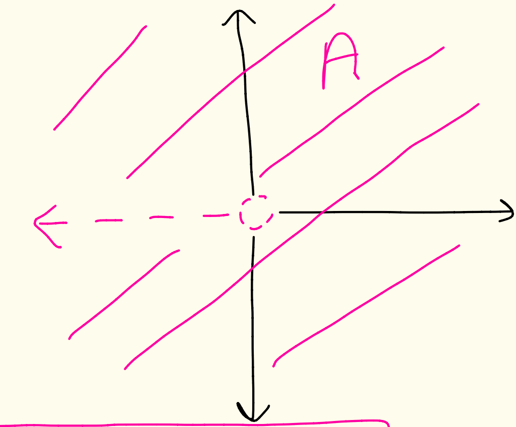
So the Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z-0)^k = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

By Taylors Thm, since $f(z) = e^z$ is analytic on all of $\mathbb{C}$, the series converges to f on all of $\mathbb{C}$.

Ex: Let $f(z) = \log(1+z)$ where we use the principal branch of log.

$\boxed{\text{4680 Recap}}$



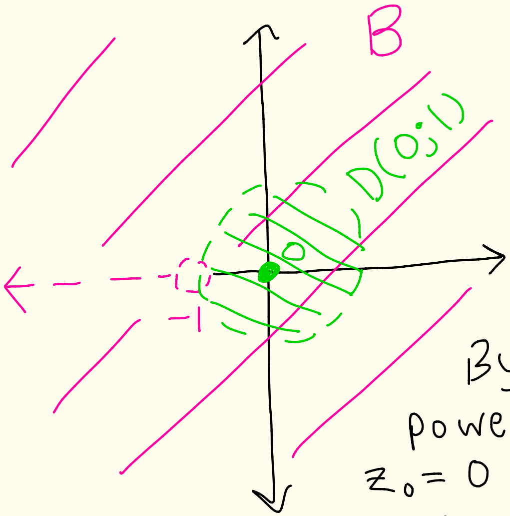
$\log(w) = \ln|w| + i\arg(w)$

where $-\pi < \arg(w) < \pi$ (principal branch)

log is analytic on

$A = \mathbb{C} - \left\{ x+iy \;\middle|\; \begin{matrix} x \le 0 \;\& \\ y = 0 \end{matrix} \right\}$

$\boxed{\frac{d}{dz}\log(z) = \frac{1}{z}}$



$f(z) = \log(1+z)$ is analytic on

$B = \mathbb{C} - \left\{ x+iy \;\middle|\; \begin{matrix} x \le -1 \;\& \\ y = 0 \end{matrix} \right\}$

By Taylor's thm the power series centered at $z_0 = 0$ will converge to $f(z)$ on $D(0;1)$

38

$$f(z) = \log(1+z)$$

$$f'(z) = (1+z)^{-1}$$

$$f''(z) = -(1+z)^{-2}$$

$$f'''(z) = 2(1+z)^{-3}$$

$$f''''(z) = -3!\,(1+z)^{-4}$$

$$f^{(5)}(z) = 4!\,(1+z)^{-5}$$

$$\vdots \qquad \vdots$$

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{(1+z)^k}$$

$k=0$

$f(0)=\log(1)=0$

$f^{(k)}(0) =$

$= (-1)^{k-1}(k-1)!$

$k \geq 1$

Thus, on $D(0;1)$ we know

$$\log(1+z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(z-0)^k$$

$k! = k\cdot(k-1)!$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k\cdot[(k-1)!]} z^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$$

Ex:

For all $z \in \mathbb{C}$ one can show using Taylor's formula that

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Theorem: Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ be power series with the same center z_0 and radii of convergence $R_1 > 0$ and $R_2 > 0$ respectively. Let $R = \min\{R_1, R_2\}$.

Let $c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$

Then, $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ has radius of convergence $\geq R$ and inside this circle of convergence we have

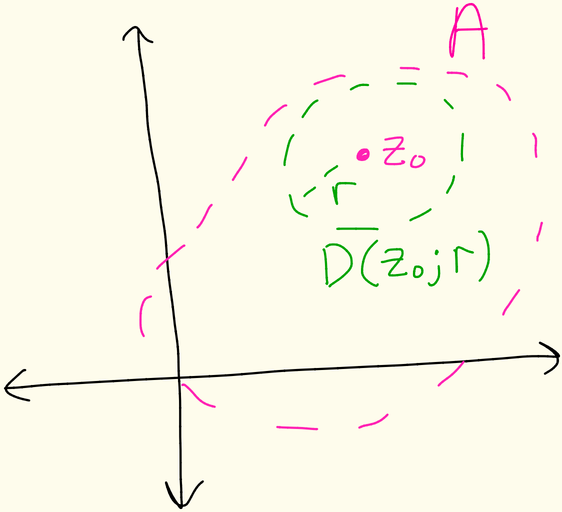
$$\left(\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-z_0)^n\right) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

Hoffman Marsden book page 215

Discussion of the zeros of an analytic function

Suppose that $f : A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is an open set.
Suppose f is analytic on A and $z_0 \in A$ with $f(z_0) = 0$.

Let $r > 0$ be such that $D(z_0; r) \subseteq A$.



By Taylor's Theorem

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$= \underbrace{}_{f(z_0)=0} f^{(1)}(z_0)(z-z_0) + \frac{f^{(2)}(z_0)}{2!}(z-z_0)^2 + \cdots$$

for all $z \in D(z_0; r)$

Case 1: Suppose $f^{(k)}(z_0) = 0 \quad \forall k \geq 1$

Then, $f(z) = 0$ on all of $D(z_0; r)$

Case 2: Otherwise there exists a smallest positive integer n with $f^{(n)}(z_0) \neq 0$

Then for $z \in D(z_0; r)$ we have

$$f(z) = \underbrace{\frac{f^{(n)}(z_0)}{n!}}_{\text{first non-zero term}} (z - z_0)^n + \frac{f^{(n+1)}(z_0)}{(n+1)!} (z - z_0)^{n+1} + \cdots$$

$$= (z - z_0)^n \left[\underbrace{\frac{f^{(n)}(z_0)}{n!}}_{\text{not zero}} + \frac{f^{(n+1)}(z_0)}{(n+1)!} (z - z_0) + \cdots \right]$$

So, for $z \in D(z_0; r) - \{z_0\}$

$$\frac{f(z)}{(z-z_0)^n} = \frac{f^{(n)}(z_0)}{n!} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0) + \cdots$$

Let

$$\varphi(z) = \frac{f^{(n)}(z_0)}{n!} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0) + \cdots \quad (*)$$

Then the power series on the right of $(*)$ converges for all $z \in D(z_0; r) - \{z_0\}$ and it also converges at z_0 since $\varphi(z_0) = \dfrac{f^{(n)}(z_0)}{n!}$

Thus, $(*)$ holds on $D(z_0; r)$.

Since power series are analytic functions we know that $\varphi(z)$ is analytic on $D(z_0; r)$

Also,

$$\varphi(z_0) = \frac{f^{(n)}(z_0)}{n!} + \underbrace{\frac{f^{(n+1)}(z_0)}{(n+1)!}(z_0 - z_0) + \cdots}_{0}$$

$$= \frac{f^{(n)}(z_0)}{n!} \neq 0.$$

Thus, in case 2,

$$f(z) = (z - z_0)^n \varphi(z)$$

where φ is analytic on $D(z_0; r)$ and $\varphi(z_0) \neq 0$.

In this case we say that f has a <u>zero of order n at z_0</u>

Ex: $f(z) = 1-\cos(z^5)$, $z_0 = 0$

Then, $f(0) = 1-\cos(0^5) = 1-1 = 0$

Then for all $z \in \mathbb{C}$ we have

$$f(z) = 1-\cos(z^5)$$

$$= 1 - \underbrace{\left[1 - \frac{(z^5)^2}{2!} + \frac{(z^5)^4}{4!} - \frac{(z^5)^6}{6!} + \cdots \right]}$$

$\cos(w) = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \cdots$ $\forall w \in \mathbb{C}$

$$= \frac{z^{10}}{2!} - \frac{z^{20}}{4!} + \frac{z^{30}}{6!} - \cdots$$

$$= z^{10} \left[\underbrace{\frac{1}{2!} - \frac{z^{10}}{4!} + \frac{z^{20}}{6!} - \cdots}_{\varphi(z)} \right]$$

$= z^{10}\varphi(z)$ where φ is analytic at 0 and $\varphi(0) = \frac{1}{2} \neq 0$. So, f has a zero of order 10 at $z_0 = 0$.

Ex: $f(z) = e^{(z-1)^2} - 1$, $z_0 = 1$

$$f(1) = e^{(1-1)^2} - 1 = e^0 - 1 = 1 - 1 = 0$$

$$\boxed{e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} \quad \forall w \in \mathbb{C}}$$

For all $z \in \mathbb{C}$ we have

$$f(z) = -1 + e^{(z-1)^2}$$

$$= -1 + \left[1 + \frac{(z-1)^2}{1!} + \frac{((z-1)^2)^2}{2!} + \frac{((z-1)^2)^3}{3!} + \cdots\right]$$

$$= (z-1)^2 + \frac{(z-1)^4}{2!} + \frac{(z-1)^6}{3!} + \cdots$$

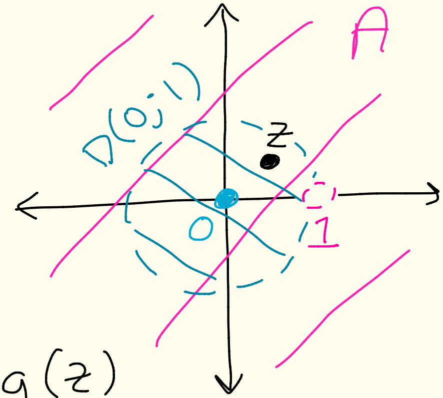
$$= (z-1)^2\left[1 + \frac{(z-1)^2}{2!} + \frac{(z-1)^4}{3!} + \cdots\right]$$

$$= (z-1)^2 \varphi(z)$$ where φ is analytic at $z_0 = 1$ and $\varphi(1) = 1 \neq 0$.

So, f has a zero of order 2 at $z_0 = 1$.

Ex: $g(z) = \frac{z}{z-1}$, $z_0 = 0$

Then, $g(0) = \frac{0}{0-1} = 0$



g is analytic on

$A = \mathbb{C} - \{1\}$

By Taylor's theorem the power series for $g(z)$ will converge on $D(0;1)$ because $D(0;1) \subseteq A$.

Let $z \in D(0;1)$. So, $|z| < 1$.

$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$

$|w| < 1$

Then,

$$g(z) = \frac{z}{z-1} = -z\left[\frac{1}{1-z}\right]$$

$$\underset{|z|<1}{=} -z\left[1 + z + z^2 + z^3 + \cdots\right]$$

$$= z\left[-1 - z - z^2 - z^3 - \cdots\right] = z\varphi(z)$$

where φ is analytic at 0 and $\varphi(0) = -1$. So, g has a zero of order 1 at $z_0 = 0$

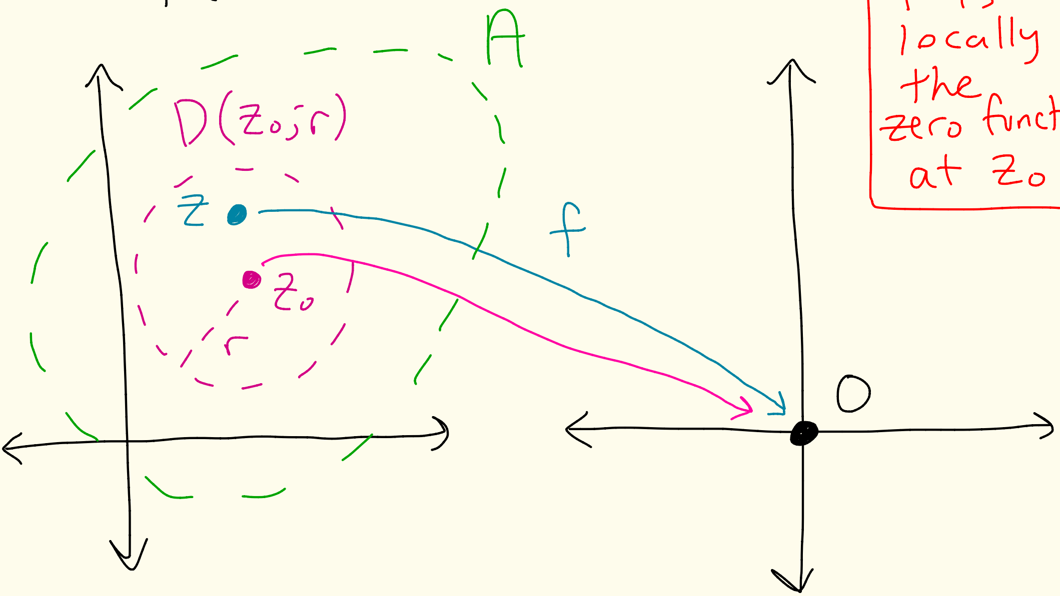
Theorem: (Isolation of zeros of analytic function)

Suppose that $f: A \to \mathbb{C}$ is analytic on an open set $A \subseteq \mathbb{C}$. And suppose $f(z_0) = 0$ at $z_0 \in A$.

Then either:

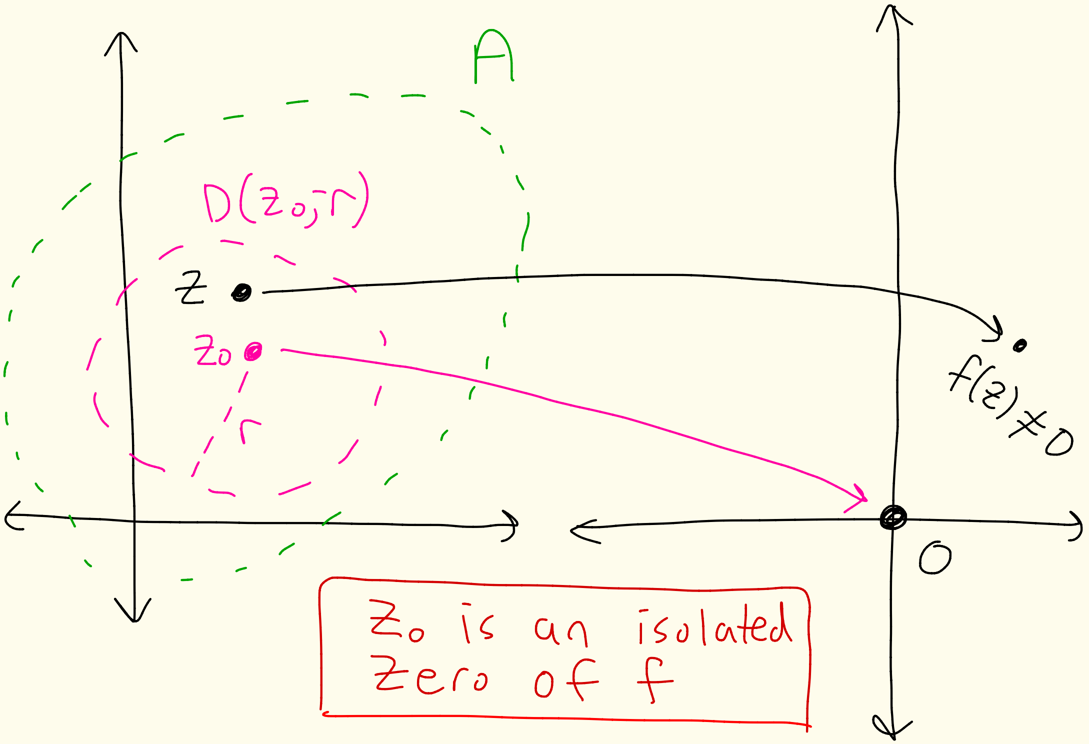
(1) There exists $r > 0$ with $D(z_0; r) \subseteq A$ where $f(z) = 0$ for all $z \in D(z_0; r)$

f is locally the zero function at z_0



OR

(2) there is an $r > 0$ such that $D(z_0; r) \subseteq A$ where $f(z) \neq 0$ for all $z \in D(z_0; r) - \{z_0\}$



$\boxed{z_0 \text{ is an isolated zero of } f}$

proof: HW ☐