Topic 4.5 – Application to Pythagorean Triples

Application to Pythagorean Triples

Consider the equation

$$\boxed{x^2 + y^2 = z^2}$$

This equation has integer solutions. For example, $x = 3, y = 4, z = 5$.

You can see there any infinitely many solutions. For example, if you set $x = 3k, y = 4k, z = 5k$ were $k \in \mathbb{Z}$, then

$$x^2 + y^2 = (3k)^2 + (4k)^2$$
$$= k^2[3^2 + 4^2] = k^2 5^2$$
$$= (5k)^2 = z^2$$

k	$x=3k$	$y=4k$	$z=5k$
1	3	4	5
2	6	8	10
-1	-3	-4	-5
10	30	40	50

Some examples
all (3,4,5) scaled

You can change the signs and get new answers:

(3,4,5), (-3,4,5), (3,-4,-5), ...
(6,8,10), (6,-8,10), (-6,-8,-10), ...

are all solutions to $x^2+y^2=z^2$

So there are infinitely many sols.

The answers we've found so far come from scaling (3,4,5) and changing signs.

You can also get some easy solutions by setting x or y to be 0.

$(x,y,z) = (2,0,2)$ is a sol. to $x^2+y^2=z^2$

<u>Def:</u> We call (x,y,z) a <u>Pythagorean triple</u> if $x,y,z \in \mathbb{Z}$, $(x,y,z) \neq (0,0,0)$, and $x^2+y^2=z^2$.

If (x,y,z) is a Pythagorean triple, we say that it is <u>positive</u> if $x>0$, $y>0$, $z>0$.

<u>Ex:</u> $(3,4,5)$ is a positive Pythagorean triple.

$(0,2,-2)$ and $(25,-60,65)$ are Pythagorean triples

<u>Ex:</u> $(25, 60, -65)$ is a Pythagorean triple because

$$(25)^2 + (60)^2 = (-65)^2$$

Let $d = \gcd(25, 60, -65) = 5$

Then,

$$\begin{aligned}(25, 60, -65) &= (5 \cdot 5,\ 5 \cdot 12,\ -5 \cdot 13) \\ &= (d \cdot 5,\ d \cdot 12,\ -d \cdot 13)\end{aligned}$$

and $(5, 12, 13)$ is a positive Pythagorean triple with $\gcd(5, 12, 13) = 1$

So, $(25, 60, -65)$ is a multiple of $(5, 12, 13)$ with a sign change in the z-spot and $\gcd(5, 12, 13) = 1$.

<u>Def:</u> We say that a Pythagorean triple (x,y,z) is <u>primitive</u> if $\gcd(x,y,z)=1$.

<u>Theorem:</u> Any Pythagorean triple is of the form $(\pm da, \pm db, \pm dc)$ where a,b,c are non-negative integers and d is a positive integer and (a,b,c) is a primitive Pythagorean triple [that is, $\gcd(a,b,c)=1$ and $a^2+b^2=c^2$]

<u>Ex:</u> $(9,-12,-15)$ ← Pythagorean triple

$$= (3\cdot 3, -3\cdot 4, -3\cdot 5)$$

$$= (d\cdot a, -d\cdot b, -d\cdot c)$$

$d=3$, $(a,b,c)=(3,4,5)$

proof of theorem:

Let (x,y,z) be a Pythagorean triple. Then $x^2+y^2=z^2$ and $(x,y,z)\neq(0,0,0)$.

Let $d=\gcd(x,y,z)$.

From class, $\boxed{\gcd\left(\frac{x}{d},\frac{y}{d},\frac{z}{d}\right)=1}$

Set

$$a=\left|\frac{x}{d}\right|,\quad b=\left|\frac{y}{d}\right|,\quad c=\left|\frac{z}{d}\right|.$$

Then,

$$(x,y,z)=(\pm da,\pm db,\pm dc)$$

The +/− choice depends on the sign of x,y,z.

And $a\geq 0$, $b\geq 0$, $c\geq 0$ with

$$\gcd(a,b,c)=\gcd\left(\pm\frac{x}{d},\pm\frac{y}{d},\pm\frac{z}{d}\right)=1$$

And

$$a^2+b^2=\left(\pm\frac{x}{d}\right)^2+\left(\pm\frac{y}{d}\right)^2=\frac{x^2+y^2}{d^2}=\frac{z^2}{d^2}=\left(\pm\frac{z}{d}\right)^2=c^2 \quad \blacksquare$$

Summary:

There are three kinds of Pythagorean triples.

① $(x, 0, x)$

$$x^2 + 0^2 = x^2$$

Ex: $(3, 0, 3)$

② $(0, y, y)$

$$0^2 + y^2 = y^2$$

Ex: $(0, -5, -5)$

③ The ones that are multiples of positive primitive triples with possible sign adjustments.

Ex of 3: $(x, y, z) = (25, 60, -65)$

$$(5, 12, 13) \xrightarrow{\times 5} (25, 60, 65) \xrightarrow[\text{adjustment}]{\text{sign}} (25, 60, -65)$$

positive primitive triple

<u>New goal:</u> Find a formula that generates all the positive, primitive Pythagorean triples, ie all (x, y, z) where $x^2 + y^2 = z^2$ where $x > 0$, $y > 0$, $z > 0$ and $\gcd(x, y, z) = 1$.

Last time we reduced the problem of finding all pythagorean triples to the problem of finding all positive, primitive Pythagorean triples.

(x,y,z) where $x>0$, $y>0$, $z>0$, $\gcd(x,y,z)=1$ and $x^2+y^2=z^2$

Ex: $(x,y,z)=(3,4,5)$

$(x,y,z)=(15,8,17)$

$(x,y,z)=(5,12,13)$

$(x,y,z)=(7,24,25)$

Suppose that (x, y, z) is a primitive, positive Pythagorean triple.

- Suppose x and y are both even. Then, in $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ we have $\bar{x} = \bar{0}$ and $\bar{y} = \bar{0}$.

Then,

$$\bar{z}^2 = \bar{x}^2 + \bar{y}^2 = \bar{0}^2 + \bar{0}^2 = \bar{0}$$

$z^2 = x^2 + y^2$

So, $\overline{(z^2)} = \overline{(x^2 + y^2)}$

Then, $\bar{z}^2 = \bar{x}^2 + \bar{y}^2$

So, $\bar{z}^2 = \bar{x}^2 + \bar{y}^2$

$\bar{z} = \bar{0}$ or $\bar{z} = \bar{1}$

but $\bar{1}^2 = \bar{1}$

So, $\bar{z} = \bar{0}$

So, $\bar{z} = \bar{0}$

Then, z is even.

But then $2|x$, $2|y$, $2|z$.
So, $\gcd(x,y,z) \geq 2$.
This contradicts $\gcd(x,y,z)=1$.
Thus, we cannot have x and y both being even.

- Suppose x and y are both odd.

In $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$
recall that a is odd
iff $\bar{a}=\bar{1}$ or $\bar{a}=\bar{3}$

previous class

Note that in $\mathbb{Z}_4$,
$\bar{1}^2 = \bar{1}$ and $\bar{3}^2 = \bar{9} = \bar{1}$.

Thus,
$$\bar{z}^2 = \bar{x}^2 + \bar{y}^2 = \bar{1} + \bar{1} = \bar{2}$$

But $\overline{z}^2 = \overline{2}$ is not possible in $\mathbb{Z}_4$ by the following table

$\overline{z}$	$\overline{z}^2$
$\overline{0}$	$\overline{0}^2 = \overline{0}$
$\overline{1}$	$\overline{1}^2 = \overline{1}$
$\overline{2}$	$\overline{2}^2 = \overline{4} = \overline{0}$
$\overline{3}$	$\overline{3}^2 = \overline{9} = \overline{1}$

Calculations are in $\mathbb{Z}_4$

Contradiction.

Thus, x and y cannot both be odd.

Thus, either

x is odd and y is even

or

x is even and y odd.

Since our equation $x^2+y^2=z^2$ is symmetric with x and y we are going to solve the case where y is even and x is odd, Let us assume now that $\boxed{y \text{ is even and } x \text{ is odd.}}$

Then in $\mathbb{Z}_2$,

$$\bar{z}^2 = \bar{x}^2 + \bar{y}^2 = \bar{1}^2 + \bar{0}^2 = \bar{1}$$

As before this implies $\bar{z} = \bar{1}$ in $\mathbb{Z}_2$

So, $\boxed{z \text{ is odd.}}$

Since x is odd and z is odd we know $z - x$ is even and $z + x$ is even.

Since $x^2+y^2=z^2$ we have

$$y^2=z^2-x^2.$$

Thus, $y^2=(z+x)(z-x)$

So, $\frac{y^2}{4}=\left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right)$

Thus, $\boxed{\left(\frac{y}{2}\right)^2=\left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right)}$ (*)

these are integers because
$2 \mid z+x$
$2 \mid z-x$
$2 \mid y$

Any common divisor of $\frac{z+x}{2}$ and $\frac{z-x}{2}$ must divide their sum

$$\frac{z+x}{2}+\frac{z-x}{2}=z$$

and their difference

$$\frac{z+x}{2}-\frac{z-x}{2}=x.$$

Note that $\gcd(x,z)=1$.

Why? In HW, $\gcd(x,z)\neq 1$ iff there is a prime p with $p \mid x$ and $p \mid z$

Suppose $\gcd(x,z)\neq 1$.
Then by HW there is a prime p with $p \mid x$ and $p \mid z$.
Then, $x = pk_1$, $z = pk_2$ where $k_1, k_2 \in \mathbb{Z}$.
So, $y^2 = z^2 - x^2 = p^2k_2^2 - p^2k_1^2$
$= p[pk_2^2 - pk_1^2]$
Then, $p \mid y^2$. So, $p \mid y \cdot y$.
Since p is prime, $p \mid y$.
But then $p \mid x$, $p \mid y$, $p \mid z$.
So, $\gcd(x,y,z) \geq p$. Contradiction.

Because any common divisor of $\frac{z+x}{2}$ and $\frac{z-x}{2}$ is a common divisor of z and x, and $\gcd(x,z)=1$ we know $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right)=1$.

Recall this thm: If A, B, C are positive integers and $\gcd(A,B)=1$ and $AB=C^n$ then there exist positive integers R and S with $A=R^n$ and $B=S^n$ (from 2/22)

Our situation is (from (*))

$$\left(\frac{y}{2}\right)^2 = \left(\frac{z+x}{2}\right)\left(\frac{z-x}{2}\right) \quad \leftarrow \boxed{C^2=AB}$$

with $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right)=1 \quad \leftarrow \boxed{\gcd(A,B)=1}$

Hence, $\frac{z+x}{2} = r^2$ and $\frac{z-x}{2} = s^2$

where r, s are positive integers

and $\gcd(r,s) = 1$ ← because $\gcd\left(\frac{z+x}{2}, \frac{z-x}{2}\right) = 1$

So, $\left(\frac{y}{2}\right)^2 = r^2 s^2$.

Then, $\boxed{\frac{y}{2} = rs.}$

Note that $r^2 = \frac{z+x}{2} > \frac{z-x}{2} = s^2$.

So, $r > s$.

Also, since $\boxed{z \text{ is odd}}$ and

$z = \frac{z+x}{2} + \frac{z-x}{2} = r^2 + s^2$

we must also have that r and s have <u>opposite parity</u> (ie one is odd and the other is even)

see by $\mathbb{Z}_2$

$\bar{r}$	$\bar{s}$	$\bar{r}^2 + \bar{s}^2$	
$\bar{0}$	$\bar{0}$	$\bar{0}$	✗
$\bar{0}$	$\bar{1}$	$\bar{1}$	
$\bar{1}$	$\bar{0}$	$\bar{1}$	
$\bar{1}$	$\bar{1}$	$\bar{0}$	✗

<u>Theorem:</u> If (x,y,z) is a positive, primitive Pythagorean triple, with y even, then

$$x = r^2 - s^2$$
$$y = 2rs$$
$$z = r^2 + s^2$$

Where r and s are positive integers of opposite parity and $r > s > 0$ and $\gcd(r,s) = 1$.

s	r	$x = r^2 - s^2$	$y = 2rs$	$z = r^2 + s^2$
1	2	3	4	5
1	4	15	8	17
1	6	35	12	37
1	8	63	16	65
2	3	5	12	13
2	5	21	20	29
2	7	45	28	53
3	4	7	24	25
3	8	55	48	73
⋮	⋮	⋮	⋮	⋮

For fun (Pythagorean triples)

Here's another way to generate all the primitive, positive Pythagorean triples using matrices.

From this paper:

"Genealogy of Pythagorean Triads"
by A. Hall
The Mathematical Gazette
Vol. 54
No. 390
Dec 1970
pg. 377 – 379

Let $A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}$,

$B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}$

$(3,4,5)$ is a positive, primitive Pythagorean triple.

$$A\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3-8+10 \\ 6-4+10 \\ 6-8+15 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 13 \end{pmatrix}$$

$$B\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3+8+10 \\ 6+4+10 \\ 6+8+15 \end{pmatrix} = \begin{pmatrix} 21 \\ 20 \\ 29 \end{pmatrix}$$

positive primitive Pythagorean triples

$$C\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -3+8+10 \\ -6+4+10 \\ -6+8+15 \end{pmatrix} = \begin{pmatrix} 15 \\ 8 \\ 17 \end{pmatrix}$$

All positive primitive Pythagorean triples are in the following infinite tree.