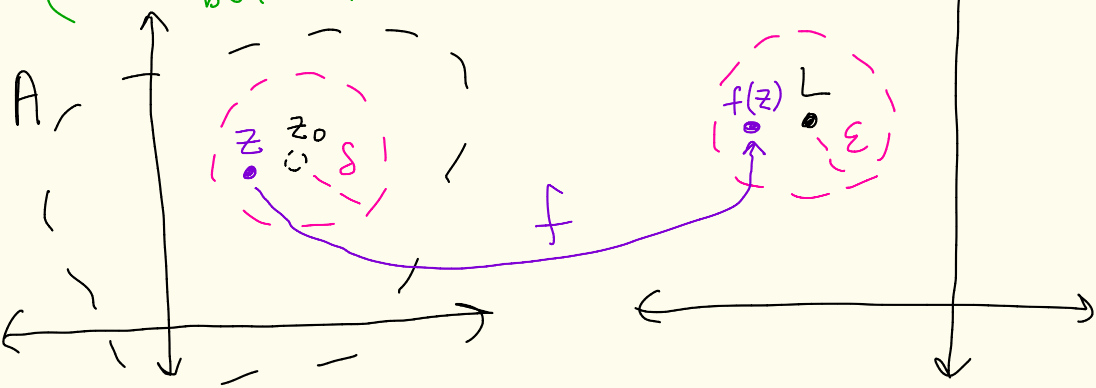
Topic 4 -
Limits

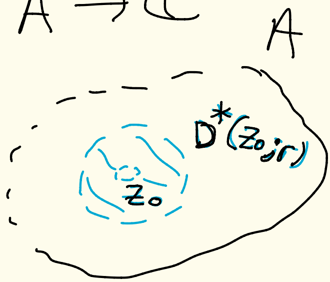
Def: Let $f : A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$. Let $z_0 \in \mathbb{C}$ where $D^*(z_0; r) \subseteq A$ for some $r > 0$ [that is, f is defined on some deleted r-neighborhood of z_0]. We say that <u>f has limit L as z approaches z_0</u>, and write $\lim_{z \to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $z \in A$ and $\underbrace{0 < |z - z_0| < \delta}_{\text{(z is within δ of z_0 but $z \neq z_0$)}}$, then $\underbrace{|f(z) - L| < \varepsilon}_{f(z) \in D(L;\varepsilon)}$



<u>Theorem:</u> If $L_1 = \lim_{z \to z_0} f(z)$ and $L_2 = \lim_{z \to z_0} f(z)$, then $L_1 = L_2$.

<u>pf:</u> HW. ▨

<u>Theorem:</u> Suppose $A \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$ with $D^*(z_0; r) \subseteq A$ for some $r > 0$. Suppose $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C}$.



Suppose $\lim_{z \to z_0} f(z) = F$ and $\lim_{z \to z_0} g(z) = G$.

Then:

① $\lim_{z \to z_0} [f(z) + g(z)] = F + G$

② $\lim_{z \to z_0} \alpha f(z) = \alpha F$ where $\alpha \in \mathbb{C}$

③ $\lim_{z \to z_0} f(z) g(z) = FG$

④ $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{F}{G}$ if $G \neq 0$

pf ① / ②:

Let's show

$$\lim_{z \to z_0} \alpha f(z) + \beta g(z) = \alpha F + \beta G$$

if $\alpha, \beta \in \mathbb{C}$.

Let $\varepsilon > 0$.

Note that

$$\begin{aligned}
&|\alpha f(z) + \beta g(z) - (\alpha F + \beta G)| \\
&= |\alpha f(z) - \alpha F + \beta g(z) - \beta G| \\
&\leq |\alpha f(z) - \alpha F| + |\beta g(z) - \beta G| \\
&= |\alpha| \underbrace{|f(z) - F|} + |\beta| \underbrace{|g(z) - G|}
\end{aligned}$$

we can control how small these are

I want to do something like this

$$< |\alpha| \frac{\varepsilon}{2|\alpha|} + |\beta| \frac{\varepsilon}{2|\beta|} = \varepsilon$$

← this idea won't work if α or β is 0

$$\leq (|\alpha| + 1)|f(z) - F| + (|\beta| + 1)|g(z) - G|$$

Since $\lim\limits_{z\to z_0} f(z) = F$, there exists $\delta_1 > 0$ where if $z \in A$ and $0<|z-z_0|<\delta_1$ then $|f(z)-F| < \dfrac{\varepsilon}{2(|\alpha|+1)}$

Since $\lim\limits_{z\to z_0} g(z) = G$, there exists $\delta_2 > 0$ where if $z \in A$ and $0<|z-z_0|<\delta_2$ then $|g(z)-G| < \dfrac{\varepsilon}{2(|\beta|+1)}$.

So, if $z \in A$ and $0<|z-z_0|< \underbrace{\min\{\delta_1,\delta_2\}}_{\text{minimum of } \delta_1 \text{ and } \delta_2}$

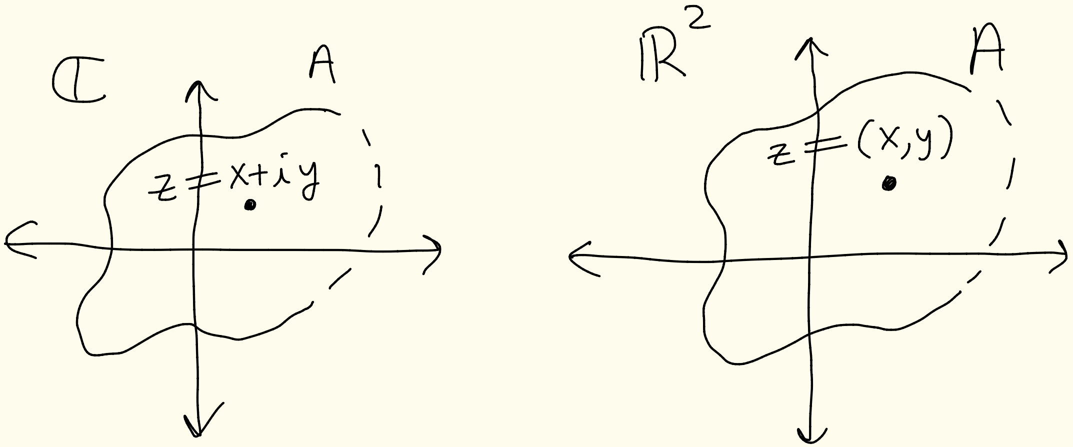
then

$$
\begin{aligned}
&|\alpha f(z)+\beta g(z) - (\alpha F + \beta G)| \\
&\leq (|\alpha|+1)|f(z)-F| + (|\beta|+1)|g(z)-G| \\
&< (|\alpha|+1)\frac{\varepsilon}{2(|\alpha|+1)} + (|\beta|+1)\frac{\varepsilon}{2(|\beta|+1)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

So, $\lim\limits_{z\to z_0}(\alpha f(z)+\beta g(z)) = \alpha F + \beta G$.

<u>Note:</u> Suppose $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$. Let $z \in A$ and $z = x + iy$.

Can think in two ways:



We will sometimes go back and forth.
So we can write

$$f(z) = f(x+iy) = f(x,y)$$
$$= u(x,y) + i\,v(x,y)$$

where $u, v : A \to \mathbb{R}^2$
where here we think $A \subseteq \mathbb{R}^2$

Ex: Let $f(z) = z^2$.

If $z = x + iy$, then

$$f(z) = f(x+iy) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y) = x^2-y^2} + i\,\underbrace{2xy}_{v(x,y) = 2xy}$$

☆☆☆☆☆☆☆☆☆☆

☆ Maybe skip this def below and dont prove next theorem and post proof online

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$\boxed{\text{Calc III limits}}$ (kind-of)

Let $g : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}^2$.

Let $(x_0, y_0) \in \mathbb{R}^2$ where $D^*((x_0,y_0); r) \subseteq A$ for some $r > 0$.

We say that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < \underbrace{|(x,y) - (x_0,y_0)|}_{\sqrt{(x-x_0)^2 + (y-y_0)^2}} < \delta$$

then $|g(x,y) - L| < \varepsilon$

Theorem: Suppose $A \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$ and $D^*(z_0; r) \subseteq A$ for some $r > 0$. Suppose $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

Then:

$$(1) \quad \lim_{z \to z_0} f(z) = \lim_{x+iy \to x_0+iy_0} f(z) = u_0 + i v_0$$

complex limit

if and only if

(2) $\mathbb{R}^2$ limits

$$(2) \quad \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0.$$

Ex:

$x+iy = 1+i$
$(x,y) = (1,1)$

$$\lim_{z \to 1+i} z^2 = \lim_{x+iy \to 1+i} \left[(x^2-y^2) + i\, 2xy \right]$$

$$= \lim_{(x,y)\to(1,1)} \left[x^2-y^2\right] + i \lim_{(x,y)\to(1,1)} \left[2xy\right]$$

$$= \left[1^2 - 1^2\right] + i\left[2(1)(1)\right] = 2i$$

<u>proof:</u> You can try $(1) \Rightarrow (2)$.

<u>$(2) \Rightarrow (1)$</u>

Let $\varepsilon > 0$.

Suppose $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

So there exist $\delta_1 > 0$ so that if

$$\underbrace{0 < \sqrt{(x-x_0)^2+(y-y_0)^2}}_{0 < |(x,y)-(x_0,y_0)| < \delta_1} < \delta_1 \quad \text{and } (x,y) \in A$$

then $|u(x,y) - u_0| < \frac{\varepsilon}{2}$.

And there exists $\delta_2 > 0$ so that if

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2 \quad \text{and } (x,y) \in A$$

then $|v(x,y) - v_0| < \frac{\varepsilon}{2}$.

Note: $\mathbb{R}^2$	$\mathbb{C}$
$\sqrt{(x-x_0)^2+(y-y_0)^2} = \lvert (x,y)-(x_0,y_0) \rvert$	$\sqrt{(x-x_0)^2+(y-y_0)^2} = \lvert z - z_0 \rvert$, $z = x+iy$, $z_0 = x_0+iy_0$

So if $z \in A$ and

$0 < \underbrace{|z - z_0|}_{\sqrt{(x-x_0)^2+(y-y_0)^2}} < \min\{\delta_1, \delta_2\}$ then

$$
\begin{aligned}
&|f(x,y) - (u_0 + i v_0)| \\
&= |u(x,y) + i v(x,y) - u_0 - i v_0| \\
&= |(u(x,y) - u_0) + i(v(x,y) - v_0)| \\
&\leq |u(x,y) - u_0| + |i(v(x,y) - v_0)| \\
&= |u(x,y) - u_0| + \underbrace{|i|}_{1}\, |v(x,y) - v_0| \\
&= |u(x,y) - u_0| + |v(x,y) - v_0| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

So,

$$\lim_{z \to z_0} f(z) = u_0 + i v_0 \qquad \blacksquare$$

Continuity

Def: Let $A \subseteq \mathbb{C}$ where A is an open set and $f : A \to \mathbb{C}$.
We say that <u>f is continuous at $z_0 \in A$</u> if $\lim_{z \to z_0} f(z)$ exists

and $\lim_{z \to z_0} f(z) = f(z_0)$.

We say that <u>f is continuous on A</u> if f is continuous at all z_0 in A.

Ex: $f(z) = z^2$ and $z_0 = 2 - i$

Let $z = x + iy$. Then,

$$\lim_{z \to 2-i} z^2 = \lim_{x+iy \to 2-i} (x+iy)^2 = \lim_{x+iy \to 2-i} \left[(x^2-y^2) + i\,2xy\right]$$

$=$ (next page)

$$= \underbrace{\lim_{(x,y)\to(2,-1)} (x^2-y^2)}_{\mathbb{R}^2 \text{ limit}} + i \underbrace{\lim_{(x,y)\to(2,-1)} 2xy}_{\mathbb{R}^2 \text{ limit}}$$

$$= \left(2^2-(-1)^2\right) + i\left(2(2)(-1)\right)$$

Calc III or analysis x^2-y^2 and $2xy$ are continuous

$$= 3-4i$$

And $f(2-i) = (2-i)^2$

$$= 4-4i+i^2$$

$$= 3-4i$$

So, $\displaystyle\lim_{z\to 2-i} z^2 = (2-i)^2$

So, z^2 is continuous at $2-i$

Corollary (to previous thm on limits):

Suppose that $f: A \to \mathbb{C}$ where A is open. Let $f(x+iy) = u(x,y) + iv(x,y)$ and $z_0 = x_0 + iy_0 \in A$.

Then f is continuous at z_0 iff both $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0)

[here the u & v continuity are the $\mathbb{R}^2$ continuous def]

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ex: $z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)} + i\underbrace{2xy}$

x^2-y^2 and $2xy$ are continuous on all of $\mathbb{R}^2$

So, z^2 is continuous on all of $\mathbb{C}$

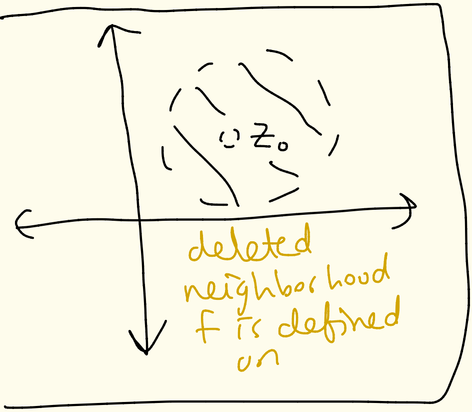
<u>Theorem:</u> Let $A \subseteq \mathbb{C}$ where A is open and $f: A \to \mathbb{C}$ and $g: A \to \mathbb{C}$. Let $z_0 \in A$. Suppose f and g are both continuous at z_0. Then $f+g$, $f-g$, αf, and fg are all continuous at z_0. Here $\alpha \in \mathbb{C}$. If $g(z_0) \neq 0$, then $\frac{f}{g}$ is continuous at z_0.

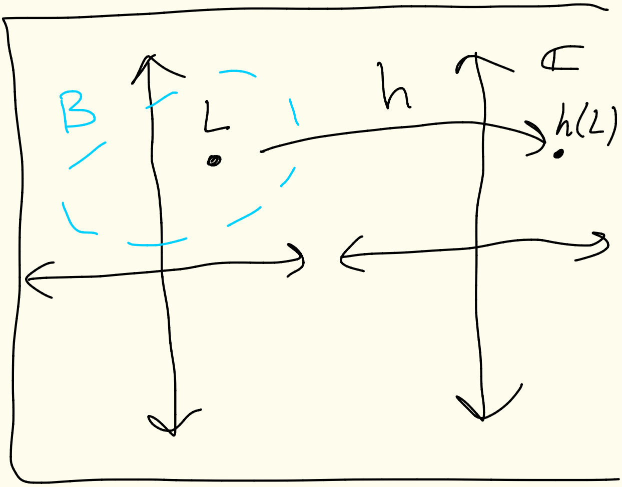
<u>pf:</u> This follows from the theorem from last week. For example since f & g are continuous at z_0 we have $\lim\limits_{z \to z_0} f(z) = f(z_0)$ and $\lim\limits_{z \to z_0} g(z) = g(z_0)$. So,

$$\lim_{z \to z_0} (fg)(z) \underset{\substack{\uparrow \\ \text{last week}}}{=} \left(\lim_{z \to z_0} f(z) \right) \cdot \left(\lim_{z \to z_0} g(z) \right) = f(z_0) g(z_0) = (fg)(z_0)$$

So, fg is continuous at z_0 $\blacksquare$

Thm: Suppose that $\lim\limits_{z \to z_0} f(z) = L$ where f is defined on a deleted neighborhood of z_0. Suppose h is defined on an open set B containing L, and h is continuous at L.

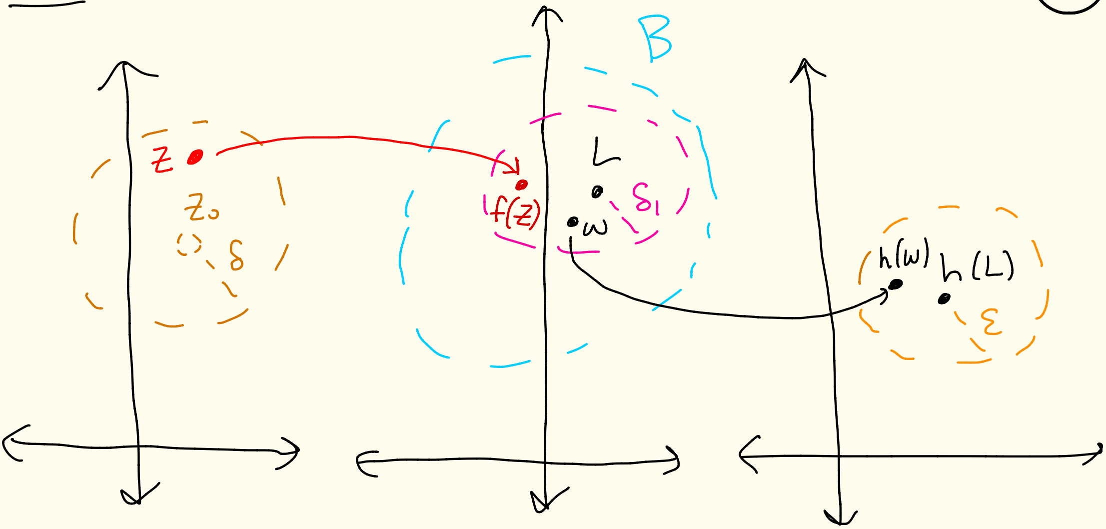




Then, $$\lim_{z \to z_0} h(f(z)) = h\left(\lim_{z \to z_0} f(z)\right) = h(L)$$

pf on next page

pf: Let $\varepsilon > 0$.



Since $\overbrace{h \text{ is continuous at } L}^{\lim_{w \to L} h(w) = h(L)}$, there exists $\delta_1 > 0$ where if $w \in B$ and $|w - L| < \delta_1$ then $|h(w) - h(L)| < \varepsilon$

To make it simpler since B is open you can make it so $D(L; \delta_1) \subseteq B$ if you want by shrinking δ_1

Since $\lim_{z \to z_0} f(z) = L$, there exists $\delta > 0$ so that if $0 < |z - z_0| < \delta$ then $|f(z) - L| < \delta_1$.

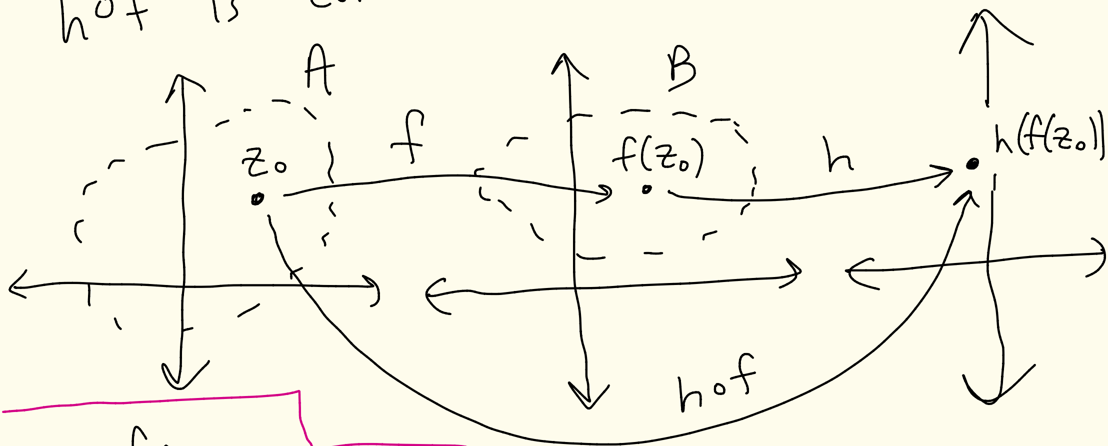
So, if $0 < |z - z_0| < \delta$

then $|h(\underbrace{f(z)}_{w})) - h(L)| < \varepsilon$.

So, $\lim\limits_{z \to z_0} h(f(z)) = h(L)$. $\blacksquare$

(Corollary to previous theorem)

Thm: Let $f : A \to \mathbb{C}$ where A is open and $z_0 \in A$. Let $h : B \to \mathbb{C}$ where B is open and $f(z_0) \in B$. If f is continuous at z_0 and h is continuous at $f(z_0)$, then $h \circ f$ is continuous at z_0.



Proof:

$$\lim_{z \to z_0} (h \circ f)(z) = \lim_{z \to z_0} h(f(z))$$

$$= h\left(\lim_{z \to z_0} f(z)\right) = h(f(z_0)) = (h \circ f)(z_0)$$

thm from Monday

f is continuous at z_0

$\blacksquare$