Topic 5 - Analytic Functions

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<u>Def:</u> Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is an open set.

1. f is said to be <u>differentiable</u> at $z_0 \in A$ if
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists. If the limit exists then we denote it by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

2. The function f is said to be <u>analytic</u> on A if f is differentiable at all $z_0 \in A$.

If someone says "Let g be analytic at z_0" what they mean is: "Let g be analytic on an <u>open set</u> containing z_0".
or neighborhood

<u>Theorem:</u> Let $A \subseteq \mathbb{C}$ be an open set and $f: A \to \mathbb{C}$.
Let $z_0 \in A$.
If f is differentiable at z_0 then f is continuous at z_0.

<u>proof:</u> We are assuming that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \text{ exists.}$$

Let's show $\lim_{z \to z_0} f(z) = f(z_0)$ and hence f will be continuous at z_0.

Note that

$$\left(\lim_{z \to z_0} f(z)\right) - f(z_0) = \lim_{z \to z_0} \left[f(z) - f(z_0) \right]$$

$$= \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)(z - z_0) = f'(z_0) \cdot 0 = 0$$

this is ok since in $\lim_{z \to z_0}$ you don't allow $z = z_0$

So, $\left(\lim_{z \to z_0} f(z)\right) - f(z_0) = 0.$ ③

Thus, $\lim_{z \to z_0} f(z) = f(z_0)$

So, f is continuous at z_0. $\blacksquare$

Theorem: Suppose that f and g are both analytic on an open set $A \subseteq \mathbb{C}$. Then:

① Let $a, b \in \mathbb{C}$. Then $af+bg$ is analytic on A. And
$$(af+bg)'(z) = af'(z) + bg'(z).$$

② fg is analytic on A and
$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

③ If $g(z) \neq 0$ for all $z \in A$, then $\frac{f}{g}$ is analytic and
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2}$$

④ Any polynomial $h(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is analytic on $\mathbb{C}$ and
$$h'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

⑤ Any rational function $\dfrac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$ is analytic on the open set consisting of all z except at most m points where the denominator is zero.

proof: We will prove 2 and 4

$\boxed{2}$ Let $z_0 \in A$.

Since f is analytic at z_0, f is also continuous at z_0.

So, $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

Then,

$$\lim_{z \to z_0} \left[\frac{(fg)(z) - (fg)(z_0)}{z - z_0}\right]$$

$$= \lim_{z \to z_0} \left[\frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0}\right]$$

$$= \lim_{z \to z_0} \frac{f(z)g(z) - f(z)g(z_0) + f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \left(f(z)\left[\frac{g(z) - g(z_0)}{z - z_0}\right] + g(z_0)\left[\frac{f(z) - f(z_0)}{z - z_0}\right]\right)$$

$$= f(z_0)\, g'(z_0) + g(z_0)\, f'(z_0).$$

$\boxed{2}$

4 We will prove that

$\frac{d}{dz} z^n = n z^{n-1}$ and $\frac{d}{dz} c = 0$ where $c \in \mathbb{C}$. Then by part 1 of this thm it will follow that

$$\frac{d}{dz}\left(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n\right)$$
$$= a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

Let's show $\frac{d}{dz} c = 0$.

Let $f(z) = c$ for all $z \in \mathbb{C}$ where $c \in \mathbb{C}$.

Then,
$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{c - c}{z - z_0}$$
$$= \lim_{z \to z_0} 0 = 0.$$

So, $\frac{d}{dz} c = 0$.

We show that $\frac{d}{dz} z^n = n z^{n-1}$ for $n \geq 1$ by induction.

Base case: At z_0 we have

$$\frac{d}{dz} z = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1,$$

Suppose $\frac{d}{dz} z^k = k z^{k-1}$ for some $k \geq 1$.

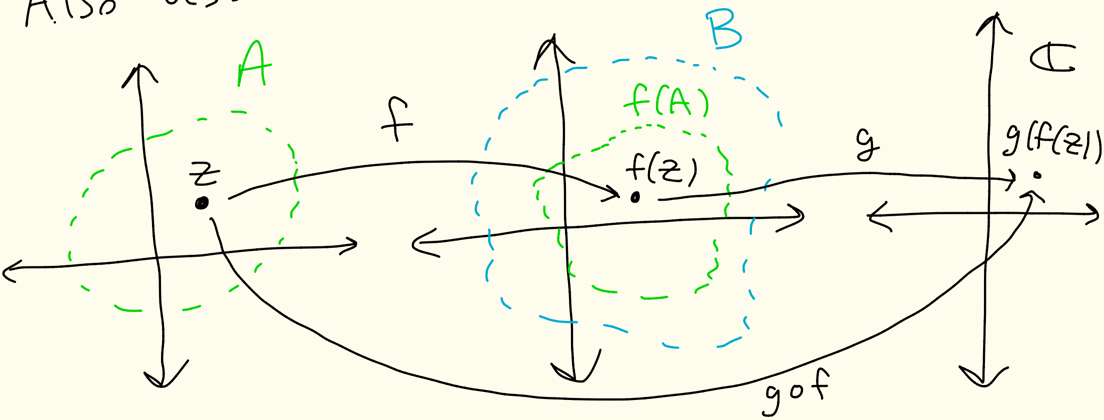
Then,

$$\begin{aligned} \frac{d}{dz} z^{k+1} &= \frac{d}{dz} z^k \cdot z \\ &\overset{2}{=} \left(\frac{d}{dz} z^k\right) z + z^k \left(\frac{d}{dz} z\right) \\ &= k z^{k-1} z + z^k \cdot 1 \\ &= k z^k + z^k = (k+1) z^k . \end{aligned}$$

So, by induction, $\frac{d}{dz} z^n = n z^{n-1}$ for all $n \geq 1$ ▨

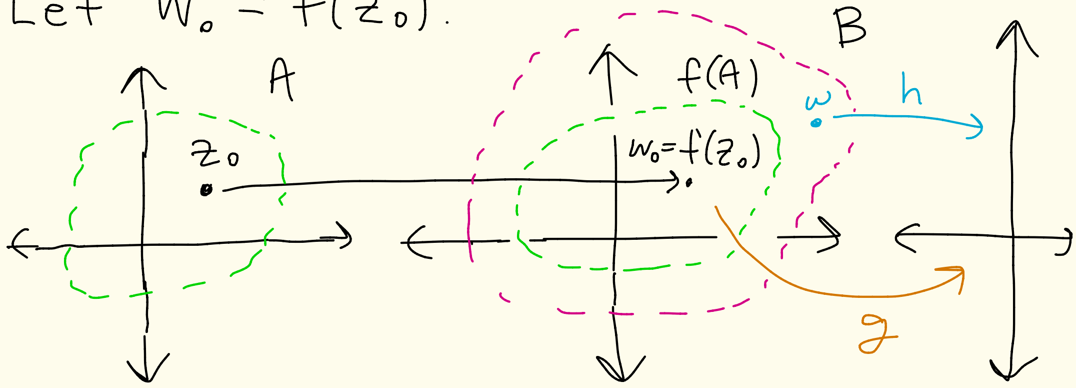
Theorem (Chain rule)

Let $A, B \subseteq \mathbb{C}$ be open sets.
Let $f : A \to \mathbb{C}$ be analytic on A
and $g : B \to \mathbb{C}$ be analytic on B.
Also assume $f(A) \subseteq B$.



Then $g \circ f : A \to \mathbb{C}$ is analytic on A
and $(g \circ f)'(z) = g'(f(z))\, f'(z)$.

<u>proof:</u> Let $z_0 \in A$.
We will look at the derivative at z_0.
Let $w_0 = f(z_0)$.



Define

$$h(w) = \begin{cases} \dfrac{g(w)-g(w_0)}{w-w_0} - g'(w_0) & \text{if } w \neq w_0 \\ 0 & \text{if } w = w_0. \end{cases}$$

for all $w \in B$.

Note that h is continuous on all of B. (why?) If $w \neq w_0$, since g is continuous on B, so is $\dfrac{g(w)-g(w_0)}{w-w_0} - g'(w_0)$.

What about at $w = w_0$? We have

$$\lim_{w \to w_0} h(w) = \lim_{w \to w_0} \left[\underbrace{\frac{g(w) - g(w_0)}{w - w_0}}_{\substack{\text{limits to} \\ g'(w_0)}} - g'(w_0) \right]$$ 10

$$= g'(w_0) - g'(w_0) = 0 = h(w_0).$$

So, h is continuous at w_0.

So,

$$\lim_{z \to z_0} h(f(z)) = h(f(z_0))$$

h is cts at $w_0 = f(z_0)$
f is cts at z_0
$h \circ f$ is cts at z_0

$$= h(w_0) = 0.$$

If $f(z) \neq w_0$ $(z \in A)$, then

$$g(f(z)) - g(w_0)$$

$$= \left[\underbrace{\frac{g(f(z)) - g(w_0)}{f(z) - w_0} - g'(w_0)}_{h(f(z)) \text{ when } f(z) \neq w_0} + g'(w_0) \right] \left[f(z) - w_0 \right]$$

$$= \left[h(f(z)) + g'(w_0) \right] \left[f(z) - w_0 \right].$$

If $f(z) = w_0$ $(z \in A)$, then

$$\left[h(f(z)) + g'(w_0) \right] \left[\underbrace{f(z) - w_0}_{0} \right]$$

$$= 0 = g(w_0) - g(w_0)$$

$$= g(f(z)) - g(w_0).$$

So, $g(f(z)) - g(w_0)$
$= \left[h(f(z)) + g'(w_0) \right] \left[f(z) - w_0 \right]$ for all $z \in A$.

Thus,

$$\lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{g(f(z)) - g(w_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{\left[h(f(z)) + g'(w_0)\right]\left[f(z) - \overset{f(z_0)}{w_0}\right]}{z - z_0}$$

$$= \lim_{z \to z_0} \left[h(f(z)) + g'(w_0)\right]\left(\frac{f(z) - f(z_0)}{z - z_0}\right)$$

$$= \left[\underbrace{h(f(z_0))}_{0} + g'(w_0)\right] \cdot f'(z_0)$$

$$= g'(w_0)\, f'(z_0) = g'(f(z_0))\, f'(z_0).$$

$\blacksquare$

Theorem: (Cauchy–Riemann equations)

Suppose $f: A \to \mathbb{C}$ where A is an open set.

Let $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$.

Let $z_0 = x_0 + iy_0 \in A$.

If $f'(z_0)$ exists, then

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist at (x_0, y_0) and they satisfy the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

and $\dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0)$.

Moreover, $f'(z_0) = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)$

proof: Suppose $f'(z_0)$ exists.

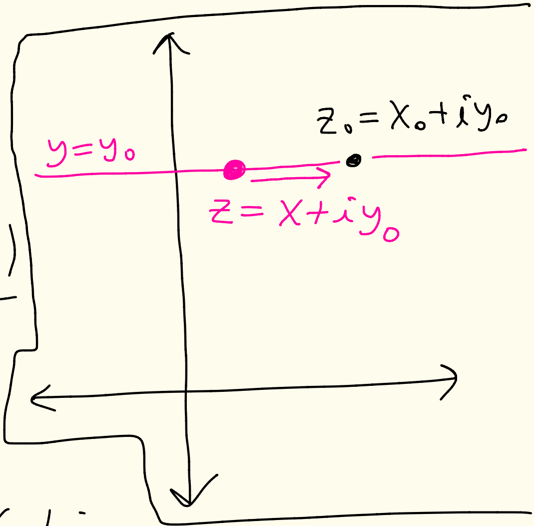
Then the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and the limit is the same no matter how z approaches z_0.

Approaching along the x-axis direction

If we approach z_0 along the line $y = y_0$ then



$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{x+iy_0 \to x_0+iy_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{(x+iy_0) - (x_0+iy_0)} =$$

$$= \lim_{x \to x_0} \left[\frac{\overbrace{u(x,y_0)+iv(x,y_0)}^{f(x+iy_0)} - \overbrace{u(x_0,y_0)-iv(x_0,y_0)}^{f(x_0+iy_0)}}{x-x_0} \right]$$

$$= \lim_{x \to x_0} \left[\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} \right]$$

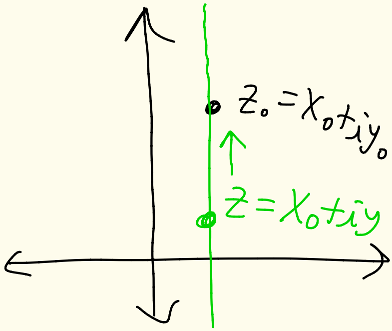
$$+ i \lim_{x \to x_0} \left[\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} \right]$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)$$

So, $f'(x_0+iy_0)$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)$$

If we instead approach z_0 along the $x = x_0$ line we get:



$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{x_0 + iy \to x_0 + iy_0} \left[\frac{u(x_0, y) + iv(x_0, y) - u(x_0, y_0) - iv(x_0, y_0)}{\underbrace{(x_0 + iy) - (x_0 + iy_0)}_{i(y - y_0)}} \right]$$

$$= \lim_{y \to y_0} \left[\frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} \right]$$

$$+ \lim_{y \to y_0} \left[\frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} \right]$$

$\frac{1}{i} = -i$

$$= -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$

So,

$$f'(x_0,y_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0) \quad \Big\}\ \text{1st part}$$

$$= -i\,\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0). \quad \Big\}\ \text{2nd part.}$$

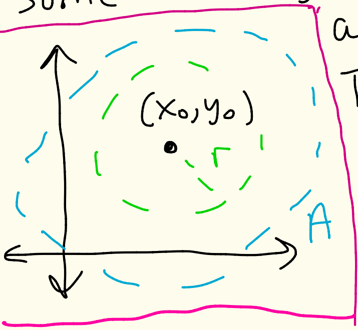
So, $\dfrac{\partial u}{\partial x}(x_0,y_0) = \dfrac{\partial v}{\partial y}(x_0,y_0)$

and $\dfrac{\partial v}{\partial x}(x_0,y_0) = -\dfrac{\partial u}{\partial y}(x_0,y_0)$

$\blacksquare$

Converse Thm: Let $f: A \to \mathbb{C}$ where A is open and $f(x+iy) = u(x,y) + iv(x,y)$

Suppose $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist in some r-neighborhood of (x_0, y_0) and are continuous at (x_0, y_0).



Then if $\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$

and $\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$

then $f'(z_0)$ exists where $z_0 = x_0 + iy_0$

Proof: See Hoffman/Marsden book $\blacksquare$

Ex: $f(z) = z^2$

$$f(x+iy) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

$$u(x,y) = x^2 - y^2$$

$$v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist and are continuous for all (x,y)

Also, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for all (x,y). So, f' exists for all z and $f'(x+iy) =$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2(x+iy).$$

that is $f'(z) = 2z$

Ex: $f(z) = \overline{z}$

Where is f analytic?
Where does f' exist?

$$f(x+iy) = \overline{x+iy} = x - iy = x + i(-y)$$

$$u(x,y) = x$$
$$v(x,y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

Cauchy Riemann

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ for all (x,y)

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for all (x,y)

The Cauchy-Riemann equations are never satisfied for any (x,y).
So, $f'(z)$ doesn't exist anywhere.

<u>Def:</u> A function $f:\mathbb{C}\to\mathbb{C}$ is <u>entire</u> if f is analytic on all of $\mathbb{C}$ [that is, $f'(z)$ exists for all $z\in\mathbb{C}$]

<u>Ex:</u> Polynomials are entire functions

<u>Ex:</u> Let $f(z)=e^z$.

We will show that f is entire and $f'(z)=e^z$ for all z.

$$
\begin{aligned}
f(x+iy) = e^{x+iy} &= e^x e^{iy} \\
&= e^x[\cos(y)+i\sin(y)] \\
&= \underbrace{e^x\cos(y)}_{u(x,y)} + i\,\underbrace{e^x\sin(y)}_{v(x,y)}
\end{aligned}
$$

$$u(x,y) = e^x \cos(y)$$
$$v(x,y) = e^x \sin(y)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos(y) = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -e^x \sin(y) = -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{Cauchy Riemann equations}$$

$$\boxed{\begin{aligned} &\frac{d}{dt}\sin(t) = \cos(t) \\ &\frac{d}{dt}\cos(t) = -\sin(t) \end{aligned}}$$

- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist for all (x,y) and are continuous for all (x,y)
- The Cauchy-Riemann equations are true for all (x,y)

Therefore, $f'(z)$ exists for all z.

And, $f'(z) = f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$

$$= e^x \cos(y) + i e^x \sin(y)$$
$$= e^x[\cos(y) + i\sin(y)] = e^z$$

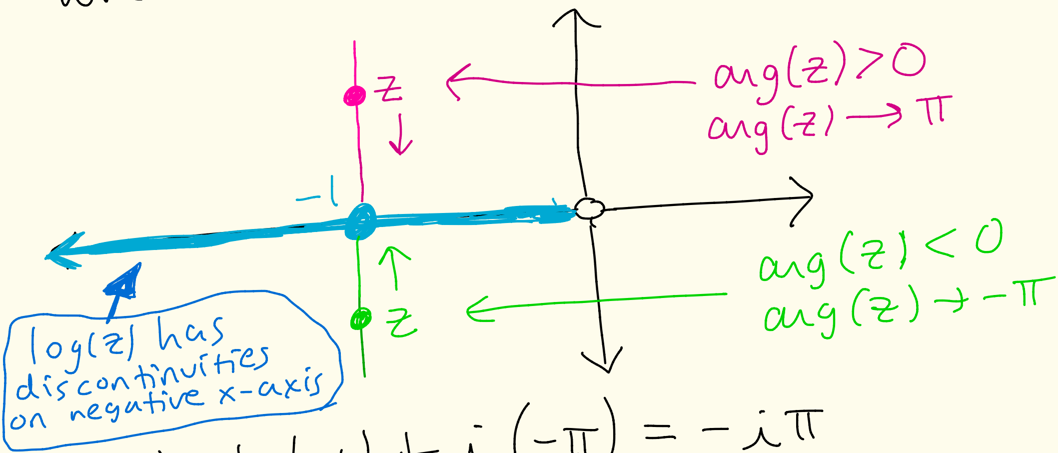
$\blacksquare$

Ex: (log has a discontinuity at its branch point)

Consider $\log : \mathbb{C} - \{0\} \to \mathbb{C}$

$$\log(z) = \ln|z| + i\arg(z)$$

where $-\pi \leq \arg(z) < \pi$.



$$\log(-1) = \ln|-1| + i(-\pi) = -i\pi$$

If you approach -1 along a vertical line from above then $\log(z) = \ln|z| + i\arg(z)$ approaches $\ln|-1| + i\pi = i\pi$

If you approach -1 along a vertical line from below then $\log(z) = \ln|z| + i\arg(z)$ approaches $\ln|-1| + i(-\pi) = -i\pi$

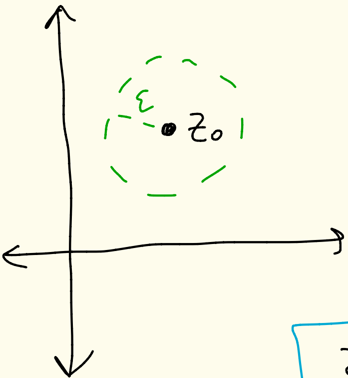
So, $\log(z)$ has a discontinuity at $z = -1$. $\log(z)$ has discontinuities on its entire branch cut (ie negative x-axis)

Theorem (Polar coordinate version of Cauchy-Riemann)

Let

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i v(r,\theta)$$

be defined on some ε-neighborhood of $z_0 = r_0 e^{i\theta_0}$. Suppose that



$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exist and are continuous on the ε-neighborhood.

If

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}}$$

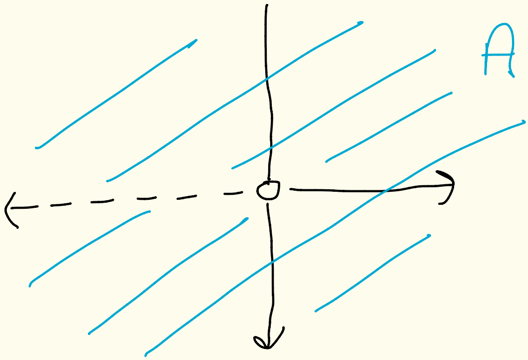
at the point (r_0, θ_0), then $f'(z_0)$ exists and

$$f'(z_0) = e^{-i\theta_0}\left(\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right)$$

proof in Hoffman/Marsden book

Ex: Let

$$A = \mathbb{C} - \{ x+iy \mid x \leq 0 \text{ and } y = 0 \}$$



Define the branch of $\log : A \to \mathbb{C}$ by

$$\log(z) = \ln|z| + i\arg(z)$$

where $-\pi < \arg(z) < \pi$.

This is called the <u>principal branch of the logarithm</u>.

We will now show that this log function is analytic on A and $\frac{d}{dz}\log(z) = \frac{1}{z}$.

[Similar statements are true for other branches of log]

Write log in polar form.

$$\log(z) \underset{\boxed{r>0}}{=} \log(re^{i\theta}) = \ln|re^{i\theta}| + i\theta$$

$$= \underbrace{\ln(r)}_{u} + i\underbrace{\theta}_{v}$$

Set $u(r,\theta) = \ln(r)$

$v(r,\theta) = \theta$

$$\underbrace{\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial \theta} = 1}_{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ for all } (r,\theta) \text{ in } A}, \quad \underbrace{\frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial r} = 0}_{-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} \text{ is true for all } (r,\theta) \text{ in } A}$$

- $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ exist and are continuous on A (since we removed the discontinuity to create A.) A.
- Cauchy-Riemann eqns are true on A.

So, $f'(z)$ exists for all $z \in A$ and

$$f'(z) = f'(re^{i\theta}) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{1}{r} + i0\right)$$

$$= \frac{1}{re^{i\theta}} = \frac{1}{z} \quad \boxtimes$$

$\boxed{\text{Ex:}}$ $\sin(z)$ and $\cos(z)$ are entire functions.

differentiable / analytic on all of $\mathbb{C}$

And $\frac{d}{dz}\sin(z) = \cos(z)$

and $\frac{d}{dz}\cos(z) = -\sin(z)$

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proof: Recall that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

We know iz has a derivative at all z.
We know e^z has a derivative at all z.
So, e^{iz} is differentiable at all z by the chain rule.

Same idea that e^{-iz} is diff. at all z.

Since sums of diff. functions are diff. and multiplying by a constant keeps differentiability, $\sin(z)$ and $\cos(z)$ are differentiable everywhere.

We can use facts about the derivative and the chain rule just like in Calculus to differentiate.

$$\frac{d}{dz}\sin(z) = \frac{d}{dz}\left[\frac{e^{iz}-e^{-iz}}{2i}\right]$$

$$= \frac{1}{2i}\frac{d}{dz}\left[e^{iz}-e^{-iz}\right]$$

$$= \frac{1}{2i}\left[e^{iz}(i)-e^{-iz}(-i)\right]$$

$$= \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$$

Similarly one can show

$$\frac{d}{dz}\cos(z) = -\sin(z)$$

$\blacksquare$

Ex: Let $a \in \mathbb{C}$, $a \neq 0$.

Define

$$f(z) = a^z = e^{z \log(a)}$$

where log is some branch of the logarithm.

[Here I mean choose $\log(w) = \ln|w| + i\arg(w)$ where $c \leq \arg(w) < c + 2\pi$ for some $c \in \mathbb{R}$]

Then, f is entire and

$$f'(z) = (\log(a))\, a^z$$

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proof: By the chain rule, since $\log(a)$ is a constant and e^w is entire, we have $e^{z\log(a)}$ is entire and

$$f'(z) = \frac{d}{dz}\left(e^{z\log(a)}\right) = e^{z\log(a)}\left(\log(a)\right)$$

$$= (\log(a))\, a^z$$

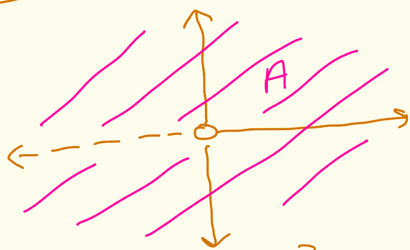
$\blacksquare$

Ex: Let $b \in \mathbb{C}$.

Let $f(z) = z^b = e^{b \log(z)}$

where $\log : A \to \mathbb{C}$ denotes a branch of the logarithm where A is a part of the domain where log is analytic.

For example if we have the principal branch of log then



$A = \mathbb{C} - \{x+iy \mid x \leq 0 \text{ \& } y = 0\}$

Then, f is analytic on A and $f'(z) = b z^{b-1}$.

proof: Since $\log(z)$ is analytic on A and e^z is analytic everywhere, by the chain rule f is analytic on A. and $f'(z) = e^{b\log(z)}\left(b \cdot \frac{1}{z}\right) = z^b \cdot b \cdot \frac{1}{z}$

$= b z^{b-1}$ ▨