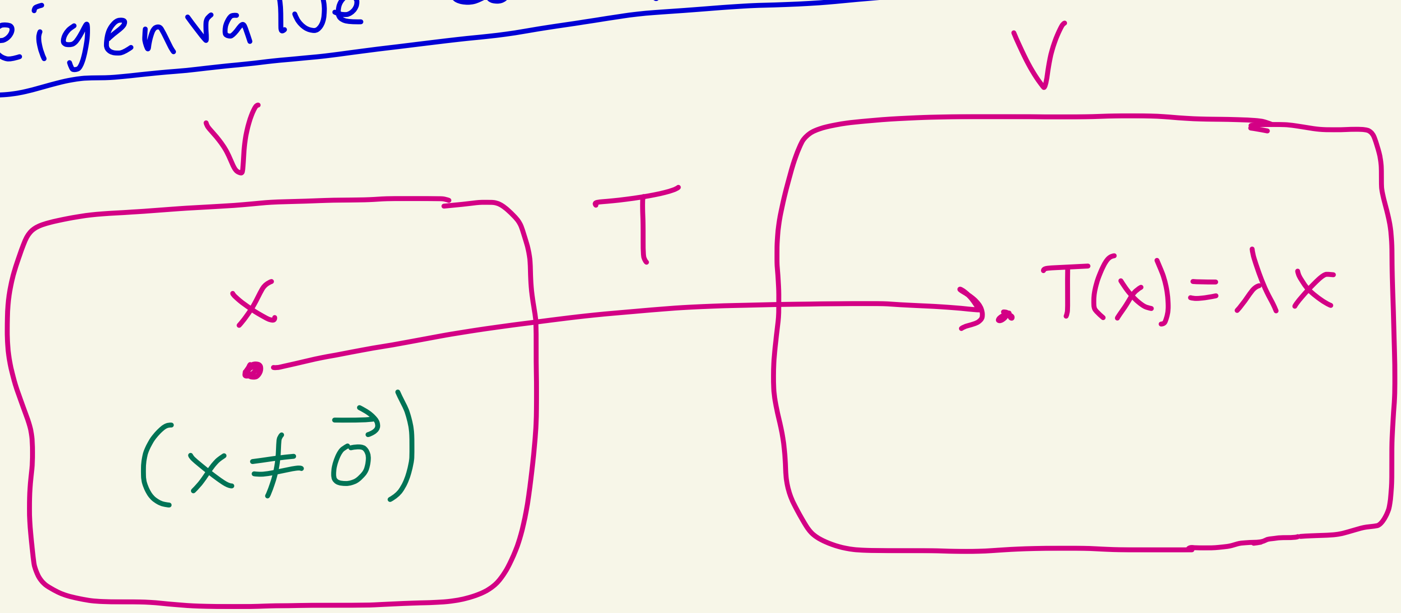
Topic 5 –

Eigenvalues, Eigenvectors, and Diagonalization

Def: Let V be a vector space over a field F. Let $T: V \to V$ be a linear transformation. If $x \in V$ with $x \neq \vec{0}$ and $T(x) = \lambda x$ where $\lambda \in F$, then we call x an **eigenvector** of T and λ the **eigenvalue corresponding to x**.



Note: $\lambda = 0$ is allowed

$x = \vec{0}$ is not allowed

Ex: Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 3b \\ 4a + 2b \end{pmatrix}$$

You can check that T is a lin. trans.

We have that

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 + 3(-1) \\ 4(1) + 2(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So, $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = -2$ [because $T(x) = -2x$]

Also,

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 + 3(4) \\ 4(3) + 2(4) \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

So, $y = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 5$ [because $T(y) = 5y$]

Ex: Let

$V = P_2(\mathbb{R}) = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$

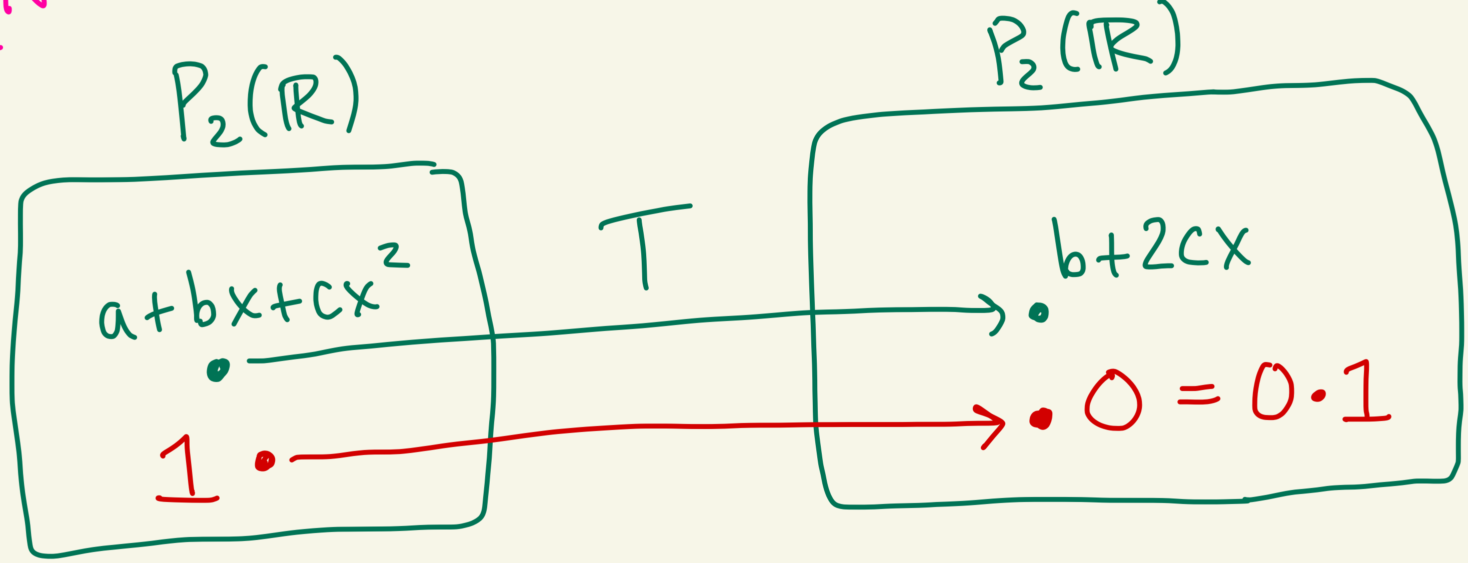
$F = \mathbb{R}$

$T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$

$T(a + bx + cx^2) = b + 2cx$

You can check this is a linear transformation

[Note that $T(f) = f'$]



Note that

$$T(1) = 0 = 0 \cdot 1$$

So, 1 is an eigenvector with eigenvalue $\lambda = 0$.

Def: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

We say that T is <u>diagonalizable</u> if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Recall: A diagonal matrix has the form

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 4a+2b \end{pmatrix}$

We saw on Monday that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are eigenvectors for T.

You can check that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are linearly independent and thus since there are two of them and $\dim(\mathbb{R}^2) = 2$ they form a basis for $\mathbb{R}^2$.

Let $\beta = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right]$.

Let's compute $[T]_\beta$. $\boxed{[T]_\beta^\beta}$

$$\underbrace{T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}}_{\text{plug } \beta \text{ into } T} = \underbrace{-2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 4 \end{pmatrix}}_{\text{write answer in terms of } \beta}$$

$$T\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Thus, $[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

So, T is diagonalizable.

Why is this useful?

Let $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. We know $\beta = [v_1, v_2]$ is a basis for $\mathbb{R}^2$.

Given any $x \in \mathbb{R}^2$ we can write $x = c_1 v_1 + c_2 v_2$. Then,

$$T(x) = T(c_1 v_1 + c_2 v_2)$$
$$\stackrel{\downarrow}{=} c_1 T(v_1) + c_2 T(v_2)$$
$$\stackrel{\downarrow}{=} c_1(-2v_1) + c_2(5v_2)$$
$$= -2c_1 v_1 + 5c_2 v_2$$

T is linear

$T(v_1) = -2v_1$
$T(v_2) = 5v_2$

In matrix notation we have

$$[T(x)]_\beta = [T]_\beta [x]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2c_1 \\ 5c_2 \end{pmatrix}$$

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

T is diagonalizable iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V consisting of eigenvectors of T. Moreover, if this is the case then

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where λ_i is the eigenvalue corresponding to v_i.

<u>proof:</u> T is diagonalizable

iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

ie $[T]_\beta$ is diagonal

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$

iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V such that

$$T(v_1) = \lambda_1 v_1 + 0v_2 + 0v_3 + \cdots + 0v_n$$
$$T(v_2) = 0v_1 + \lambda_2 v_2 + 0v_3 + \cdots + 0v_n$$
$$T(v_3) = 0v_1 + 0v_2 + \lambda_3 v_3 + \cdots + 0v_n$$
$$\vdots$$
$$T(v_n) = 0v_1 + 0v_2 + 0v_3 + \cdots + \lambda_n v_n$$

iff

iff there exists an ordered basis $\beta = [v_1, v_2, \ldots, v_n]$ of V consisting of eigenvectors of T where $T(v_i) = \lambda_i v_i$ [So each λ_i is an eigenvalue for v_i]. ∎

Why is this useful?

Let $T: V \to V$ be a linear transformation and $\beta = [v_1, v_2, \ldots, v_n]$ be an ordered basis of eigenvectors with eigenvalues λ_i.

Let $x \in V$.

Express $x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

So,
$$T(x) = T(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)$$
$$= c_1 T(v_1) + c_2 T(v_2) + \ldots + c_n T(v_n)$$
$$= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \ldots + c_n \lambda_n v_n$$

T linear

$T(v_i) = \lambda_i v_i$

Let's learn how to find the eigenvalues and eigenvectors

<u>Theorem:</u> Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Let β and γ be ordered bases for V. Then,

$$\det([T]_\beta) = \det([T]_\gamma)$$

$I: V \to V$
$I(x) = x$
identity transformation

<u>proof:</u> [HW 5 #4] We have that

$$\det([T]_\beta) = \det\left([I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma\right)$$

$$= \det([I]_\gamma^\beta)\det([T]_\gamma)\det([I]_\beta^\gamma)$$

$$= \det([T]_\gamma)\det([I]_\gamma^\beta)\det([I]_\beta^\gamma)$$

$$= \det([T]_\gamma)\det\left([I]_\gamma^\beta [I]_\beta^\gamma\right)$$

$\det(AB) = \det(A)\det(B)$

$$=$$

$$= \det([T]_\gamma)\det\big(\underbrace{[I]_\gamma^\beta}_{[I]_\gamma^\beta = \left([I]_\beta^\gamma\right)^{-1}}[I]_\beta^\gamma\big)$$

these are matrices $n\times n$

$$= \det([T]_\gamma)\det(I_n)$$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

n = # of elements in β and γ

$$= \det([T]_\gamma)\cdot 1$$

$$= \det([T]_\gamma).$$

$\blacksquare$

The previous theorem makes the next definition well-defined.

Def: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. The determinant of T is defined to be

$$\det(T) = \det([T]_\beta)$$

where β is any ordered basis for V.

Ex: Recall

$P_2(\mathbb{R}) = \{ a+bx+cx^2 \mid a,b,c \in \mathbb{R} \}$

Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$

be given by $T(a+bx+cx^2) = b+2cx$

T is a linear transformation.

Let's calculate $\det(T)$.

Let's pick $\beta = [1, x, x^2]$

(ie the standard basis)

$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

Thus, $[T]_\beta = [T]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Then,

$$\det(T) = \det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

expand on column 1

If a matrix has a row or column of zeros, then its determinant is zero

We will need the following:

Let V be a finite-dimensional vector space over a field F and $T: V \to V$ be a linear transformation. T is 1-1 iff $\det(T) \neq 0$.

<u>proof:</u> By HW 3 #6(b), since $T: V \to V$ we know T is 1-1 iff T is onto. By HW 5 #5(a), $\det(T) \neq 0$ iff T is 1-1 and onto. $\blacksquare$

Theorem: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. (15)

Then, $\underbrace{\text{the following are equivalent}}_{\text{TFAE}}$:

(1) There exists an eigenvector $x \in V$, $x \neq \vec{0}$, of T with eigenvalue λ.

(2) $\det(T - \lambda I) = 0$

(3) $N(T - \lambda I) \neq \{\vec{0}\}$

$T - \lambda I : V \to V$

$(T - \lambda I)(x)$

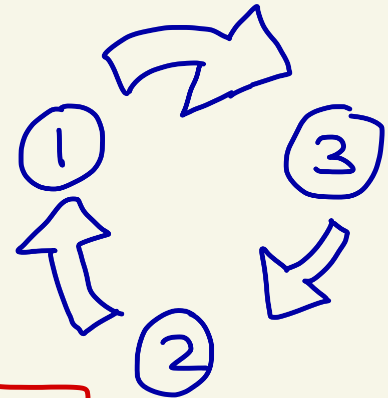
$= T(x) - \lambda I(x) = T(x) - \lambda x$

$I: V \to V$ is the identity transformation

TFAE means if one of (1), (2), or (3) is true then they are all true

Proof:

We will prove this like this



Proof that 1 $\Rightarrow$ 3:

Suppose 1 is true. That is, there exists $x \in V$, $x \neq \vec{0}$, where $T(x) = \lambda x$ and $\lambda \in F$.

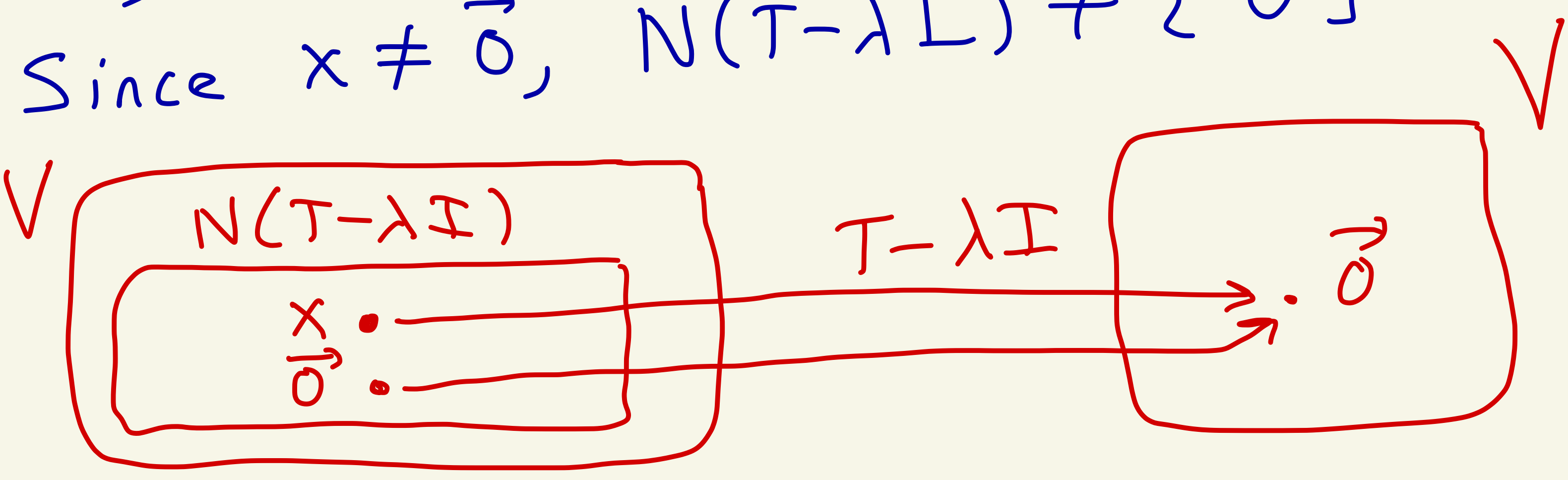
Then, $T(x) = \lambda I(x)$ ← $I(x) = x$

So, $T(x) - \lambda I(x) = \vec{0}$.

Thus, $(T - \lambda I)(x) = \vec{0}$.

So, $x \in N(T - \lambda I)$.

Since $x \neq \vec{0}$, $N(T - \lambda I) \neq \{\vec{0}\}$



$\boxed{\text{proof that } ③ \Rightarrow ②}$:

Suppose ③ is true, that is $N(T-\lambda I) \neq \{\vec{0}\}$ for some $\lambda \in F$.

Recall that $\vec{0} \in N(T-\lambda I)$ because $T-\lambda I$ is a linear transformation and so by HW 3 #1(a), $(T-\lambda I)(\vec{0}) = \vec{0}$.

Since $N(T-\lambda I) \neq \{\vec{0}\}$ there exists $x \in V$ with $x \neq \vec{0}$ and $x \in N(T-\lambda I)$.

Then, $(T-\lambda I)(x) = \vec{0}$.

Thus, $(T-\lambda I)(x) = \vec{0} = (T-\lambda I)(\vec{0})$.

Since $x \neq \vec{0}$ this shows that $T-\lambda I$ is not one-to-one.

By our earlier discussion,
$\det(T-\lambda I) = 0$.

proof that 2 $\Rightarrow$ 1 :

Suppose 2 is true, that is $\det(T-\lambda I) = 0$ for some $\lambda \in F$.

By our previous discussion $T-\lambda I$ is not one-to-one.

This will lead to $N(T-\lambda I) \neq \{\vec{0}\}$.

Why?

Since $T-\lambda I$ is not one-to-one there exists x_1, x_2 with $x_1 \neq x_2$ and $(T-\lambda I)(x_1) = (T-\lambda I)(x_2)$.

Then, $(T-\lambda I)(x_1) - (T-\lambda I)(x_2) = \vec{0}$

Since $T-\lambda I$ is a linear transformation, $(T-\lambda I)(x_1 - x_2) = \vec{0}$

Thus, $x_1 - x_2 \in N(T-\lambda I)$ and since $x_1 \neq x_2$ we have $x_1 - x_2 \neq \vec{0}$.

Let $x = x_1 - x_2$.

Then, $x \neq \vec{0}$ and $(T - \lambda I)(x) = \vec{0}$.

So, $T(x) - \lambda I(x) = \vec{0}$.

Thus, $T(x) = \lambda I(x)$

$I(x) = x$

Hence, $T(x) = \lambda x$

So, $x \neq \vec{0}$ is an eigenvector of T with eigenvalue λ.

$\blacksquare$

Theorem: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation.

Let β be an ordered basis for V.

Then,

$$\det(T - \lambda I) = \det\left([T]_\beta - \lambda I_n\right)$$

where I_n is the identity matrix with $n = \dim(V)$.

Recall $I: V \to V$ where $I(x) = x$ for all $x \in V$.

<u>proof:</u> We have that

def of det

$$\det(T-\lambda I) = \det([T-\lambda I]_\beta)$$

$$= \det([T]_\beta + [-\lambda I]_\beta)$$

$$= \det([T]_\beta - \lambda [I]_\beta)$$

$$= \det([T]_\beta - \lambda I_n)$$

$\blacksquare$

HW 4 #2

$[T+S]_\beta = [T]_\beta + [S]_\beta$

$[cT]_\beta = c[T]_\beta$

HW 5 #2

$[I]_\beta = I_n$

Def: Let V be a finite-dimensional vector space over a field F and let $T: V \to V$ be a linear transformation. Let λ be an eigenvalue of T.

Define

$$E_\lambda(T) = \{ x \in V \mid T(x) = \lambda x \}$$

$$= N(T - \lambda I)$$

$$T(x) = \lambda x$$
$$T(x) - \lambda x = \vec{0}$$
$$T(x) - \lambda I(x) = \vec{0}$$
$$(T - \lambda I)(x) = \vec{0}$$

$E_\lambda(T)$ is called the <u>eigenspace of T corresponding to λ</u>. The dimension of $E_\lambda(T)$ is called the <u>geometric multiplicity of λ</u>.

- $E_\lambda(T)$ is a subspace of V [HW 5]
- $E_\lambda(T)$ consists of $\vec{0}$ and all the eigenvectors corresponding to λ.

Def: Let V be a finite-dimensional vector space over a field F. Let $T: V \to V$ be a linear transformation. Let β be an ordered basis for V. Let $n = \dim(V)$. Then the function

$$f_T(\lambda) = \det(T - \lambda I) = \det([T]_\beta - \lambda I_n)$$

is called the <u>characteristic polynomial of T</u>. The roots of $f_T(\lambda)$ are the eigenvalues of T. If λ_0 is a root of $f_T(\lambda)$ then it's multiplicity as a root is called the <u>algebraic multiplicity of λ_0</u>.

That is, the alg. mult. of λ_0 is the largest positive integer k such that $(\lambda - \lambda_0)^k$ is a factor of $f_T(\lambda)$

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}$

You can that T is a linear transformation.

Let's find the eigenvalues, eigenvectors, etc for T.

Let's find the eigenvalues first, ie the roots of $f_T(\lambda)$.

We need to pick a basis for $V = \mathbb{R}^3$.

Let $\beta = [v_1, v_2, v_3]$ where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

β is the standard basis for $\mathbb{R}^3$.

Let's calculate $[T]_\beta$

We have

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\1\\3\end{pmatrix} = \underbrace{-2\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{put in terms of } \beta}$$

$$\boxed{T\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-2c\\a+2b+c\\a+3c\end{pmatrix}}$$

Thus, $[T]_\beta = \begin{pmatrix}0 & 0 & -2\\1 & 2 & 1\\1 & 0 & 3\end{pmatrix}$

So,

$$f_T(\lambda) = \det\left([T]_\beta - \lambda I_3\right)$$

I_3 since $\dim(\mathbb{R}^3) = 3$

$$= \det\left(\begin{pmatrix}0 & 0 & -2\\1 & 2 & 1\\1 & 0 & 3\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & \boxed{0} & -2 \\ 1 & \boxed{2-\lambda} & 1 \\ 1 & \boxed{0} & 3-\lambda \end{pmatrix}$$

expand on column 2

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= -0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix} + (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} - 0 \cdot \begin{vmatrix} -\lambda & -2 \\ 1 & 1 \end{vmatrix}$$

$$\begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} \quad \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} \quad \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}$$

$$= 0 + (2-\lambda)\Big[\underbrace{(-\lambda)(3-\lambda) - (-2)(1)}_{-3\lambda+\lambda^2+2}\Big] + 0$$

$$= -6\lambda + 2\lambda^2 + 4 + 3\lambda^2 - \lambda^3 - 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Recall the rational roots theorem

Let
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
where $a_n, a_{n-1}, \ldots, a_1, a_0$ are integers, $a_n \neq 0$, $a_0 \neq 0$. If a rational number $\frac{p}{q}$ is a root of $f(x)$, then p divides a_0 and q divides a_n

This theorem gives you a list of the possible rational roots

The possible rational roots of

$$f_T(\lambda) = \underline{\underline{-}}\,\lambda^3 + 5\lambda^2 - 8\lambda + \underline{\underline{4}}$$

are $\frac{p}{q}$ where p divides 4 and q divides -1.

So, $p = \pm 1, \pm 2, \pm 4$ and $q = \pm 1$.

This gives that possible rational roots are

$$\frac{p}{q} = \pm 1, \pm 2, \pm 4.$$

<u>check:</u>

$f_T(1) = -(1)^3 + 5(1)^2 - 8(1) + 4 = 0$

$f_T(-1) = -(-1)^3 + 5(-1)^2 - 8(-1) + 4 = 16 \neq 0$

$f_T(2) = 0$

$f_T(-2) \neq 0$

$f_T(\pm 4) \neq 0$

So the only rational roots of $f_T(\lambda)$ are $\lambda = 1$ and $\lambda = 2$.

Since $\lambda = 1$ is a root of $f_T(\lambda)$ we know $(\lambda - 1)$ is a factor of $f_T(\lambda)$. Let's divide!

$$
\begin{array}{r|l}
 & -\lambda^2 + 4\lambda - 4 \\
\hline
\lambda - 1 & -\lambda^3 + 5\lambda^2 - 8\lambda + 4 \\
 & -(-\lambda^3 + \lambda^2) \\
\hline
 & 4\lambda^2 - 8\lambda + 4 \\
 & -(4\lambda^2 - 4\lambda) \\
\hline
 & -4\lambda + 4 \\
 & -(-4\lambda + 4) \\
\hline
 & 0
\end{array}
$$

no remainder

Thus,

$$\underbrace{-\lambda^3 + 5\lambda^2 - 8\lambda + 4}_{f_T(\lambda)} = (\lambda - 1)(-\lambda^2 + 4\lambda - 4)$$

$\boxed{\begin{array}{l}\underline{\text{Recall:}} \text{ If } r_1, r_2 \text{ are roots of } ax^2+bx+c=0 \\ \text{then} \\ ax^2+bx+c = a(x-r_1)(x-r_2)\end{array}}$

The roots of $-\lambda^2+4\lambda-4$ are

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(-1)(-4)}}{2(-1)} = \frac{-4 \pm \sqrt{0}}{-2} = 2$$

Thus, 2 is a root twice!

So, $-\lambda^2+4\lambda-4 = \boxed{-(\lambda-2)(\lambda-2)}$.

Thus,

$$\begin{aligned} f_T(\lambda) &= (\lambda-1)(-\lambda^2+4\lambda-4) \\ &= -(\lambda-1)(\lambda-2)^2 \end{aligned}$$

Ex:

From last time:

$T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix}$

$$f_T(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$
$$= -(\lambda-1)(\lambda-2)^2$$

eigenvalue of T	$\lambda = 1$	$\lambda = 2$
algebraic multiplicity	1	2

multiplicity as a root of $f_T(\lambda)$

Let's calculate $E_1(T)$

$$E_1(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{T(x) = 1 \cdot x} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$$

add $\begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}$ to both sides

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -a-2c \\ a+b+c \\ a+2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} -a-2c=0 \\ a+b+c=0 \\ a+2c=0 \end{matrix} \right\}$$

Let's solve the following system:

$$\boxed{\begin{array}{rcrcrcl} -a & & & - & 2c & = & 0 \\ a & + & b & + & c & = & 0 \\ a & & & + & 2c & = & 0 \end{array}}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \underbrace{\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\text{reduced}}$$

This gives

$$\boxed{\begin{aligned} a \quad\quad + 2c &= 0 \\ b - c &= 0 \\ 0 &= 0 \end{aligned}}$$

leading variables
a, b

free variable
c

Give free variable new name.

Let $c = t$.

Solve eqns for leading variables.

$$\boxed{\begin{aligned} a &= -2c \quad (1) \\ b &= c \quad (2) \end{aligned}}$$

Back substitute:

$$c = t$$

(2) $b = c = t$

(1) $a = -2c = -2t$

Thus,

$$E_1(T) = \left\{ \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Let $\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$.

Then β_1 spans $E_1(T)$ and since β_1 consists of one non-zero vector, β_1 is a lin. ind. set. HW

So, β_1 is a basis for $E_1(T)$

The geometric multiplicity of $\lambda = 1$ is $\dim(E_1(T)) = 1$ ← size of β_1

Let's calculate $E_2(T)$.

$$E_2(T) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \underbrace{T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{T(x) = 2x} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2c \\ a+2b+c \\ a+3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -2a & -2c \\ a & +c \\ a & +c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

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add $\begin{pmatrix} -2a \\ -2b \\ -2c \end{pmatrix}$ to both sides

Let's solve

$$\boxed{\begin{aligned} -2a \quad -2c &= 0 \\ a \quad +c &= 0 \\ a \quad +c &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This becomes

$$\boxed{\begin{aligned} a \quad + c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading variables
a

free variable
b, c

Set $b = t$
$c = s$

Then,

$$\boxed{\begin{aligned} a &= -c = -s \\ b &= t \qquad \text{where } s, t \in \mathbb{R} \\ c &= s \end{aligned}}$$

So,

$$E_2(T) = \left\{ \begin{pmatrix} -s \\ t \\ s \end{pmatrix} \middle| \ s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} -s \\ 0 \\ s \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} \middle| \ s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \ s, t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$$

Let $\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$.

Since $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are not multiples of each other, by HW they form a linearly independent set.

So, β_2 is a basis for $E_2(T)$.

Thus, the geometric multiplicity of $\lambda = 2$ is $\dim(E_2(T)) = 2$

Eigenvalues	$\lambda = 1$	$\lambda = 2$
algebraic multiplicity	1	2
geometric multiplicity	1	2
basis for $E_\lambda(T)$	$\beta_1 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right]$	$\beta_2 = \left[\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

Let $\beta = \beta_1 \cup \beta_2 = \left[\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$

One can show β is a basis for $\mathbb{R}^3$.

What is $[T]_\beta$?

$$T\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So,

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus, T is diagonalizable

Ex: Let

$T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$

$T(f) = f'$

$T(a + bx + cx^2) = b + 2cx$

Let's find the eigenvalues of T.

Let $\gamma = [1, x, x^2]$

Then,

$$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$
$$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

Thus,

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$f_T(\lambda) = \det\left([T]_\gamma - \lambda I_3\right)$$

$$= \det\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}$$

expand on this column

$$\boxed{\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}}$$

$$= -\lambda \cdot \underbrace{\begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix}}_{\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}} + 0 + 0$$

$$= -\lambda\left[\lambda^2 - 0\right]$$

$$= -\lambda^3$$

$$= -(\lambda - 0)^3$$

Since $f_T(\lambda) = -(\lambda - 0)^3$,

$\lambda = 0$ is the only eigenvalue of T and it has algebraic multiplicity 3.

Let's calculate $E_0(T)$.

$$E_0(T) = \left\{ a+bx+cx^2 \in P_2(\mathbb{R}) \;\middle|\; T(a+bx+cx^2) = 0(a+bx+cx^2) \right\}$$

$$= \left\{ a+bx+cx^2 \in P_2(\mathbb{R}) \;\middle|\; b+2cx = 0 \right\}$$

$b = 0$
$2c = 0$

or

$b = 0$
$c = 0$

$$= \left\{ a \;\middle|\; a \in \mathbb{R} \right\}$$

$$= \left\{ a \cdot 1 \;\middle|\; a \in \mathbb{R} \right\} = \operatorname{span}(\{1\})$$

Thus, $\beta = [1]$ is a basis for $E_0(T)$.

So, $\lambda = 0$ has geometric multiplicity $\dim(E_0(T)) = 1$

elements in β

Eigenvalue	$\lambda = 0$
algebraic multiplicity	3
geometric multiplicity	1
basis for $E_\lambda(T)$	$[1]$

note:
geo. mult. $\leq$ alg. mult.

In this example there aren't enough eigenvectors to diagonalize T. It turns out that T is not diagonalizable. We need 3 lin. ind. eigenvectors and we only have 1.

<u>Lemma:</u> Let $T: V \to V$ be a linear transformation where V is a vector space over a field F.
Let $v_1, v_2, \ldots, v_r$ be eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\lambda_i \neq \lambda_j$ when $i \neq j$.
Then, $v_1, v_2, \ldots, v_r$ are linearly independent.

[So, eigenvectors from different/distinct eigenspaces are linearly independent]

<u>proof:</u> We prove by induction on r.

<u>Base case:</u> Suppose $r = 1$.
Suppose v_1 is an eigenvector of T.
By def of eigenvector $v_1 \neq \vec{0}$
By HW 2 # 6, $\{v_1\}$ is a linearly independent set.

<u>Induction hypothesis:</u> Suppose any k eigenvectors of T with distinct eigenvalues are linearly independent.

<u>Now we prove for $k+1$:</u>

Suppose $v_1, v_2, \ldots, v_k, v_{k+1}$ are eigenvectors of T with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}$ where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Consider the equation

$$\boxed{c_1 v_1 + c_2 v_2 + \ldots + c_k v_k + c_{k+1} v_{k+1} = \vec{0}} \quad (*)$$

where $c_1, c_2, \ldots, c_{k+1}$ can be in F.

Apply T to $(*)$ and use the formulas $T(v_i) = \lambda_i v_i$ and $T(\vec{0}) = \vec{0}$.

This gives

We get

$$T(c_1 v_1 + \dots + c_{k+1} v_{k+1}) = T(\vec{0})$$

which becomes

$$c_1 T(v_1) + \dots + c_{k+1} T(v_{k+1}) = \vec{0}$$

which becomes

$$\boxed{c_1 \lambda_1 v_1 + \dots + c_k \lambda_k v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0}} \quad (**)$$

Multiply $(*)$ by λ_{k+1} to get:

$$c_1 \lambda_{k+1} v_1 + \dots + c_k \lambda_{k+1} v_k + c_{k+1} \lambda_{k+1} v_{k+1} = \vec{0} \quad (***)$$

Computing $(**) - (***)$ we get

$$\boxed{c_1(\lambda_1 - \lambda_{k+1}) v_1 + c_2(\lambda_2 - \lambda_{k+1}) v_2 + \dots + c_k(\lambda_k - \lambda_{k+1}) v_k = \vec{0}} \quad (****)$$

Since we have k eigenvectors $v_1, \ldots, v_k$ with distinct eigenvalues we can apply the induction hypothesis and get that $v_1, v_2, \ldots, v_k$ are lin. ind.

Thus $(****)$ gives

$$c_1(\lambda_1 - \lambda_{k+1}) = 0$$
$$c_2(\lambda_2 - \lambda_{k+1}) = 0$$
$$\vdots$$
$$c_k(\lambda_k - \lambda_{k+1}) = 0$$

Since

$$\lambda_1 - \lambda_{k+1} \neq 0, \lambda_2 - \lambda_{k+1} \neq 0, \ldots, \lambda_k - \lambda_{k+1} \neq 0$$

we must have

$$c_1 = c_2 = \ldots = c_k = 0.$$

Plug this back into $(*)$ and get

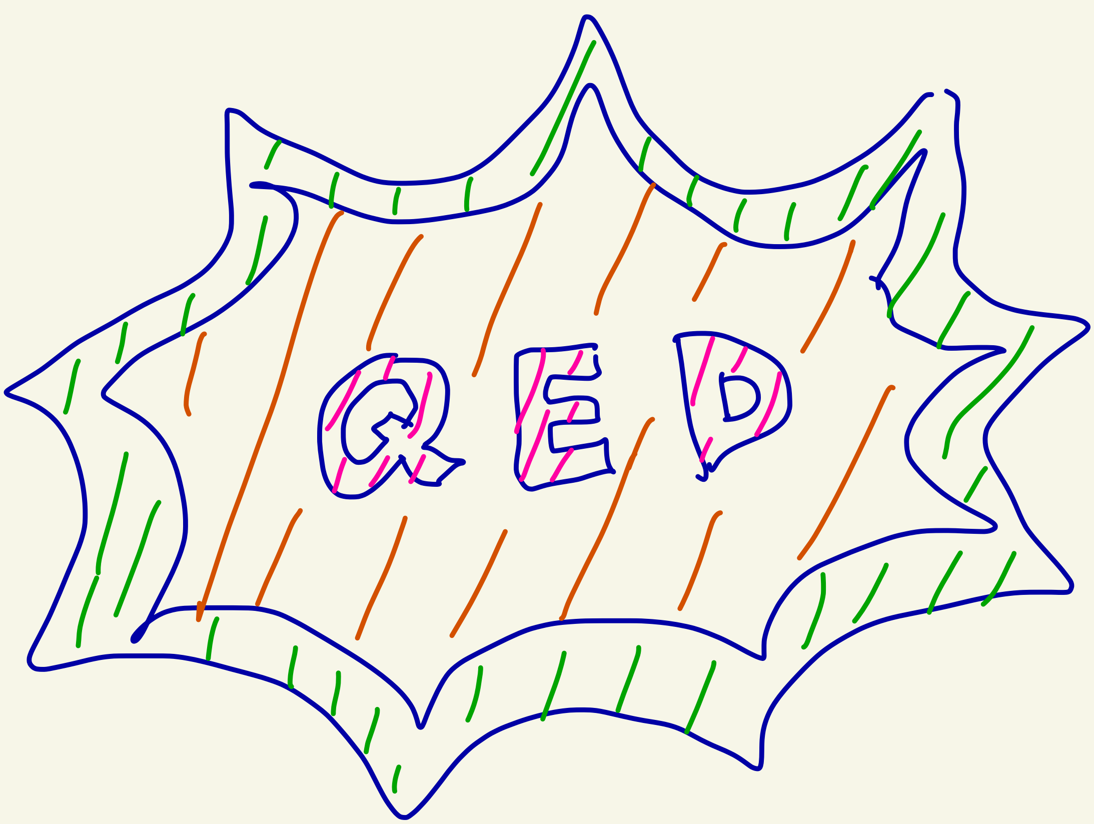
$$c_{k+1} v_{k+1} = \vec{0}$$

Since $v_{k+1} \neq \vec{0}$ the above equation gives $c_{k+1} = 0$.

Thus, $c_1 = c_2 = \dots = c_k = c_{k+1} = 0$ are the only solutions to

$$c_1 v_1 + \dots + c_k v_k + c_{k+1} v_{k+1} = \vec{0}.$$

So, $v_1, v_2, \dots, v_k$ are linearly independent.



Theorem: Let V be a finite-dimensional vector space over a field F. Let $n = \dim(V)$.
Let $T : V \to V$ be a linear transformation.
Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of T.
Let $n_1, \ldots, n_r$ be their geometric multiplicities, ie $n_i = \dim(E_{\lambda_i}(T))$

For each i, let

$$\beta_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}]$$

be an ordered basis for $E_{\lambda_i}(T)$

Let

$$\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_r$$

$$= [\, v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}, \leftarrow \text{basis for } E_{\lambda_1}(T)$$

$$v_{2,1}, v_{2,2}, \ldots, v_{2,n_2}, \leftarrow \text{basis for } E_{\lambda_2}(T)$$

$$\vdots$$

$$v_{r,1}, v_{r,2}, \ldots, v_{r,n_r} \,] \leftarrow \text{basis for } E_{\lambda_r}(T)$$

Then, β is a linearly independent set.

[However, β might not be a basis for V.]

Moreover,

β is a basis for V

iff $n_1 + \cdots + n_r = |\beta| = n$

iff T is diagonalizable.

<u>proof:</u>

We first show β is a lin. ind. set.

Suppose

$$\boxed{\sum_{i=1}^{r} \sum_{k=1}^{n_i} c_{i,k} v_{i,k} = \vec{0} \qquad (*)}$$

where $c_{i,k} \in F$.

<u>Goal:</u> Show $c_{i,k} = 0$ for all i,k.

By HW 5 #6, $E_{\lambda_i}(T)$ is a subspace of V.

Thus, since $v_{i,1}, \ldots, v_{i,n_i} \in E_{\lambda_i}(T)$

we know

$$w_i = \sum_{k=1}^{n_i} c_{i,k} v_{i,k}$$

is in $E_{\lambda_i}(T)$.

So, (*) becomes

$$\boxed{\underbrace{w_1}_{\text{in } E_{\lambda_1}(T)} + \underbrace{w_2}_{\text{in } E_{\lambda_2}(T)} + \dots + \underbrace{w_r}_{\text{in } E_{\lambda_r}(T)} = \vec{0}} \quad (**)$$

We will now show that

$$w_1 = w_2 = \dots = w_r = \vec{0}.$$

Suppose this isn't the case. By reordering/renumbering if necessary, there must then exist m with $1 \le m \le r$ and $w_i \ne \vec{0}$ if $1 \le i \le m$ and $w_i = \vec{0}$ if $m < i \le r$

$$\boxed{\underbrace{w_1, w_2, \dots, w_m}_{\text{all } \ne \vec{0}}, \underbrace{w_{m+1}, \dots, w_r}_{\text{all } = \vec{0}}}$$

Thus (**) becomes

$$\boxed{w_1 + w_2 + \dots + w_m = \vec{0} \qquad (***)}$$

But then since each w_i is in $E_{\lambda_i}(T)$ and non-zero, we have m eigenvectors $w_1, \dots, w_m$ with distinct eigenvalues $\lambda_1, \dots, \lambda_m$ satisfying the dependency relation (***)

ie $1 \cdot w_1 + 1 \cdot w_2 + \dots + 1 \cdot w_m = \vec{0}$.

This would contradict the previous lemma.

Thus, $w_1 = w_2 = \dots = w_r = \vec{0}$

So, $$\boxed{w_i = \sum_{k=1}^{n_i} c_{i,k} v_{i,k} = \vec{0} \qquad (****)}$$

for each i

But by assumption,

$$\beta_i = [v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}]$$

is a basis for $E_{\lambda_i}(T)$ and hence β is a lin. ind. set.

Thus from $(\ast\ast\ast\ast)$,

$$c_{i,k} = 0 \quad \text{for all } i, k.$$

Thus, we've done it!

$\beta = \beta_1 \cup \cdots \cup \beta_r$ is a lin. ind. set.

<u>Moreover part</u>:

Since β is a lin. ind. set and $n = \dim(V)$, β will be a basis for V iff $\underbrace{|\beta|}_{n_1+n_2+\cdots+n_r} = n = \dim(V)$

Now we will show $n = n_1 + \cdots + n_r$ iff T is diagonalizable.

[Recall: $n_i = \dim(E_{\lambda_i}(T))$, $n = \dim(V)$]

($\Leftarrow$) Suppose T is diagonalizable.

This means there exists an ordered basis γ of V of eigenvectors of T.

Let $\gamma_i = \gamma \cap E_{\lambda_i}(T)$ for $i = 1, \ldots, r$.

Then, $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_r$.

Then,

$$\underset{\dim(V)}{\underset{\uparrow}{n}} = \dim(\underbrace{\operatorname{span}(\gamma)}_{V}) = \sum_{i=1}^{r} \dim(\operatorname{span}(\gamma_i))$$

And $\dim(\underbrace{\operatorname{span}(\gamma_i)}_{\text{subspace of } E_{\lambda_i}(T)}) \leq \dim(E_{\lambda_i}(T)) = n_i$

Thus,

$$\boxed{n = \sum_{i=1}^{r} \dim(\text{span}(\gamma_i)) \leq \sum_{i=1}^{r} n_i = n_1 + \ldots + n_r}$$

But since β is a lin. ind. set with $n_1 + n_2 + \ldots + n_r$ elements and they sit inside V with $\dim(V) = n$ we must have

$$\boxed{n_1 + n_2 + \ldots + n_r \leq n.}$$

By the above two equations

$$n = n_1 + n_2 + \ldots + n_r.$$

($\Rightarrow$) Suppose that

$$\underbrace{n}_{\dim(V)} = \underbrace{n_1 + \dots + n_r}_{\#\text{ elements in } \beta}.$$

Then, β is a basis for V consisting of eigenvectors of T.

[Because we know β is a lin. ind. set. and if $|\beta| = \dim(V)$ it must span V also.]

Thus T is diagonalizable.

One more thing about eigenvalues

Let V be a finite-dimensional vector space over a field F.
Let $T: V \to V$ be a linear transformation.
Then:

① Let λ be an eigenvalue of T.

Then,

$$1 \leq \underbrace{\begin{matrix}\text{geometric} \\ \text{multiplicity} \\ \text{of } \lambda\end{matrix}}_{\dim(E_\lambda(T))} \leq \underbrace{\begin{matrix}\text{algebraic} \\ \text{multiplicity} \\ \text{of } \lambda\end{matrix}}_{\substack{\text{multiplicity of } \lambda \\ \text{as a root} \\ \text{of characteristic} \\ \text{polynomial of } T}}$$

② T is diagonalizable iff

$$\left(\begin{matrix}\text{geometric mult.} \\ \text{of } \lambda\end{matrix}\right) = \left(\begin{matrix}\text{algebraic} \\ \text{mult. of } \lambda\end{matrix}\right)$$

for all eigenvalues λ.

$\boxed{\text{HW 5 (1)(e)}}$

$T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$

$T(f) = f' + f''$

You can check this is a linear transformation

$\boxed{\text{Find eigenvalues}}$

Pick a basis for $P_3(\mathbb{R})$

$\beta = [1, x, x^2, x^3]$

$\boxed{\text{standard basis}}$

Make $[T]_\beta$

$T(1) = 0 + 0 = 0 \cdot 1 + 0x + 0x^2 + 0x^3$

$T(x) = 1 + 0 = 1 \cdot 1 + 0x + 0x^2 + 0x^3$

$T(x^2) = 2x + 2 = 2 \cdot 1 + 2x + 0x^2 + 0x^3$

$T(x^3) = 3x^2 + 6x = 0 \cdot 1 + 6x + 3x^2 + 0x^3$

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$f_T(\lambda) = \det\left([T]_\beta - \lambda I_4\right)$$

$$= \det\left(\begin{pmatrix}0&1&2&0\\0&0&2&6\\0&0&0&3\\0&0&0&0\end{pmatrix} - \begin{pmatrix}\lambda&0&0&0\\0&\lambda&0&0\\0&0&\lambda&0\\0&0&0&\lambda\end{pmatrix}\right)$$

$$= \det\begin{pmatrix}-\lambda&1&2&0\\0&-\lambda&2&6\\0&0&-\lambda&3\\0&0&0&-\lambda\end{pmatrix}$$

expand on column 1

$$= -\lambda\underbrace{\begin{vmatrix}-\lambda&2&6\\0&-\lambda&3\\0&0&-\lambda\end{vmatrix}}_{\begin{pmatrix}\cancel{-\lambda}&\cancel{1}&\cancel{2}&\cancel{0}\\\cancel{0}&-\lambda&2&6\\\cancel{0}&0&-\lambda&3\\\cancel{0}&0&0&-\lambda\end{pmatrix}} + 0 + 0 + 0$$

$$= (-\lambda)(-\lambda)\underbrace{\begin{vmatrix}-\lambda&3\\0&-\lambda\end{vmatrix}}_{\begin{pmatrix}\cancel{-\lambda}&\cancel{2}&\cancel{6}\\\cancel{0}&-\lambda&3\\\cancel{0}&0&-\lambda\end{pmatrix}} + 0 + 0$$

$$= (-\lambda)(-\lambda)\left[(-\lambda)(-\lambda) - (3)(0)\right]$$

$$= \lambda^4 = (\lambda - 0)^4$$

So, $\lambda = 0$ is the only eigenvalue with algebraic multiplicity of 4.

$\boxed{\text{Eigenspace time!}}$

$$E_0(T) = \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} T(a+bx+cx^2+dx^3) \\ = 0\cdot(a+bx+cx^2+dx^3) \end{array} \right\}$$

$$= \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} (b+2cx+3dx^2)+(2c+6dx) \\ = 0+0x+0x^2+0x^3 \end{array} \right\}$$

$$= \left\{ a+bx+cx^2+dx^3 \;\middle|\; \begin{array}{l} (b+2c)+(2c+6d)x+3dx^2 \\ = 0+0x+0x^2+0x^3 \end{array} \right\}$$

We need to solve

$$\boxed{\begin{array}{rl} b+2c & = 0 \\ 2c+6d & = 0 \\ 3d & = 0 \end{array}}$$

$$\boxed{\begin{aligned} b + 2c \quad\quad &= 0 \\ 2c + 6d &= 0 \\ 3d &= 0 \end{aligned}}$$

divide
R_2 by 2
R_3 by 3

$$\Rightarrow \boxed{\begin{aligned} b + 2c \quad\quad &= 0 \quad (1) \\ c + 3d &= 0 \quad (2) \\ d &= 0 \quad (3) \end{aligned}}$$

leading variables
b, c, d
free variable
a

$a = t$

(3) $d = 0$

(2) $c = -3d = -3(0) = 0$

(1) $b = -2c = -2(0) = 0$

Solutions:

$$\boxed{\begin{aligned} a &= t \\ b &= 0 \\ c &= 0 \\ d &= 0 \end{aligned}}$$

$$E_0(T) = \{ t \mid t \in \mathbb{R} \} = \{ t \cdot 1 \mid t \in \mathbb{R} \}$$

$$= \operatorname{span}(\{ 1 \})$$

So, $\beta = [1]$ is a basis for $E_0(\lambda)$

Thus, geometric mult. of λ is 1.

eigenvalues	$\lambda = 0$
alg. mult.	4
basis for $E_0(\lambda)$	$[1]$
geometric mult.	1

not equal

Is T diagonalizable?

Not enough eigenvectors.

We only have 1 lin. ind. eigenvector. We need 4 to diagonalize T because $\dim(P_3(\mathbb{R})) = 4$.