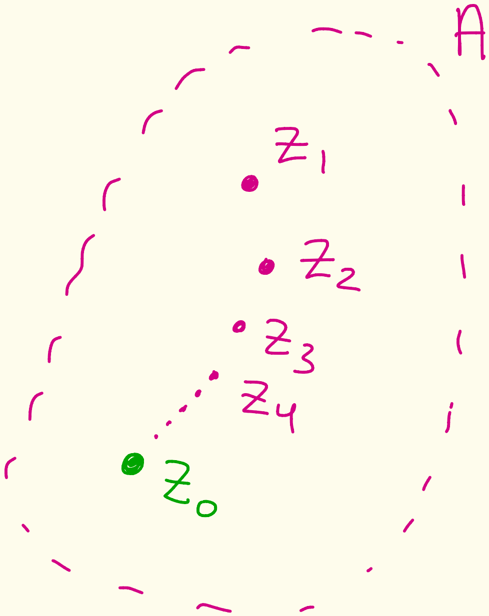
TOPIC 7 –

The Identity Theorem

Identity Theorem: Let f and g be analytic in a region A. (region = open and path-connected) Suppose that there exists a sequence $z_1, z_2, z_3, \ldots$ of distinct points in A converging to z_0 in A, such that

$$f(z_n) = g(z_n)$$

for all $n = 1, 2, 3, \ldots$



Then,

$$f(z) = g(z) \text{ for all } z \text{ in } A.$$

Proof: Will prove later

Corollary: Let f and g be analytic in a region A. Suppose $f(z) = g(z)$ for all z in some disc inside of A. Then $f(z) = g(z)$ for all z in A.

proof: Suppose $f(z) = g(z)$ for all $z \in D(z_0; r) \subseteq A$.

Let $z_n = z_0 + \frac{r}{n+1}$, $n \geq 1$.

Then each z_n is in $D(z_0; r) \subseteq A$ and $z_n \to z_0 \in A$ as $n \to \infty$

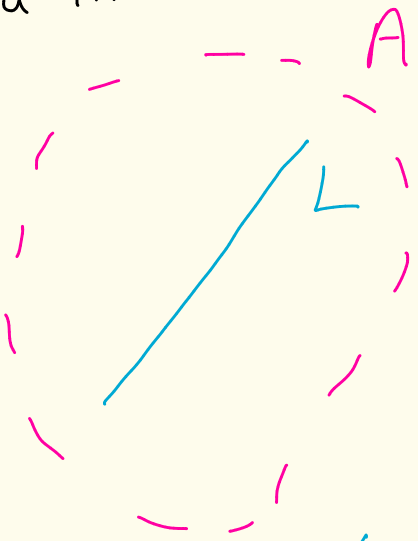
And $f(z_n) = g(z_n)$, $\forall n \geq 1$.

Thus, by the identity theorem $f(z) = g(z)$ for all z in A. ▨

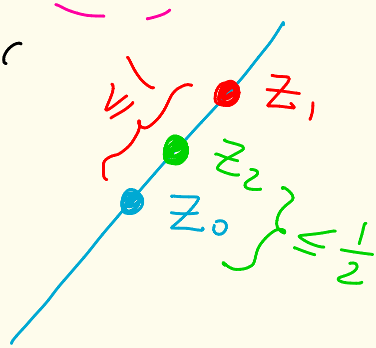
Corollary: Let f and g be analytic in a region A. Suppose there is a line segment L contained in A. Suppose $f(z) = g(z)$ for all z on L. Then

$$f(z) = g(z)$$

for all z in A.



proof: Let z_0 be near the middle of the line segment. For each $n \geq 1$ pick a point z_n on L where $|z_n - z_0| < \frac{1}{n}$



Then, $z_n \in A$ and $z_n \to z_0$.
By assumption $f(z_n) = g(z_n)$, $\forall n \geq 1$.
By the identity thm, $f(z) = g(z)$ $\forall z \in A$. ▨

Ex: Suppose f is an entire function [So, f is analytic on all of $\mathbb{C}$.]

Suppose $f(x+0i) = e^x$ for all $x \in \mathbb{R}$.

[So, f equals the real-valued exponential function on the real-line]

Claim: $f(z) = e^z$ for all $z \in \mathbb{C}$.

proof: Let L be the real axis. Then, if z is on L, we have $z = x + i0$ and

$$f(z) = f(x+i0) = e^x = e^{x+i0} = e^z$$

So, f equals e^z on L. By the identity thm, $f(z) = e^z$ on all of $\mathbb{C}$. ▨

So, there is only one way to extend the e^x from calculus / real analysis to an entire function.

It's this function

$$
\begin{aligned}
f(z) &= f(x+iy) \\
&= e^x\left[\cos(y) + i\sin(y)\right] \\
&= e^z
\end{aligned}
$$

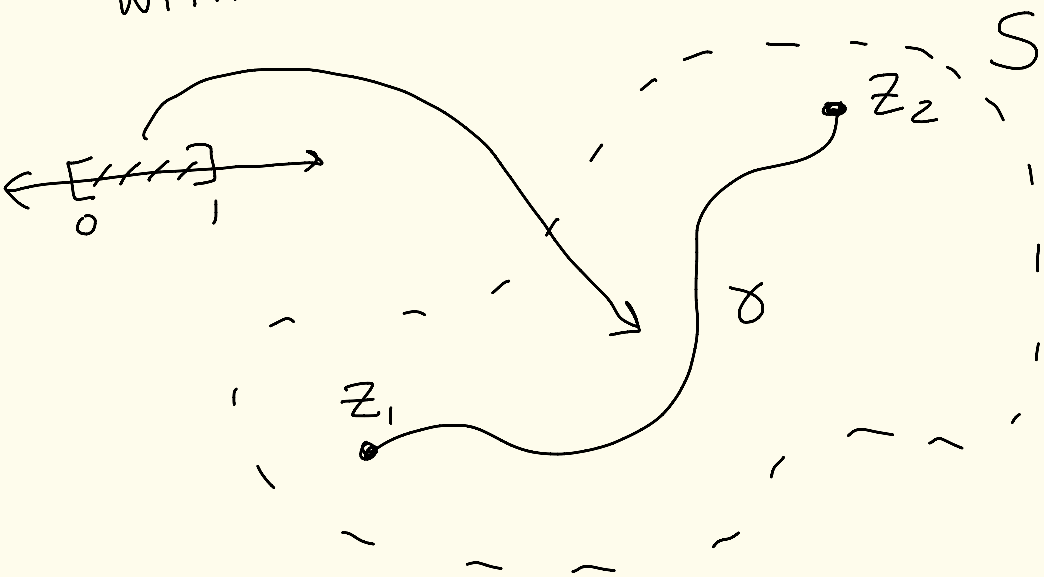
Same idea for $\sin(z)$ and $\cos(z)$.

What follows is the proof of the identity theorem

Note for Tony: Do proof after covering Rouché's thm and examples then come back and prove this and prove Rouché's thm after

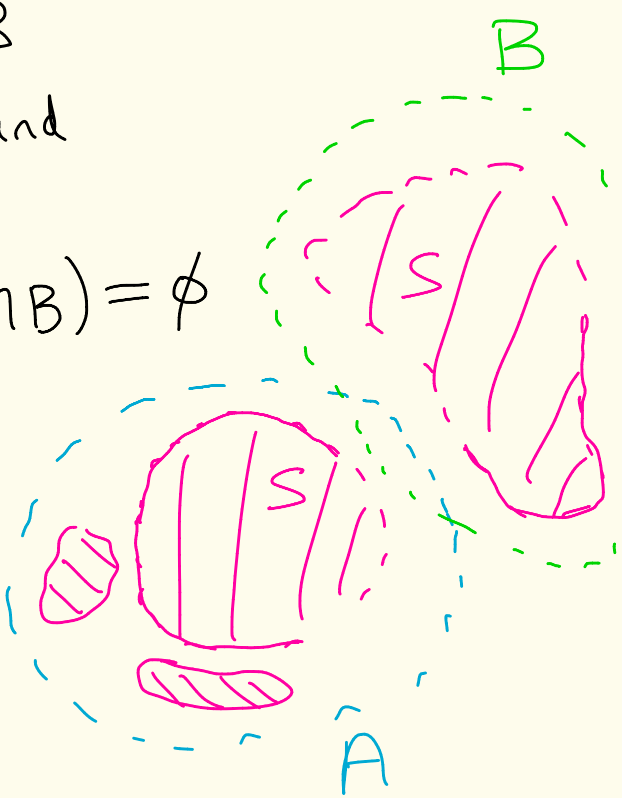
In 4680:

- **domain** is open and path-connected
- $S \subseteq \mathbb{C}$ is **path-connected** if for every pair of points $z_1, z_2 \in S$ there exists a piecewise-smooth curve $\gamma : [0,1] \to S$ with $\gamma(0) = z_1$ and $\gamma(1) = z_2$



Def: A set $S \subseteq \mathbb{C}$ is <u>disconnected</u> if there exist open sets A and B such that the following three conditions are true:

1. $S \subseteq A \cup B$
2. $S \cap A \neq \emptyset$ and $S \cap B \neq \emptyset$
3. $(S \cap A) \cap (S \cap B) = \emptyset$



If S is not disconnected then we say that S is <u>connected</u>.

<u>Theorem:</u> Let $S \subseteq \mathbb{C}$ be an open set. Then, S is connected iff S is path-connected.

<u>proof:</u>

($\Rightarrow$) Suppose S is open and connected.

Fix some arbitrary point $a \in S$.

Let

$$A = \left\{ x \in S \;\middle|\; \begin{array}{l} \text{there exists a piecewise} \\ \text{smooth curve } \gamma : [0,1] \to S \\ \text{where } \gamma(0) = a \text{ and } \gamma(1) = x \end{array} \right\}$$

Goal: Show $A = S$.

This would show that S is path-connected since $a \in S$ was arbitrary.

Suppose to the contrary that $A \neq S$. (10)

Let $B = S - A$.

(1) $S = A \cup (S-A) = A \cup B$

(2) $S \cap A = A \neq \emptyset$ because $a \in A$.

$S \cap B = B \neq \emptyset$ because we assumed $A \neq S$ so $S - A \neq \emptyset$.

(3) $(S \cap A) \cap (S \cap B) = A \cap B$

$= A \cap (S-A) = \emptyset$

(4) We next will show that A and B are both open.

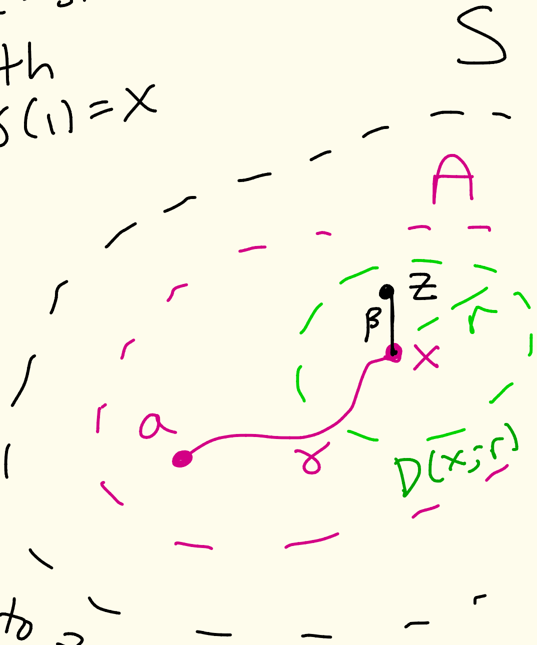
This will be a contradiction since then S would be disconnected.

A is open:

Let $x \in A$.

We will show that x is an interior point of A. Then A will be open.

Because x is in A, there exists a piecewise-smooth $\gamma : [0,1] \to S$ with $\gamma(0) = a$ and $\gamma(1) = x$



Since S is open $\exists r > 0$ where $D(x;r) \subseteq S$.

Let $z \in D(x;r)$.

Let β be the straightline from x to z.

Then β lies inside of $D(x;r) \subseteq S$.

Thus, $\gamma + \beta$ is a piecewise-smooth curve going from a to z and contained in S.

Thus, $z \in A$. So, $D(x;r) \subseteq A$. Thus, x is an interior pt of A.

$B = S - A$ is open

Let $z \in B = S - A$.

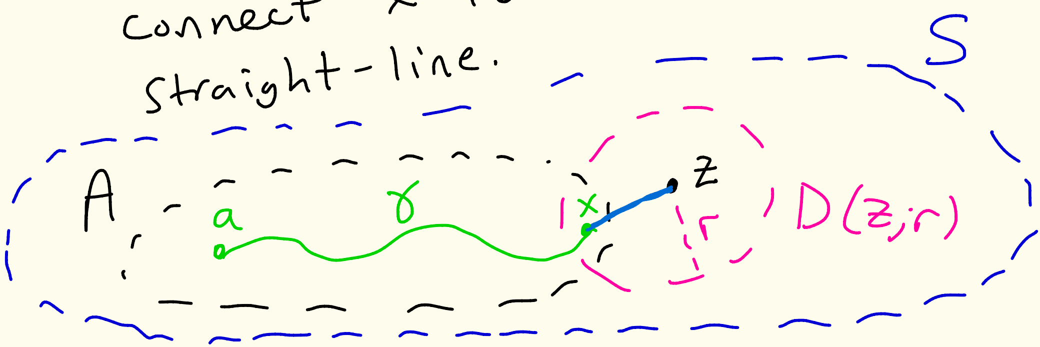
Since S is open we can find a $r > 0$ where $D(z;r) \subseteq S$.

We want to show $D(z;r) \subseteq B$ making z an interior pt of B.

Suppose not.

Then there exists $x \in A \cap D(z;r)$

Then as in the previous page we could first connect a to x (since x is in A) and then connect x to z with a straight-line.



This would imply that $z \in A$.

Can't happen.

Thus, $D(z;r) \subseteq B$.

So, z is an interior pt of B.

The above shows S would be disconnected if $A \neq S$.

Thus, $A = S$.

So, S is path-connected.

($\Leftarrow$) In Hoffman's book. $\blacksquare$

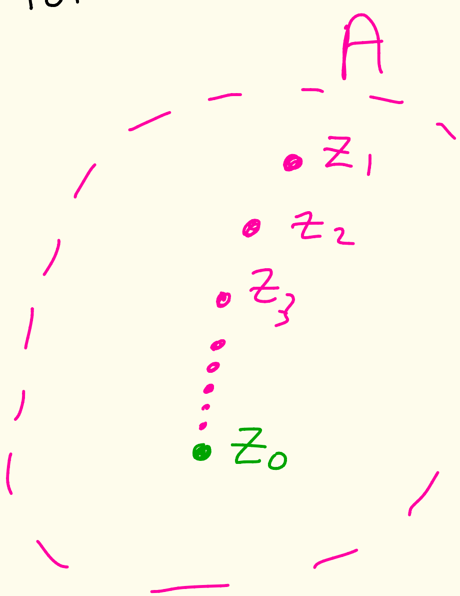
Identity Theorem

Let f and g be analytic on a region (open & path connected / connected) A. Suppose there exists a infinite sequence $z_1, z_2, z_3, \dots$ of distinct points in A converging to $z_0 \in A$. Suppose $f(z_n) = g(z_n)$ for all $n \geq 1$.

Then,

$$f(z) = g(z)$$

for all $z \in A$.



<u>proof:</u> Let $h(z) = f(z) - g(z)$.

We want to show that $h(z) = 0$, $\forall z \in A$.

We know h is analytic on A and

$$h(z_n) = 0 \quad \text{for all } n \geq 1.$$

Since h is continuous on A we have

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} h(z_n) = h\left(\lim_{n\to\infty} z_n\right) = h(z_0)$$

So, $h(z_0) = 0$.

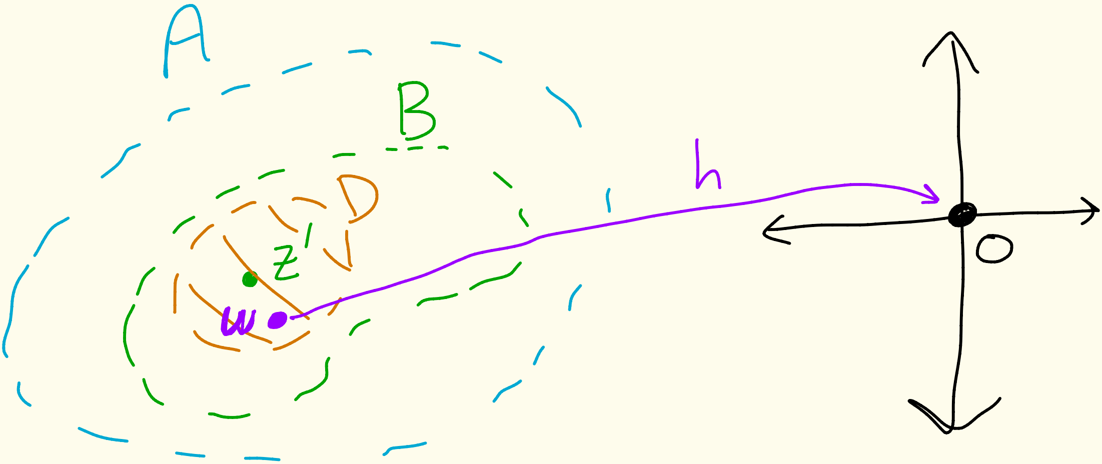
So, z_0 is not an isolated zero of h.

By HW 3 #7, there must exist a disc $D \subseteq A$ where $h(z) = 0$ $\forall z \in D$.

Let

$$B = \{ z' \in A \mid \text{there exists a disc } D \subseteq A \text{ with } z' \in D \text{ and } h(w) = 0 \text{ for all } w \in D \}$$

We know $z_0 \in B$.



Our goal is to show that $B = A$ and then $h(w) = 0$ for all $w \in A$.

Suppose $B \neq A$.

Note $B \neq \phi$ because $z_0 \in B$.

And $A - B \neq \phi$ because we assumed $A \neq B$

So, $A = B \cup (A-B)$ and

$$B \cap (A-B) = \phi.$$

If we can show that both A and $A-B$ are open then this will disconnect A and be a contradiction.

Then we will have $A = B$ and we are done.

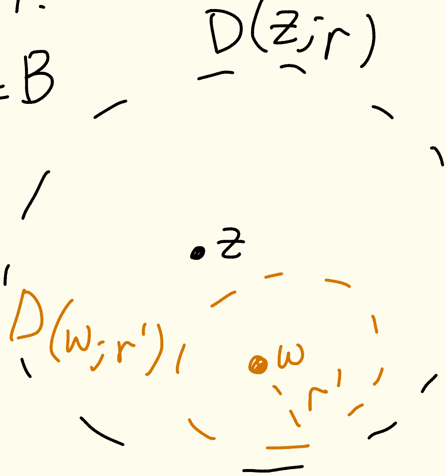
Let's do this.

B is open :

Let $z \in B$. We need to show that z is an interior point of B.

Since $z \in B$ there exists $D(z;r) \subseteq A$ with $h(w) = 0$ for all $w \in D(z;r)$.

We now show $D(z;r) \subseteq B$



Let $w \in D(z;r)$.

Pick a smaller disc $D(w;r')$ where $D(w;r') \subseteq D(z;r)$

Then h is zero on $D(w;r')$ because h is zero on $D(z;r)$.

So, $w \in B$.

Thus, $D(z;r) \subseteq B$. So, z is an interior point of B.

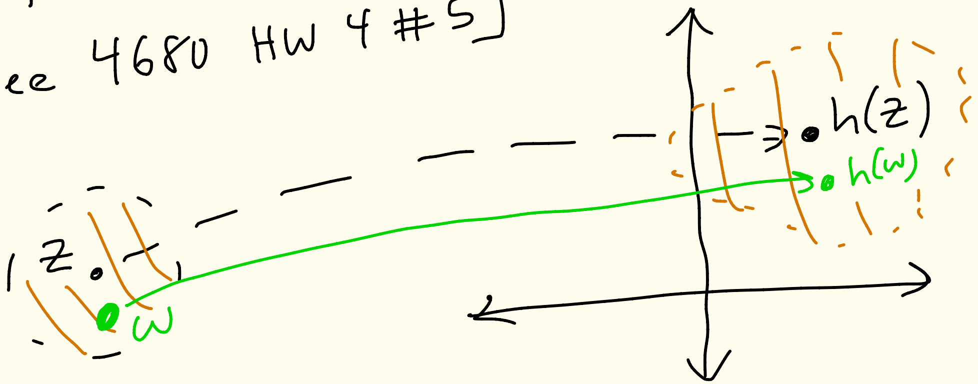
A − B is open

Let $z \in A - B$. We want to show z is an interior point of $A - B$.

Case 1: Suppose $h(z) \neq 0$.

Since h is continuous there is a disc $D \subseteq A$, with $z \in D$, and $h(w) \neq 0$ for all $w \in D$.

[See 4680 HW 4 #5]



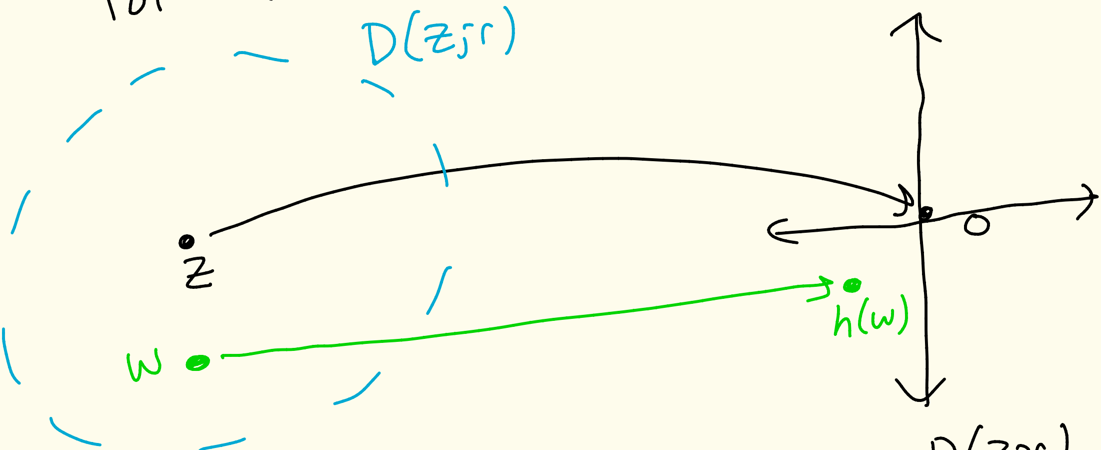
Thus, $D \subseteq A - B$. [Because if you pick $w \in D$ there is no disc around w where the whole disc goes to 0]

So, z is an interior point of $A - B$.

Case 2: Suppose $h(z)=0$

Since $z \notin B$, this means that z is an isolated zero of f.

By HW 3 #7, there is a disc $D(z;r) \subseteq A$ where $h(w) \neq 0$ for all $w \in D(z;r) - \{z\}$



Also because of this, each $w \in D(z;r)$ satisfies $w \notin B$.

Thus, $D(z;r) \subseteq A-B$.

So, z is an interior point of $A-B$.

Thus, by case 1 and case 2, $A - B$ is open.

So, if $A \neq B$, then A is disconnected. Contradiction.

So, $A = B$.

Thus, $h(w) = 0 \quad \forall w \in A$. $\blacksquare$