Topic 8 - Sequences

Sequences (HW 8)

Def: A sequence $(z_n)_{n=1}^{\infty}$ is an ordered list of complex numbers.

n is an integer

Ex: $z_n = i^n$

$z_1 = i^1 = i$

$z_2 = i^2 = -1$

$z_3 = i^3 = -i$

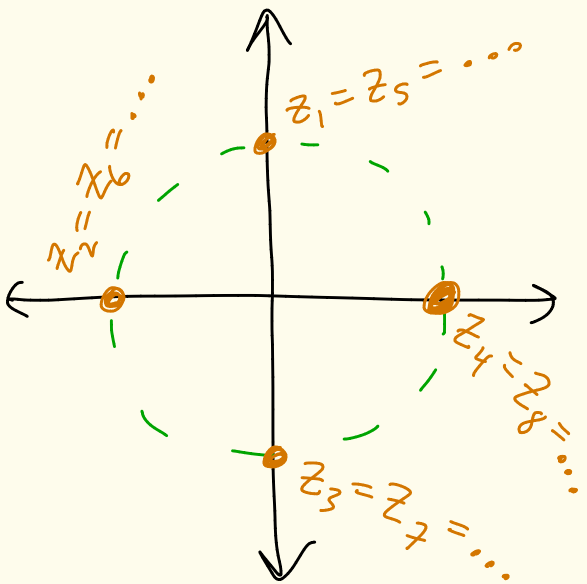
$z_4 = i^4 = 1$

$z_5 = i^5 = i$

$z_6 = i^6 = -1$

$z_7 = i^7 = -i$

$\vdots \qquad \vdots$

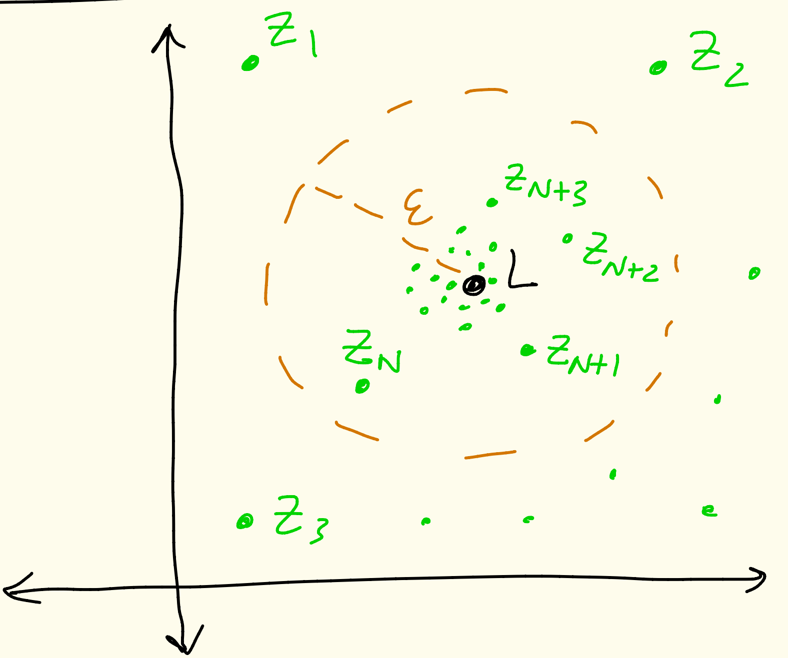


<u>Def:</u> A sequence $(z_n)_{n=1}^{\infty}$ of complex numbers <u>converges</u> to $L \in \mathbb{C}$ if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that if $n \geq N$ then $|z_n - L| < \varepsilon$.

If this is the case, then we write $\lim_{n\to\infty} z_n = L$ or $z_n \to L$

$\boxed{\text{picture}}$

Given $\varepsilon > 0$ and $N > 0$ corresponding to ε.



Thm: Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Let $L \in \mathbb{C}$.
Suppose $z_n = x_n + i y_n$ for $n \geq 1$.
Suppose $L = X + iY$.
We have that $\lim\limits_{n\to\infty} z_n = L$

iff $\underbrace{\lim\limits_{n\to\infty} x_n = X \text{ and } \lim\limits_{n\to\infty} y_n = Y}_{\text{4650/Calculus limits}}$.

proof: $(\Leftarrow)$ Suppose $\lim\limits_{n\to\infty} x_n = X$ and $\lim\limits_{n\to\infty} y_n = Y$. Let's show $\lim\limits_{n\to\infty} z_n = L$.

Let $\varepsilon > 0$.
Since $x_n \to X$, there exists $N_1 > 0$ where if $n \geq N_1$ then $|x_n - X| < \varepsilon/2$.
Since $y_n \to Y$, there exists $N_2 > 0$ where if $n \geq N_2$ then $|y_n - Y| < \varepsilon/2$.

4650 def

Let $N = \max\{N_1, N_2\}$.

If $n \geq N$, then

$$
\begin{aligned}
|z_n - L| &= |(x_n + iy_n) - (X + iY)| \\
&= |(x_n - X) + i(y_n - Y)| \\
&\leq |x_n - X| + |i(y_n - Y)| \\
&= |x_n - X| + \underbrace{|i|}_{1}\,|y_n - Y| \\
&= |x_n - X| + |y_n - Y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Thus, if $n \geq N$, then $|z_n - L| < \varepsilon$.

So, $\lim\limits_{n\to\infty} z_n = L$.

(5)

($\Rightarrow$) Suppose $\lim\limits_{n\to\infty} z_n = L = X + iY$

Let's show $\lim\limits_{n\to\infty} x_n = X$ and $\lim\limits_{n\to\infty} y_n = Y$

Let $\varepsilon > 0$.

Since $\lim\limits_{n\to\infty} z_n = L$ we have that there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \varepsilon$.

So if $n \geq N$, then

$$|x_n - X| = |\mathrm{Re}(z_n - L)| \leq |z_n - L| < \varepsilon.$$

$\boxed{|\mathrm{Re}(w)| \leq |w|}$

So $\lim\limits_{n\to\infty} x_n = X$.

$\boxed{|\mathrm{Im}(w)| \leq |w|}$

Also if $n \geq N$, then

$$|y_n - Y| = |\mathrm{Im}(z_n - L)| \leq |z_n - L| < \varepsilon.$$

So, $\lim\limits_{n\to\infty} y_n = Y$

$\blacksquare$

Ex: $z_n = \underbrace{\frac{1}{n}}_{x_n} + i\underbrace{\left(\frac{\sin(n)}{n} + 5\right)}_{y_n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(\frac{\sin(n)}{n} + 5\right) = 0 + 5$$

$$= 5$$

$$\boxed{-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}, \quad -\frac{1}{n} \to 0, \quad \frac{1}{n} \to 0}$$

by squeeze thm $\frac{\sin(n)}{n} \to 0$

So, $\lim_{n\to\infty} z_n = 0 + i(5) = 5i$

$\boxed{\lim_{n\to\infty} x_n}$ $\boxed{\lim_{n\to\infty} y_n}$

Def: A sequence of complex numbers (z_n) is **Cauchy** if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that if $n, m \geq N$ then $|z_n - z_m| < \varepsilon$

Theorem: A sequence (z_n) of complex numbers is Cauchy iff there exists $w \in \mathbb{C}$ with $\lim_{n\to\infty} z_n = w$.

proof: $(\Leftarrow)$ Suppose $\lim_{n\to\infty} z_n = w$ where $w \in \mathbb{C}$.

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} z_n = w$, there exists $N > 0$ where if $n \geq N$, then $|z_n - w| < \frac{\varepsilon}{2}$.

Then if $n, m \geq N$ we have

$$
\begin{aligned}
|z_n - z_m| &= |z_n - w + w - z_m| \\
&\leq |z_n - w| + |w - z_m| \\
&= |z_n - w| + |-(z_m - w)| \\
&= |z_n - w| + |z_m - w| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

$|-z| = |z|$

So, (z_n) is Cauchy.

($\Leftarrow$) Suppose that (z_n) is Cauchy. 9

Let $z_n = x_n + i y_n$ for all n.

By HW 8 #3, since (z_n) is Cauchy, both (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$.

From 4650, since (x_n) and (y_n) are Cauchy and $\mathbb{R}$ is complete, there exists x and y in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Let $w = x + iy$. So, $w \in \mathbb{C}$.

And, $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$

$= x + iy = w$. $\blacksquare$