Topology Comprehensive Examination, Fall 2021.

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Do five out of seven problems. If you attempt more than five, the best five will be considered. The symbol $\mathbb{R}^2$ denotes the plane with the standard (Euclidian) topology.

1. Let X be a set, and let $\mathcal{T} = \{U \subseteq X\colon \text{the complement } X\backslash U \text{ is finite or } U = \emptyset\}$.

a. Show $\mathcal{T}$ is a topology for X. We call it the co-finite topology.

b. Let X be equipped with the co-finite topology, and let $A \subseteq X$. Describe the boundary ∂A of A.

Hint: Consider all cases where A or $X\backslash A$ is finite or infinite.

2. Suppose X is a compact Hausdorff space. Prove a subset A of X is compact, if and only if, A is a closed subset of X.

3. Let $D = \{(x, y)\colon x^2 + y^2 \leq 1\}$, be the closed unit disk in the plane $\mathbb{R}^2$. Prove or disprove that the open punctured disk $\{(x, y)\colon 0 < x^2 + y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2\backslash D$.

4. Let X be a topological space. A component of $A \subseteq X$ is a connected subset of A that is not a proper subset of a connected subset of A.

a. Show that each point of X belongs to one and only one component of X.

b. Prove or disprove that each component of an open set $G \subseteq X$ is open.

c. Prove or disprove that each component of a closed set $F \subseteq X$ is closed.

5. A topological space X is Lindelof if every open cover of X has a countable subcover. Prove that the plane $\mathbb{R}^2$ is Lindelof.

6. Let X be a set, and let $\mathcal{T}_0, \mathcal{T}_1$ be topologies on X. If $\mathcal{T}_0 \subseteq \mathcal{T}_1$, we say that $\mathcal{T}_1$ is finer than $\mathcal{T}_0$ (or equivalently $\mathcal{T}_0$ is coarser than $\mathcal{T}_1$).

a. Suppose Y is a set with topologies $\mathcal{T}_0, \mathcal{T}_1$ such that the identity map $id_Y: (Y, \mathcal{T}_0) \to (Y, \mathcal{T}_1)$ is continuous. What is the relationship between $\mathcal{T}_0$ and $\mathcal{T}_1$ (is one finer than the other?). Justify your claim. Note: $id_Y(a) = a$, for all $a \in Y$.

b. Suppose Y is a set with topologies $\mathcal{T}_0, \mathcal{T}_1$ where $\mathcal{T}_1$ is finer than $\mathcal{T}_0$. What does connectedness in one topology imply about connectedness in the other?

c. Under the same assumptions as in b, what does convergence in one topology imply about convergence in the other.

7. Let $(X, \mathcal{T}_0), (Y, \mathcal{T}_1)$ denote topological spaces. Recall that a function $f: X \to Y$ is continuous, if $f^{-1}(U) \in \mathcal{T}_0$, whenever $U \in \mathcal{T}_1$. Show $f: X \to Y$ is continuous, if and only if, for every $A \subseteq X$, $f(p) \in \overline{f(A)}$ whenever $p \in \bar{A}$.

Note: $\bar{A}$ denotes the closure of A in X, and $\overline{f(A)}$ denotes the closure of $f(A)$ in Y.