# Topology Comprehensive Examination - Spring 2021

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**Do 5 out of 7:**

1. Let $X$ be a Hausdorff space, and let $p \in X$.

   a. Prove that the boundary of $\{p\}$ is either the empty set or $\{p\}$.

   b. Give an example where the boundary of $\{p\}$ is the empty set.

   c. Give an example where the boundary of $\{p\}$ is $\{p\}$.

2. Consider the set of all real numbers endowed with the lower-limit topology, whose neighborhood basis consists of all half-open intervals of the form $[a, b) = \{x: a \leq x < b\}$. What are the connected components of this topological space? Prove your answer is correct.

3. We say a topological space $X$ is Lindelof if every open cover of $X$ has a countable subcover. Prove that every second-countable space is Lindelof.

4. Let $\mathbb{R}$ denote the set of real numbers endowed with the standard (Euclidean) topology. Either exhibit a homeomorphism between the following subsets of $\mathbb{R}$, or prove that one does not exist.

   a. $[0,2] = \{x: 0 \leq x \leq 2\}$ and $[1,3] = \{x: 1 \leq x \leq 3\}$

   b. $[0,2] = \{x: 0 \leq x \leq 2\}$ and $(-\infty, 0] = \{x: x \leq 0\}$

   c. $\mathbb{R}$ and $\mathbb{R}^2$ (the 2-dimensional Euclidean space)

5. Let $(X,d)$ be a metric space.

a. Let $p \in X$, and let $\langle x_n \rangle \subseteq X$ be a sequence converging to $x \in X$. Prove that $\lim_{n\to\infty} d(x_n,p) = d(x,p)$.

b. Suppose $A \subseteq X$ is compact, and $x \in X \setminus A$. Prove that there exists $a_0 \in A$, such that, $d(x,a_0) = \inf_{a\in A} d(x,a)$.

6. A topological space is said to have the *fixed-point property* if for every continuous function $f: X \to X$ there exists $p \in X$, such that, $f(p) = p$. Prove that if $X$ has the fixed-point property, then $X$ must be connected.

7. Suppose $X$ is a nonempty set, and let $p \in X$. Consider the family of sets $\tau = \{G \subseteq X : p \in G\} \cup \{\emptyset\}$. Prove the following:

a. The family $\tau$ is a topology for $X$.

b. The closure of $\{p\}$ is $X$.

c. $(X,\tau)$ is separable.

d. If $X$ is uncountable, then $X \setminus \{p\}$ (with the subspace topology) is not separable.