Comprehensive Examination – Topology

Fall 2000

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Do any five of the problems that follow. Each problem is worth 20 points. All spaces are assumed to be Hausdorff.

1. (a) Let B be a nonempty subset of a topological space X. Show that the set of accumulation (limit) points of B is a closed set.

(b) Show that each finite subset E of X is closed.

(c) Let F be a closed subset of X and $h : X \to Y$ be continuous. Must $h(F)$ be closed in Y? Either prove that this is true or present a counterexample.

2. (a) Let X be a compact Hausdorff space. Prove that X is regular.

(b) Let X be a metrizable space. Prove that X is regular.

3. Let C be a (connected) component of the product space $X \times Y$. Prove that $C = A \times B$, where A is a component of X and B is a component of Y.

4. Let $f : X \to Y$ and let $g : X \to Y$ be continuous.

(a) Prove that $h : X \to Y \times Y$ defined by $h(x) = (f(x), g(x))$ is continuous.

(b) Prove that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

5. Let X be a second countable space.

(a) Show that X is separable, i.e., that X has a countable dense subset.

(b) Show that each open cover $\{V_i,\ i \in I\}$ of X has a countable subcover.

6. Let (X, d) be a metric space. If $E \subset X$, define the diameter of E by the formula

$$\operatorname{diam}(E) = \sup\{d(x,y) : x \in E,\ y \in E\}.$$

(a) Prove that for any subset E of X, $\operatorname{diam}(E) = \operatorname{diam}(\overline{E})$.

(b) Suppose (E_n) is a decreasing sequence of nonempty closed sets in a complete metric space (X, d) with $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$. Prove that $\bigcap_{n=1}^{\infty} E_n$ is nonempty.

7. Let $f : X \to Y$ be continuous.

(a) Show that the graph of f defined by $\Gamma(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

(b) Suppose X is compact. Prove that $\Gamma(f)$ is also compact.