Comprehensive Examination – Topology

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G. Beer, P. Chabot, A. Verona*

Do five problems, including the first one. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of rationals by Q, and the set of real numbers by R. The notation A^c means the complement of the set A with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Explain carefully the following concepts:

(a) Locally connected space.

(b) Uniformly continuous function between two metric spaces.

(c) Normal space.

(d) Separable space and second countable space.

(e) Compact topological space.

(f) Basis for a topology.

2. Let d_1 and d_2 be metrics on the same space X and let τ_1 and τ_2 be the corresponding topologies. Prove that the following are equivalent:

(a) Whenever $(x_n) \xrightarrow{d_1} x$ then $(x_n) \xrightarrow{d_1} x$ (that is whenever a sequence converges with respect to d_1 it also converges with respect to d_2, and to the same limit).

(b) $\tau_1 \subseteq \tau_2$

3. Let X and Y be topological spaces with X connected and let $f : X \to Y$ be continuous.

(a) Prove that $f(X)$ is connected.

(b) Prove that $\Gamma(f) = \{(x, f(x)) : x \in X\}$ is connected as a subspace of $X \times Y$.

4. On R consider the following family of subsets:

$$\tau = \{D \subseteq R : D = \emptyset, \text{ or } D = R \text{ or the complement of } D \text{ is countable}\}.$$

(a) Prove that τ is a topology on R.

(b) Show that any convergent (in τ) sequence is eventually constant (that is, whenever $\lim_{n\to\infty} x_n = x$ there exists N_0 such that $x_n = x$ if $n > N_0$).

5. Prove that a topological space is (Hausdorff) if and only if $D = \{(x, x) \in X \times X : x \in X\}$ is closed as a subspace of $X \times X$ endowed with the product topology.

6. Prove that every (Hausdorff) compact regular space is normal.