Topology Comprehensive Exam Fall 2008

(Beer, Krebs, Verona)

Do 5 of the 7 problems below; each is worth 20 points.

1. Explain precisely each of the following
   (a) regular topological space $\langle X, \tau \rangle$
   (b) Urysohn's Lemma
   (c) Cauchy sequence $\langle x_n \rangle$ in a metric space $\langle X, d \rangle$
   (d) boundary of a subset A of a topological space $\langle X, \tau \rangle$
   (e) product topology for a product of topological spaces $\{\langle X_i, \tau_i \rangle : i \in I\}$
   (f) path connected topological space $\langle X, \tau \rangle$
   (g) bounded subset B of a metric space $\langle X, d \rangle$.

2. In the plane $\mathbb{R}^2$ with the usual metric, put $A := \{(x, y) : x^2 + y^2 = 1\}$ and put $B := \{(x, y) : x^2 + y^2 = 1 \text{ and } x \geq 0\}$.
   (a) Explain why A and B are both connected.
   (b) Explain why A and B are not homeomorphic.

3. Give an example of
   (a) a connected topological space that is not locally connected;
   (b) a topological space that is not Hausdorff;
   (c) a continuous bijection between topological spaces that is not a homeomorphism.

4. Consider $\mathbb{R}^2$ equipped with the topology τ having as a subbase all sets of the form $U_a := \{(x, y) : x \neq a\}$ where $a \in \mathbb{R}$.
   (a) Prove that $V \in \tau$ iff for some finite subset F of $\mathbb{R}$, $V = \{(x, y) : x \notin F\}$.
   (b) Prove that $\langle \mathbb{R}^2, \tau \rangle$ is compact.

5. Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces.
   (a) Show that if $f : X \to Y$ is continuous, and $A \subseteq X$, then $f|A$ is continuous.
   (b) Suppose A, B are closed subsets of X with union X and $f|A$ and $f|B$ are both continuous. Prove f is continuous.

   In the above both A and B are of course equipped with the relative (subspace) topology.

6. Suppose we equip the $[0, 1] \times [0, 1]$ with the dictionary order:

   $$(x_1, y_1) \preceq (x_2, y_2) \text{ if either } x_1 < x_2, \text{ or } x_1 = x_2 \text{ and } y_1 \leq y_2.$$

   Giving the square the order topology, show

   (a) $\{(x, y) : 0 < x < 1\}$ is open; (b) $\{(x, y) : y = \frac{1}{2}\}$ is not closed.

7. Let $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ be metric spaces. Equip $X \times Y$ with the metrics $\rho_1((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$ and $\rho_2((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$. Prove that ρ_1 and ρ_2 determine the same open sets in the product.